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ω_1 and $-\omega_1$ May Be the Only Minimal Uncountable Linear Orders

Justin Tatch Moore

For Stephanie

1. Introduction

In 1971 Laver proved the following result, confirming a long-standing conjecture of Fraïssé.

Theorem 1.1 [10]. *If L_i $(i < \omega)$ is a sequence of σ-scattered linear orders, then there exist $i < j$ such that L_i is embeddable into L_j. In particular, the σ-scattered orders are well-founded when given the quasi-order of embeddability.*

Here a linear order is *scattered* if it does not contain a copy of the rationals; a linear order is *σ-scattered* if it is a countable union of scattered suborders.

Around the same time, Baumgartner proved the following theorem. (As usual, ZFC is used to denote "Zermelo–Fraenkel set theory with the axiom of choice" and MA to denote "Martin's axiom".)

Theorem 1.2 [3]. *It is relatively consistent with ZFC—and follows from the proper forcing axiom (PFA)—that any two $\aleph_1$-dense sets of reals are order-isomorphic.* (A linear order is $\aleph_1$-*dense* if every proper interval contains $\aleph_1$ elements.) *In particular, PFA implies that any subset of $\mathbb{R}$ of cardinality $\aleph_1$ is minimal with respect to not being σ-scattered.*

It is therefore consistent that Laver's theorem is not sharp. However, it is unclear whether ZFC alone implies that there is a linear order that is minimal with respect to not being σ-scattered.

Question 1.3. Is there a linear order that is not σ-scattered and that is minimal in this regard?

Two important classes of linear orders that are not σ-scattered are the real types and the Aronszajn types: the *real types* are those uncountable dense linear orders that are separable; the *Aronszajn types* are those linear orders that are uncountable and yet have no uncountable suborders that are scattered or real types. This latter class was considered—and proved nonempty—long ago by Aronszajn and Kurepa in the context of Souslin's problem. (The existence of Aronszajn lines was later

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rediscovered by Specker and thus they are sometimes referred to as *Specker types* in the literature; see [18, 5.15] for a historical discussion of this point.) Both of these classes have the property that they are closed under taking uncountable suborders. Hence a model of set theory in which Question 1.3 has a negative answer cannot contain minimal real or Aronszajn types. This motivates the following question.

QUESTION 1.4. Is there a ZFC example of a minimal uncountable linear order other than ω_1 or $-\omega_1$?

Here $-\omega_1$ (often denoted ω_1^* in the literature) signifies the reverse of ω_1. By the following classical result of Sierpiński, the continuum hypothesis (CH) implies that there are no minimal real types.

THEOREM 1.5 [16]. *If $X \subseteq \mathbb{R}$ and $|X| = |\mathbb{R}|$, then there is a $Y \subseteq X$ with $|Y| = |\mathbb{R}|$ such that, if $f \subseteq Y^2$ is a monotonic function, then f differs from the identity on a set whose cardinality is less than $|\mathbb{R}|$.*

The picture was less clear for Aronszajn lines. An important class of Aronszajn lines considered by Countryman and proved nonempty by Shelah [14] consists of the *Countryman lines*: uncountable linear orders C such that the coordinatewise partial order on C^2 is the union of countably many nondecreasing relations. The following theorem shows that, under a fairly mild set-theoretic assumption, such linear orders are minimal. (It is not entirely clear when this was first known or who discovered the proof; a weaker statement was conjectured in [14] and repeated again in [4], and a proof can be obtained using techniques in [20].)

THEOREM 1.6. *$MA(\aleph_1)$ implies that every Countryman line is minimal.*

In fact, this argument can be adapted to show that if a pair of Countryman lines are $\aleph_1$-dense then they are either isomorphic or reverse-isomorphic.

It is natural to suspect that there should be an analog of Sierpiński's result for Aronszajn lines—that CH (or a stronger enumeration principle such as $\diamondsuit$) implies that there are no minimal Aronszajn lines. In a somewhat surprising twist, however, Baumgartner proved the following result, which rules out the conventional method for proving that Question 1.3 is independent.

THEOREM 1.7 [4]. *$\diamondsuit^+$ implies that there is a minimal Aronszajn line.*

Baumgartner also noted that his construction in Theorem 1.7 produces a Souslin line and asked whether this is necessarily the case.

In this paper I will prove that the answer to Question 1.4 is negative. This can be viewed as a companion to the following result.

THEOREM 1.8 [12]. *PFA implies that the class of uncountable linear orders has a five-element basis consisting of $X, \omega_1, -\omega_1, C$, and $-C$, where X is an arbitrary set of reals of cardinality $\aleph_1$ and C is a Countryman line.*

The members of this basis are each minimal and canonical (assuming PFA). The main result of this paper shows that some hypothesis is needed to draw this conclusion in all but the trivial cases.

The main ingredient in the proof is a variation on the notion of uniformizing a ladder system coloring. A *ladder system coloring* is a sequence $\langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ such that, for each $\alpha \in \lim(\omega_1)$, the domain of f_α is a ladder C_α on α and the range of f_α is contained in ω. Here a *ladder* on a countable limit ordinal α is a cofinal subset of α that has ordertype ω.

Whether all ladder system colorings can be *uniformized*—when there is a $\varphi\colon \omega_1 \to \omega$ such that $f_\alpha =^* \varphi \restriction C_\alpha$ for all relevant α—turns out to be of interest both in pure combinatorial set theory and in applications. (The notation $=^*$ will be explained shortly.) For instance, a variation on this theme played a crucial role in the solution of Whitehead's problem (see [13]).

Devlin and Shelah [6] have shown that the assertion

(U) all ladder system colorings can be uniformized

implies $2^{\aleph_0} = 2^{\aleph_1}$. This is rather remarkable because, for any given ladder system coloring, there is a proper forcing that uniformizes it yet does not introduce real numbers. The obstruction to obtaining the consistency of (U) with CH is therefore in the inability to iterate these forcings without introducing reals.

In this paper we will consider a weaker variant (A) of (U) that is consistent with CH:

(A) every ladder system coloring can be T-uniformized for every Aronszajn tree T

(see Section 2). This variant is of interest because, in the presence of a minimal Aronszajn line, it implies that $2^{\aleph_0} = 2^{\aleph_1}$. Hence the conjunction of (A) and CH implies that there are no minimal Aronszajn lines and so, by Sierpiński's result (Theorem 1.5), there are no minimal uncountable linear orders other than ω_1 and $-\omega_1$.

The main results of this paper are as follows.

THEOREM 1.9. *There is a proper forcing extension in which the continuum hypothesis and* (A) *are both true.*

THEOREM 1.10. *If* (A) *is true and there is a minimal Aronszajn line, then* $2^{\aleph_0} = 2^{\aleph_1}$.

THEOREM 1.11. *It is consistent that* ω_1 *and* $-\omega_1$ *are the only minimal uncountable linear orders.*

It is worth noting at this point that, although the major technical difficulty of this paper is in proving Theorem 1.9, the key idea for proving the main result was the realization that Theorem 1.10 is true. A more direct approach to Theorem 1.11 would be to introduce, for a given Aronszajn line L, a suborder X of L into which L cannot embed and then argue that this procedure can be iterated *while preserving that L does not embed into X.* In fact, though hindsight will suggest that such preservation may be possible, it still seems to be a daunting task. This is especially true if one wishes to obtain models satisfying Theorem 1.11 together with other hypotheses (see the discussion in Section 6). Theorem 1.10 allows us to take a less direct approach by linking the introduction of embeddings to the introduction of real numbers, a phenomenon that has already been extensively studied and has a relatively well-developed theory (see [15]).

In fact, the discovery of the proof is a direct consequence of studying the possibility of maximizing Π_2-sentences for $H(\aleph_1^+)$ in the presence of CH. An example related to my original motivations is discussed in Section 6. I would like to thank Ali Enayat for bringing Question 1.4 to my attention at just the right time. I have also taken this paper as an opportunity to present a framework for showing that an iteration of proper forcings does not introduce reals, which I feel makes the tasks at hand more transparent. This is the content of Section 4.

Some attempt has been made to keep this paper fairly self-contained—provided the reader is fluent in modern set-theoretic techniques. The reader is assumed to have some knowledge of proper forcing in addition to the usual proficiency in set theory. For the most part I will follow the notation in [9], to which the reader is referred for background in basic set theory and forcing. I will use the language from category theory in parts of the paper; [11] will be used as the standard reference. Further information on iterated proper forcing can be found in [15]. All ordinals considered in this paper are von Neumann ordinals: they are the set of their predecessors. In particular, ω is the set of all finite ordinals and ω_1 is the set of all countable ordinals. If two functions f and g have a common domain D and if $\{x \in D : f(x) \neq g(x)\}$ is finite, then I will write $f =^* g$; if f is constantly i except on a finite set, then I will write $f \equiv^* i$. Throughout the paper, θ will always refer to an uncountable regular cardinal. For a given θ, $H(\theta)$ will denote the collection of all sets of hereditary cardinality less than θ; hence $H(\aleph_1^+)$ consists of all sets of hereditary cardinality at most $\aleph_1$. (The reader may be puzzled by the use of $\aleph_1^+$ instead of $\aleph_2$: the point is that $H(\aleph_1^+)$ is more suggestive of the typical cardinality of its members.) These structures are of interest since, for a given θ, $H(\theta)$ satisfies all of the axioms of ZFC except the power set axiom and is closed under taking subsets. Such structures will always be tacitly equipped with a fixed well-ordering that is used to generate Skolem functions for the structure.

One major departure from the norm will be the emphasis on countable transitive set models. It will be convenient to utilize the following specialized notation. If $\hat{M}$ is a set, then there is a unique transitive set M and a unique *collapsing isomorphism* from $(\hat{M}, \in)$ to $(M, \in)$. (A set M is *transitive* if every element of M is also a subset of M.) If an object is first named as $\hat{M}$, then M will denote its transitive collapse. Furthermore, if X is an element of $\hat{M}$, then X^M will denote the result of applying $\hat{M}$'s collapsing isomorphism to X. If ε is a function and X is a subset of the domain of ε, then εX will denote the image of X under ε.

2. Background on Trees

In this section I will present some background on trees and fix some notation. The reader who is familiar with trees may wish to skip this section and refer to it if any of the notation is unfamiliar. Further reading as well as some historical discussion can be found in [18].

Recall that a *tree* is a partial ordering $(T, <)$ in which every set of the form $\{s \in T : s < t\}$ for t in T is well-ordered by $<$. The ordertype of this set is called the *height* of t. All trees considered in this paper are, moreover, *Hausdorff*: if $t \neq t'$ and if both have limit height then each has a different set of predecessors.

The set of all elements of T of a given height δ is denoted T_δ and is called the δth *level* of T. This allows us to make the following definitions.

DEFINITION 2.1. If t is in T and if α is an ordinal, then $t \upharpoonright \alpha$ is t if α is at least the height of t; otherwise, $t \upharpoonright \alpha$ is the element s of T such that $s < t$ and the height of s is α.

DEFINITION 2.2. If s and t are incomparable elements of T, then $\Delta(s,t)$ is the greatest ordinal ζ such that $s \upharpoonright \zeta = t \upharpoonright \zeta$.

We can also use restriction to define an abstract notion of a lexicographical ordering on a tree.

DEFINITION 2.3. If $(T, <)$ is a tree, then a linear ordering $\leq_{\text{lex}}$ on T is a *lexicographical ordering* if, whenever s and t are incomparable elements of T, $s \leq_{\text{lex}} t$ is equivalent to $s \upharpoonright (\zeta + 1) \leq_{\text{lex}} t \upharpoonright (\zeta + 1)$, where $\zeta = \Delta(s,t)$.

Our interest in trees will be limited to *Aronszajn trees*—those that are uncountable but have countable levels and branches.

THEOREM 2.4 (see [18, 5.1]). *Every Aronszajn line is embeddable into a lexicographical ordering on an Aronszajn tree.*

I will also need the notion of a subtree of an Aronszajn tree. (There is no universal definition of the term "subtree"; the following is the most appropriate definition for the discussion in this paper.)

DEFINITION 2.5. A *subtree* of T is an uncountable subset U of T that is downwards closed: if u is in U and $t \leq u$, then t is in U. If every element of U has uncountably many extensions in U, then U is said to be *pruned*.

REMARK 2.6. If T is an Aronszajn tree and U is a subtree of T, then it is well known and readily verified that the set U' of all u in U that have uncountably many extensions in U is a pruned subtree of U. In particular, every subtree of an Aronszajn tree contains a pruned subtree.

Now we can formulate the statement (A) introduced in Section 1. Let T be an Aronszajn tree and let $\vec{f} = \langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ be a ladder system coloring.

DEFINITION 2.7. The coloring $\vec{f}$ can be T-*uniformized* if there is a subtree U of T and a function $\varphi\colon U \to \omega$ such that, if u is an element of U of limit height α, then $f_\delta =^* \varphi[u] \upharpoonright C_\alpha$. Here $\varphi[u]\colon \alpha \to \omega$ is defined by $\xi \mapsto \varphi(u \upharpoonright \xi)$.

The statement (A) is then the assertion that every ladder system coloring can be T-uniformized for every Aronszajn tree T. Observe that this is a weaker statement than (U); it becomes equivalent if we require that the upper bound on the error in $\varphi[u]$ depend only on the height of u.

I will finish this section with a lemma relating the minimality of Aronszajn lines to a more combinatorial notion of minimality that will be easier to work with. Recall the following definition.

DEFINITION 2.8. If S and T are two Aronszajn trees, then S is said to be *club-embeddable* into T if there is a closed unbounded set $E \subseteq \omega_1$ and an order-preserving function from $S \restriction E$ into $T \restriction E$. Here $S \restriction E = \bigcup_{\delta \in E} S_\delta$.

LEMMA 2.9. *Suppose that T is an Aronszajn tree and that $X \subseteq T$ is dense in the tree order that is a minimal Aronszajn line in some lexicographical order on T.* (A subset X of T is *dense* in T if every element of T has an extension in X.) *If S is a subtree of*

$$T' = \{t \in T : \exists t_0, t_1 \in T (t \leq t_0, t_1) \text{ and } (t_0 \perp t_1)\},$$

then T' club-embeds into S. In particular, if there is a minimal Aronszajn line, then there is an Aronszajn tree that club-embeds into all of its subtrees.

I could not find this mentioned specifically in the literature, although a related discussion can be found in [4]. I will leave the proof to the interested reader.

3. Coding Using (A) and a Minimal Aronszajn Line

In this section I will prove Theorem 1.10. I will need the following theorem of Devlin and Shelah.

THEOREM 3.1 [6]. *The inequality $2^{\aleph_0} < 2^{\aleph_1}$ implies the following statement. For every $F: 2^{<\omega_1} \to 2$ there is a $g: \omega_1 \to 2$ such that, for every $f: \omega_1 \to 2$,*

$$\{\delta < \omega_1 : g(\delta) = F(f \restriction \delta)\}$$

is stationary.

Applications of this theorem frequently involve some encoding and decoding of countable structures as countable binary sequences. This involves a fairly standard argument (cf. e.g. [9, II, Exer. (51)]; see also the proof of Lemma 4.11 to follow), but for concreteness I will state an equivalent formulation of Theorem 3.1.

THEOREM 3.2. *The inequality $2^{\aleph_0} < 2^{\aleph_1}$ implies the following statement. For every $F: H(\aleph_0^+) \to 2$ there is a $g: \omega_1 \to 2$ such that, for every $\mathfrak{A}$ in $H(\aleph_1^+)$, there exists a countable elementary submodel $\hat{M}$ of $H(\aleph_1^+)$ with $\mathfrak{A}$ in $\hat{M}$ such that $g(\omega_1^M) \neq F(\mathfrak{A}^M)$.*

The following lemma, taken with Lemma 2.9, now completes Theorem 1.10.

LEMMA 3.3. *Suppose there is an Aronszajn tree T with the following properties*:
(i) *T is club-embeddable into all of its subtrees, and*
(ii) *every ladder system coloring can be T-uniformized.*
Then $2^{\aleph_0} = 2^{\aleph_1}$.

Proof. Suppose that T satisfies the hypotheses of the lemma. By replacing T with an isomorphic tree if necessary, we may assume for simplicity that the elements of T are in $H(\aleph_0^+)$. Fix a ladder system $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ and a function $\tau : \omega_1 \to T$ such that $\tau(\alpha)$ has height α for each $\alpha < \omega_1$. We will define a function $F : H(\aleph_0^+) \to 2$ that shows the conclusion of Theorem 3.2 to be false and hence that $2^{\aleph_0} = 2^{\aleph_1}$.

Suppose h is a club-endomorphism of T, as evidenced by a club $E \subseteq \omega_1$, and suppose φ is a function (defined on the range of h) that takes values in ω. If $\hat{M}$ is a countable elementary submodel of $H(\aleph_1^+)$ that contains T, E, h, and φ and if $\delta = \hat{M} \cap \omega_1$, then

$$T^M = \bigcup_{\alpha<\delta} T_\alpha,$$

$$(E, h, \varphi)^M = (E \cap \delta, h \restriction T^M, \varphi \restriction T^M).$$

If $\mathfrak{A} = (E, h, \varphi)$ for some E, h, φ, and $\hat{M}$ as before, set $F(\mathfrak{A}^M) = i$ if and only if (iff) $\varphi[h(\delta)] \restriction C_\delta \equiv^* i$. Notice that, since E is a club and h is an endomorphism of $T \restriction E$, this depends only on $\mathfrak{A}^M = (E \cap \delta, h \restriction T^M, \varphi \restriction T^M)$. If F is defined in this way then we will say that F is defined nontrivially. On the rest of $H(\aleph_0^+)$, set F equal to 0.

In order to finish the proof, I will show that for every $g : \omega_1 \to 2$ there is an $\mathfrak{A} = (E, h, \varphi)$ such that, if $\hat{M}$ is a countable elementary submodel of $H(\aleph_1^+)$ with $\mathfrak{A}$ in $\hat{M}$, then $F(\mathfrak{A}^M) = g(\omega_1^M)$. Toward this end, let g be given and define $\vec{f}$ by letting f_α be the function with domain C_α that takes the constant value $g(\alpha)$. Now apply (A) to find a subtree S of T and a function $\varphi : S \to 2$ that uniformizes $\vec{f}$. By assumption, there is a club $E \subseteq \omega_1$ and an order-preserving map h of $T \restriction E$ into $S \restriction E$. Put $\mathfrak{A} = (E, h, \varphi)$ and let $\hat{M}$ be a countable elementary submodel of $H(\aleph_1^+)$ such that $\mathfrak{A}$ is in $\hat{M}$. It follows from our definitions and the choices made previously that $F(\mathfrak{A}^M) = g(\omega_1^M)$. □

4. Iterating Proper Forcings without Adding Reals

In this section I will present the preservation lemmas to be used in Section 5. The approach will seem different than that used in the literature but is equivalent for our purposes. Part of my motivation for this departure is the hope that it makes the tasks at hand more transparent.

First I review some definitions and theorems from [15]. Recall that a *forcing* is a transitive relation $\leq$ on a set $\mathcal{Q}$ that has a greatest element. Typically the same letter is used to denote both the forcing and the underlying set. Elements of $\mathcal{Q}$ are referred to as *conditions* and should be viewed as approximating a generic object that is being created by the forcing. In this paper, $p \leq q$ will mean that p is an extension of q; that is, p is a better approximation than q.

Shelah's notion of a *completeness system* has served as a staple in proofs showing that certain countable support iterations do not introduce reals.

DEFINITION 4.1 [15, V.5.2]. A *completeness system* for a forcing $\mathcal{Q}$ is a function $\mathbb{D}$ such that the following statements hold.

(1) For a sufficiently large θ, the domain of $\mathbb{D}$ consists of pairs (M, q), where M is a countable elementary submodel of $H(\theta)$ containing $\mathcal{Q}$ as an element and q is in $\mathcal{Q} \cap M$.
(2) For every (M, q) in the domain of $\mathbb{D}$, $\mathbb{D}(M, q)$ is a collection of subsets of

$$\mathrm{Gen}(M, \mathcal{Q}, q) = \{G \subseteq \mathcal{Q} \cap M : G \text{ is an } M\text{-generic filter}\}.$$

DEFINITION 4.2 [15, V.5.2]. If λ is a cardinal, then $\mathbb{D}$ is a λ*-completeness system* if, for every (M, q) in the domain of $\mathbb{D}$, the intersection of fewer than $1 + \lambda$ elements is nonempty.

DEFINITION 4.3 [15, V.5.4]. A completeness system $\mathbb{D}$ for $\mathcal{Q}$ is said to be *simple* if there is a second-order formula ψ such that $\mathbb{D}(M, q) = \{\mathcal{G}_X : X \subseteq M\}$, where

$$\mathcal{G}_X = \{G \in \mathrm{Gen}(M, q) : (M, \in, \mathcal{Q} \cap M) \models \psi[G, X]\}.$$

(A *second-order* formula allows quantification over both elements and subsets.)

DEFINITION 4.4 [15, V.5.3]. Suppose that $\mathbb{D}$ is a simple completeness system for a forcing $\mathcal{Q}$. Then $\mathcal{Q}$ is said to be $\mathbb{D}$-*complete* if, for every (M, q) in the domain of $\mathbb{D}$,

$$\mathrm{Gen}^+(M, \mathcal{Q}, q) = \{G \in \mathrm{Gen}(M, \mathcal{Q}, q) : M \models G \text{ has a lower bound}\}$$

contains an element of $\mathbb{D}(M, q)$.

THEOREM 4.5 [15, VIII.4.5]. *A countable support iteration of forcings that are α-proper for all $\alpha < \omega_1$ and $\mathbb{D}$-complete with respect to a simple 2-completeness system does not introduce reals.*

REMARK 4.6. The concept of α-properness is defined in [15, V.3.1]. An equivalent formulation for forcings that do not introduce reals will be given in Definition 4.15.

I will now define an abstract completeness system and argue that it captures much of the generality of the foregoing definitions. It will be useful to define a certain category to facilitate the discussion. First, expand the language of ZFC to add a predicate Q for a distinguished forcing. Let $\mathrm{ZFC}^{\mathcal{Q}}$ be the axioms of ZFC but with the power set axiom replaced by "$\mathcal{P}(\mathcal{P}(Q))$ exists". The objects of the category $\mathfrak{M}$ are those countable transitive sets, together with a distinguished element $\mathcal{Q}^M$, that satisfy $\mathrm{ZFC}^{\mathcal{Q}}$ when Q is interpreted as $\mathcal{Q}^M$. Note that if $\mathcal{Q}$ is a set and $\mathcal{P}(\mathcal{P}(\mathcal{Q}))$ is in $H(\theta)$ for some θ, then $H(\theta)$ would be an element of $\mathfrak{M}$ except that it is not countable.

An *arrow* $\overrightarrow{MN}$ in $\mathfrak{M}$ is an elementary embedding $\varepsilon\colon M \to N$ with the property that ε is in N and N satisfies "$M = \mathrm{dom}(\varepsilon)$ is countable". The notation $M \to N$ will be used to denote $\overrightarrow{MN}$ and also to assert the statement "$\overrightarrow{MN}$ is an arrow in $\mathfrak{M}$". Observe that arrows fix hereditarily countable sets. Also, notation such as $M \to N$ is meaningful even if N is uncountable.

We will mostly consider commutative diagrams in $\mathfrak{M}$, so there will be at most one arrow between two given objects. If $M \to N$ and if X is a subset of $\mathcal{Q}^M$, then X will simultaneously viewed as a subset of $\mathcal{Q}^N$.

DEFINITION 4.7. Suppose that $\hat{N}$ is a model of $\mathrm{ZFC}^{\mathcal{Q}}$ and that $\hat{M}$ is an elementary submodel of $\hat{N}$ such that $\hat{M}$ is in $\hat{N}$ and $\hat{N}$ satisfies "$\hat{M}$ is countable". Then there is a unique *induced arrow* $M \to N$ that commutes with the collapsing maps.

I am now ready to define the simple completeness system of interest. Suppose that $\mathcal{Q}$ is a given forcing.

DEFINITION 4.8. If θ is a regular cardinal, then a $(\mathcal{Q}, \theta)$*-diagram* is a diagram in $\mathfrak{M}$ such that there exist (i) a minimum M in the order induced from the arrows and (ii) an elementary embedding $x \mapsto \hat{x}$ from M into $H(\theta)$ that sends $\mathcal{Q}^M$ to $\mathcal{Q}$. The range of this embedding is a countable elementary submodel of $H(\theta)$, which will be denoted $\hat{M}$. A diagram is a $\mathcal{Q}$*-diagram* if it is a $(\mathcal{Q}, \theta)$-diagram for some θ.

DEFINITION 4.9. Let $M \to N$ be a $\mathcal{Q}$-diagram. If $G \subseteq \mathcal{Q}^M$, then we will say that G is $\overrightarrow{MN}$*-prebounded* if, whenever $N \to \tilde{N}$ and G is in $\tilde{N}$, $\tilde{N}$ satisfies "G is bounded in $\mathcal{Q}^{\tilde{N}}$".

DEFINITION 4.10. A forcing $\mathcal{Q}$ is *completely proper* if there is a θ such that, for every $(\mathcal{Q}, \theta)$-diagram of the form $M \to N_i$ $(i < 2)$ and q in $\mathcal{Q}^M$, there exists a $G \subseteq \mathcal{Q}^M$ that is M-generic, contains q, and is $\overrightarrow{MN_i}$-prebounded for both $i < 2$.

LEMMA 4.11. *Every completely proper forcing is* 2*-complete with respect to some simple completeness system* $\mathbb{D}$.

REMARK 4.12. Of course, there is a canonical definition of λ-completely proper for each $\lambda \leq \aleph_1$. The results in the remainder of this section are easily adapted to this greater generality, but at present I see no reason to seek such generality. The terminology *completely proper* is sometimes given the same meaning as *totally proper*. I have chosen the present usage since it is more closely tied to Shelah's usage of "completeness" and since "totally proper" seems more established in the literature.

Proof of Lemma 4.11. Let $\mathcal{Q}$ be a completely proper forcing and let θ be large enough to exhibit this. Suppose $\hat{M}$ is a countable elementary submodel of $H(\theta)$ that has $\mathcal{Q}$ as an element. If $M \to N$, define $\mathcal{G}_{\overrightarrow{MN}}$ to be the set of all $G \in \mathrm{Gen}(\hat{M}, \mathcal{Q}, q)$ such that πG is $\overrightarrow{MN}$-prebounded. Here π is the collapsing isomorphism for $\hat{M}$, the image of M under its embedding into $H(\theta)$. Define $\mathbb{D}(\hat{M}, q)$ to be the collection of all $\mathcal{G}_{\overrightarrow{MN}}$ as $\overrightarrow{MN}$ ranges over the arrows in $\mathfrak{M}$.

Clearly $\mathbb{D}$ is a completeness system and $\mathcal{Q}$ is 2-complete with respect to it; hence it suffices to show that $\mathbb{D}$ is simple. Though we are not allowed explicit quantification over arrows in the definition of "simple", this can be achieved by appropriate coding. For instance, there is a second-order formula ψ_0 such that, if $X \subseteq M$ and

$$(M, \in, \mathcal{Q}^M) \models \psi_0[X]$$

and we define $R_X = \{p \in M : (0, p) \in X\}$ and $\iota_X = \{p \in M : (1, p) \in X\}$, then (ω, R_X, Q) is a well-founded model of $\mathrm{ZFC}^{\mathcal{Q}}$ and ι_X is an elementary embedding from $(M, \in, \mathcal{Q}^M)$ into (ω, R_X, Q). This takes care of the assertion that N is in $\mathfrak{M}$ and $M \to N$; one can similarly handle the quantification over $N \to \tilde{N}$. □

It is also worth noting that it is possible to prove a partial converse to Lemma 4.11.

DEFINITION 4.13. $\mathcal{Q}$ is said to have the *effective bounding property* if, whenever $\mathcal{Q}_0$ is a countable subset of $\mathcal{Q}$, the set of all $G \subseteq \mathcal{Q}_0$ such that G has a lower bound in $\mathcal{Q}$ is a Borel subset of $\mathcal{P}(\mathcal{Q}_0)$.

Many forcings (including those in the next section) have this property, and this condition is readily verified by inspection.

LEMMA 4.14. *If $\mathcal{Q}$ has the effective bounding property and is $\mathbb{D}$-complete with respect to some simple 2-completeness system $\mathbb{D}$, then $\mathcal{Q}$ is completely proper.*

Proof. Let ψ be the formula used to define $\mathbb{D}$ and let θ_0 be the sufficiently large regular cardinal witnessing that $\mathbb{D}$ is a 2-completeness system. Let θ be such that $H(\theta_0)$ is in $H(\theta)$. Now suppose that $M \to N_i$ $(i < 2)$ is a given $(\mathcal{Q}, \theta)$-diagram and that q is in $\mathcal{Q}^M$. Notice that, for a given $i < 2$, N_i satisfies "ψ defines a simple 2-completeness system witnessing that $\mathcal{Q}$ is $\mathbb{D}$-complete". In particular, if $\hat{M}^{N_i}$ is the image of M under the embedding $M \to N_i$, then $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{N_i}$ contains an element Y^{N_i} of $\mathbb{D}(\hat{M}, q)^{N_i}$ as a subset. Furthermore, $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{N_i}$ is a Borel subset of $(\mathcal{Q} \cap \hat{M})^{N_i}$ because N_i satisfies "$\mathcal{Q}^{N_i}$ has the effective bounding property".

There are two sets in $H(\theta)$ that correspond to $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{N_i}$. One is a (countable) subset of N_i and the other is a (typically uncountable) Borel set $\mathcal{X}^{N_i}$ obtained by interpreting N_i's Borel code for $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{N_i}$. Let $\varepsilon_i : \hat{M}^{N_i} \to \hat{M}$ for $i < 2$ be the unique elementary isomorphisms, and put $\mathcal{X}_i = \{\varepsilon_i H : H \in \mathcal{X}^{N_i}\}$ and $\mathcal{Y}_i = \{\varepsilon_i H : H \in \mathcal{Y}^{N_i}\}$. Observe that $\mathcal{Y}_i$ is in $\mathbb{D}(\hat{M}, q)$ by the elementarity of ε_i applied to ψ. Since $\mathbb{D}$ is a 2-completeness system, it follows that $\mathcal{Y}_0 \cap \mathcal{Y}_1$ contains an element $H \subseteq \mathcal{Q} \cap \hat{M}$. Let $G \subseteq \mathcal{Q}^M$ be the image of H under $\hat{M}$'s collapsing isomorphism.

Now suppose that $i < 2$. To see that G is $\overrightarrow{MN_i}$-prebounded, let $N_i \to \tilde{N}$ be such that G is in $\tilde{N}$. For any $X \subseteq (\mathcal{Q} \cap \hat{M})^{N_i}$ that is in $\tilde{N}$, $\varepsilon_i X$ is in $\mathcal{X}_i$ iff X is in $\mathcal{X}^{N_i}$ iff $\tilde{N}$ satisfies "X is in $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{\tilde{N}}$". Therefore, $\tilde{N}$ satisfies "$\varepsilon_i^{-1} H$ is in $\mathrm{Gen}^+(\hat{M}, \mathcal{Q}, q)^{\tilde{N}}$" and hence also that "$G$ is bounded in $\mathcal{Q}^{\tilde{N}}$". (In the latter quotation I am identifying G with its image under the embedding of M into $\tilde{N}$; this is not the case in the former quotation.) □

Traditionally, completeness with respect to a simple completeness system needs to be supplemented with $<\omega_1$-properness in order to obtain preservation results for not adding reals. While total $<\omega_1$-properness is a standard notion in this context [15, V.3.1], the following definition gives an equivalent formulation in terms of diagrams.

DEFINITION 4.15. A forcing $\mathcal{Q}$ is *totally $<\omega_1$-proper* if, whenever $M_\xi \to M_\eta$ ($\xi < \eta \le \gamma$) is an *amenable* $\mathcal{Q}$-diagram for $\gamma < \omega_1$ and q is in $\mathcal{Q}^{M_0}$, there exists a $\bar{q}$ in $\mathcal{Q}$ such that $\bar{q} \le q$ and $\bar{q}$ is *totally* $(M, \mathcal{Q})$*-generic* for all $\xi \le \gamma$—that is, the filter $\{p \in \mathcal{Q}^{M_\xi} : \bar{q} \le p\}$ is M_ξ-generic. (Here, *amenable* means that, whenever $\eta \le \gamma$ is a limit, $\lim_{\xi\to\eta} M_\xi = M_\eta$ and $\langle M_\xi : \xi < \eta\rangle \in M_{\eta+1}$ if $\eta + 1 \le \gamma$.)

The following lemma is now an immediate consequence of the previous observations and [15, VIII.4.5].

LEMMA 4.16. *A countable support iteration of completely proper, $<\omega_1$-proper forcings does not introduce reals.*

I will now finish this section with a condition that is essentially a reformulation of Shelah's properness isomorphism condition. The condition is made useful by the following theorem.

THEOREM 4.17 [15, VIII.2.4]. *If $\mathcal{P}$ is a countable support iteration of length at most ω_2 whose iterands satisfy the properness isomorphism condition (p.i.c.), then $\mathcal{P}$ satisfies the $\aleph_2$-chain condition.*

Theorem VIII.2.4 of [15] states something stronger and more general; in particular, the $\aleph_2$-p.i.c. is a consequence of the conventional properness isomorphism condition stated in Definition 4.18.

DEFINITION 4.18. A forcing $\mathcal{Q}$ satisfies the *properness isomorphism condition* if, whenever

(i) $\hat{M}_i$ ($i < 2$) are countable elementary submodels of $H(\theta)$ for θ a sufficiently large cardinal number,
(ii) $\varphi\colon \hat{M}_0 \to \hat{M}_1$ is an isomorphism that fixes $\hat{M}_0 \cap \hat{M}_1$ and $\mathcal{Q}$ in $\hat{M}_0 \cap \hat{M}_1$, and
(iii) q is in $\mathcal{Q} \cap \hat{M}_0$,

there is a $\bar{q}$ that extends both q and $\varphi(q)$ and that is $(\hat{M}_i, \mathcal{Q})$-generic for each $i < 2$.

The following condition is formally stronger than the properness isomorphism condition, but it seems likely to be the same in practice.

DEFINITION 4.19. $\mathcal{Q}$ is said to have the *strong chain condition* (this is an ad hoc name and not intended for long-term use) if (a) whenever $M \xrightarrow{\varepsilon_i} N$ ($i < 2$) is a $\mathcal{Q}$-diagram and q is in $\mathcal{Q}^M$, there is a $G \subseteq \mathcal{Q}^M$ that is M-generic and (b) whenever $N \to \tilde{N}$ is an arrow in $\mathfrak{M}$ and G is in $\tilde{N}$, there is a $\bar{q}$ in $\mathcal{Q}^{\tilde{N}}$ that is a lower bound for $\varepsilon_i G$.

REMARK 4.20. The difference between the strong chain condition and complete properness is that in the strong chain condition only pairs of "top models" that are equal are considered—but with the added requirement that, if this common model is extended to pick up G, then there is a single bound for both images of G in this extension.

LEMMA 4.21. *The strong chain condition implies the properness isomorphism condition.*

Proof. This is similar to the arguments already given. I will leave the proof to the interested reader. □

Combining Theorem 4.17 and Lemma 4.21, we now have the following lemma.

LEMMA 4.22. *A countable support iteration of length ω_2 of forcings with the strong chain condition over a ground model that satisfies CH has the $\aleph_2$ chain condition and, in particular, preserves cardinals that are at least $\aleph_2$.*

5. How to Uniformize Colorings Relative to an Aronszajn Tree and Not Introduce Reals

In this section I will prove Theorem 1.9. For the moment, let T be a fixed Aronszajn tree, let $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$ be a fixed ladder system, and let $\vec{f} = \langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ be a coloring of $\langle C_\alpha : \alpha \in \lim(\omega_1)\rangle$. For simplicity we may and will assume that T is a subtree of $\omega^{<\omega_1}$, the collection of all countable length sequences from ω ordered by extension. This has the added benefit of causing elements of T to be fixed by the arrows discussed in the previous section.

DEFINITION 5.1. If $n < \omega$, let $T^{[n]}$ denote the subset of T^n of all σ such that, for all $i < j < n$, $\sigma(i)$ has the same height as $\sigma(j)$ and $\sigma(i) \leq \sigma(j)$ in the lexicographical ordering. If an element of $T^{[n]}$ is one-to-one, then it will be identified with its range without further mention. A *finite power of T* is a set of the form $T^{[n]}$ for some $n < \omega$.

DEFINITION 5.2. Let σ be in a finite power of T and let X be a subset of T consisting of elements of height at most α with $X \cap T_\alpha \neq \emptyset$. Then σ *is consistent with X* if $\sigma \restriction \alpha$ is a subset of X. Also, we say that two functions f and g are *consistent* if they agree on the intersection of their domains.

DEFINITION 5.3. Let $\mathcal{Q} = \mathcal{Q}(\vec{f}; T)$ be the collection of all $q = (\varphi, \mathcal{U})$ such that the following statements hold.

(1) There is a minimal $\alpha = \alpha_q$ such that the domain of φ (denoted $X = X_q$) is a downwards closed subset of T consisting of elements of height at most α.
(2) $\mathcal{U}$ is a nonempty countable collection such that, if U is in $\mathcal{U}$, then U is a pruned subtree of a finite power of T.
(3) For every U in $\mathcal{U}$, there is a σ in U of height α that is consistent with X.
(4) φ is a function from the elements of X of height less than α into ω such that, if s is an element of X of limit height ν, then $f_\nu =^* \varphi[s] \restriction C_\nu$.

NOTATION 5.4. If q is in $\mathcal{Q}$, then $(\varphi_q, \mathcal{U}_q)$ will be used to denote q. Here $\mathcal{Q}$ is made into a forcing notion by saying that q extends p if φ_p is an initial part of φ_q and $\mathcal{U}_p \subseteq \mathcal{U}_q$.

LEMMA 5.5. *Suppose that $\hat{M}$ is a countable elementary submodel of $H(2^{\aleph_1^+})$ and let $\delta = \hat{M} \cap \omega_1$. For every q in $\mathcal{Q} \cap \hat{M}$, every dense open $D \subseteq \mathcal{Q}$ in $\hat{M}$, and every finite $\sigma \subseteq T_\delta$ that is consistent with q, there exists a $\bar{q} \leq q$ in D such that σ is consistent with $\bar{q}$.*

Proof. Suppose this is not the case and let $\hat{M}$, q, D, and σ be counterexamples. Let n denote the cardinality of σ. Since T_δ is countable, there is a function $\tau : \omega_1 \to T^{[n]}$ in $\hat{M}$ such that $\tau(\delta) = \sigma$ and $\tau(\xi)$ has height ξ for all $\xi < \omega_1$. Let Ξ be the set of all $\xi < \omega_1$ such that, if $\bar{q} \leq q$, $\bar{q}$ is in D, and $\alpha_{\bar{q}} < \xi$, then $\tau(\xi)$ is not consistent with $\bar{q}$. Observe that Ξ is in $\hat{M}$. I now claim that δ is in Ξ. To see this, suppose $\bar{q} \leq q$ is in D and $\nu = \alpha_{\bar{q}} < \delta$. Then σ is consistent with $\bar{q}$ iff $\sigma \restriction \nu$ is. Since $\sigma \restriction \nu$ is in $\hat{M}$, if σ were consistent with $\bar{q}$ then we could find such a $\bar{q}$ in $\hat{M}$. But this would contradict our assumptions on $\hat{M}$, q, D, and σ.

Now let U be the set of all υ such that, for uncountably many ξ in Ξ, $\upsilon \leq \tau(\xi)$. It is routine to verify that U is uncountable, pruned, and downwards closed. I shall obtain a contradiction by arguing that $(\varphi_q, \mathcal{U}_q \cup \{U\})$ has no extension in D. If it did, let $\bar{q}$ be such an extension and pick a $\sigma_0 \subseteq X_{\bar{q}} \cap U$ of height $\alpha_{\bar{q}}$. By construction, there is a ξ in Ξ such that $\sigma_0 = \tau(\xi) \restriction \alpha_{\bar{q}}$. But this would imply that $\tau(\xi)$ is consistent with $\bar{q}$, contradicting the definition of Ξ. □

DEFINITION 5.6. Suppose that q and $\bar{q}$ are in $\mathcal{Q}$. Then $\bar{q}$ is said to be a *conservative extension* of q if $\bar{q} \leq q$ and, whenever σ is a finite subset of T that is consistent with q, σ is consistent with $\bar{q}$.

LEMMA 5.7. *For every β, every q in $\mathcal{Q}$, and every finite partial function $\psi : T \to \omega$ that is consistent with φ_q, there is a conservative extension $\bar{q}$ of q such that $\beta \leq \alpha_{\bar{q}}$ and ψ is consistent with $\varphi_{\bar{q}}$. In particular, every condition in $\mathcal{Q}$ forces that:*

(5) *the union $\dot{U}$ of X_q $(q \in \dot{G})$ is uncountable and hence is a subtree of T; and*
(6) *the union $\dot{\varphi}$ of the first coordinates of elements of $\dot{G}$ is a uniformizing function for $\vec{f}$ that is defined on $\dot{U}$.*

Proof. Let β and ψ be fixed and suppose that q is in $\mathcal{Q}$. By making β larger if necessary, we may assume that β is an upper bound on the heights of elements of the domain of ψ. If $\beta \leq \alpha_q$, then $\bar{q} = q$ works. Now suppose that $\alpha_q < \beta$ and let $r : \lim(\beta + 1) \to \beta$ be a regressive function such that

$$\{C_\xi \setminus r(\xi) : \xi \in \lim(\beta + 1)\}$$

is a pairwise disjoint family whose union does not contain the heights of any element of the domain of ψ. Let

$$X_{\bar{q}} = \{s \in T_{\leq \beta} : s \restriction \alpha \in X_q\},$$
$$\mathcal{U}_{\bar{q}} = \mathcal{U}_q,$$

and define $\varphi_{\bar{q}}$ on those elements of $X_{\bar{q}}$ of height less than β by

(7) $\varphi_{\bar{q}}(s) = \varphi_q(s)$ if s is in X_q,
(8) $\varphi_{\bar{q}}(s) = f_\nu(\xi)$ if $\xi \geq \alpha_q$ is the height of s and if ν satisfies $\xi \in C_\nu \setminus r(\nu)$,
(9) $\varphi_{\bar{q}}(s) = \psi(s)$ if s is in the domain of ψ, and
(10) $\varphi_{\bar{q}}(s) = 0$ otherwise.

It is left to the reader to verify that $\bar{q}$ is a condition in $\mathcal{Q}$ and that it is a conservative extension of q. Observe, however, that this is where we need the requirement that the elements of $\mathcal{U}_q$ be pruned. □

The following lemma will be useful in demonstrating that a given generic filter has a lower bound in a larger model.

LEMMA 5.8. *Suppose that $M \to N$ is a $\mathcal{Q}$-diagram and that $G \subseteq \mathcal{Q}^M$ is M-generic. Then G is $\overrightarrow{MN}$-prebounded if the following conditions are satisfied: for every U in $\bigcup_{p \in G} \mathcal{U}_p$, there exist a σ in U^N of height $\delta = \omega_1^M$ and a $\delta_0 < \delta$ such that, for every p in G, σ is consistent with p and $\varphi_p(s \restriction \xi) = f_\delta^N(\xi)$ whenever s is in σ and ξ is in C_δ^N with $\delta_0 < \xi \leq \alpha_p$.*

Proof. Let M, N, and G be as given in the statement of the lemma and suppose that the conditions are satisfied by G. Suppose that $N \to \tilde{N}$ is given. Working in $\tilde{N}$, set

$$X = \bigcup_{p \in G} X_p, \quad \mathcal{U}_{\bar{q}} = \bigcup_{p \in G} \mathcal{U}_p, \quad \varphi_{\bar{q}} = \bigcup_{p \in G} \varphi_p.$$

Let $X_{\bar{q}}$ be the union of X with the set of all s in T_δ such that every predecessor of s is in X and, for all but finitely many ξ in C_δ, $\varphi(s \restriction \xi) = f_\delta(\xi)$. If U is in $\mathcal{U}_{\bar{q}}$ then, by assumption, there is a σ in U of height δ such that $\sigma \subseteq X_{\bar{q}}$. Hence $\bar{q}$ is a condition in $\mathcal{Q}^{\tilde{N}}$ that, moreover, is clearly a lower bound for G. □

Observe that the bound $\bar{q}$ produced by Lemma 5.8 has the following property: whenever σ is an element of a finite power of T that is consistent with every p in G, then σ is consistent with $\bar{q}$ provided that, for all but finitely many ξ in $C_\delta^{\tilde{N}} \setminus \delta_0$,

$$\varphi_{\bar{q}}(s \restriction \xi) = f_\delta^{\tilde{N}}(\xi)$$

for all $s \in \sigma$. If, moreover, $\alpha_{\bar{q}} = \delta$, then $\bar{q}$ is unique and will be referred to as the *conservative bound* for G.

The relevant properties of $\mathcal{Q}$ will be proved by iterating the following lemma with appropriate "bookkeeping".

LEMMA 5.9. *Suppose that $M \to N_i$ $(i \leq k)$ is a $\mathcal{Q}$-diagram, q is in $\mathcal{Q}^M$, U is in $\mathcal{U}_q^M$, $D \subseteq \mathcal{Q}$ is dense open and in $\hat{M}$, and σ_i $(i < k)$ and ν_i $(i \leq k)$ satisfy the following conditions*:

(11) *for each $i < k$, ν_i is an ordinal with $\omega_1^M \leq \nu_i < \omega_1^{N_i}$;*
(12) *for each $i < k$, σ_i is an element of a finite power of T^{N_i} of height ν_i that is consistent with q; and*
(13) *$\{\sigma_i \restriction \alpha_q : i < k\}$ is a pairwise disjoint family.*

Then there exist a σ_k and a $\bar{q} \le q$ in D^M such that:

(14) *σ_k is in U^{N_k} and has height ν_k*;
(15) *for all $i \le k$, σ_i is consistent with $\bar{q}$*;
(16) *$\{\sigma_i \restriction \alpha_{\bar{q}} : i \le k\}$ is a pairwise disjoint family*; *and*
(17) *if $i < k$, ξ is in $C^{N_i}_{\nu_i}$ with $\alpha_q \le \xi < \alpha_{\bar{q}}$, and s is in σ_i, then $\varphi_{\bar{q}}(s \restriction \xi) = f^{N_i}_{\nu_i}(\xi)$.*

Proof. Let $M \to N_i$ $(i \le k)$, q, U, D as well as σ_i $(i < k)$ and ν_i $(i \le k)$ be as given in the statement of the lemma, and let α denote α_q. Select a $P \in M$ such that M satisfies "P is an elementary submodel of $H(2^{\aleph_1^+})$ such that q and D^M are in P". Let F be the set of all $s \restriction \xi$ such that, for some $i < k$, s is in σ_i and ξ is in $P \cap C^{N_i}_{\nu_i} \setminus \alpha$. Let ψ denote the function with domain F defined by

$$s \restriction \xi \mapsto f^{N_i}_{\nu_i}(\xi)$$

if s is in σ_i. Note that, by (13), ψ is well-defined.

Applying Lemma 5.7 in P to ψ and to a β that bounds the heights in F, there is a $q' \le q$ in P that is a conservative extension of q such that $\varphi_{q'}$ extends ψ. Putting

$$\upsilon = \bigcup_{i<k} \sigma_i \restriction (P \cap \omega_1),$$

we can apply Lemma 5.5 to obtain a $q'' \le q'$ in P such that q'' is in $D^M \cap P$ and υ is consistent with q''.

I now need to construct σ_k. Let τ be an element of U of height $\alpha_{q''}$ that is in $X_{q''}$. Let n denote the cardinality of υ. By [19, Lemma 5.9] applied in P, there exist υ_j $(j < n+1)$ in $U \cap P$ such that (a) each extends τ and (b) if $j \ne j' < n+1$, then no element of υ_j is comparable with any element of $\upsilon_{j'}$. Notice that if s is in υ then there is at most one $j < n+1$ such that s is comparable with an element of υ_j. Hence there is an $l < n+1$ such that no element of υ is compatible with any element of υ_l. Since U is pruned, there is a σ_k in $U^{N_k}_{\nu_k}$ that extends $\upsilon^{N_k}_l$. Finally, use Lemma 5.7 to find a $\bar{q}$ in P that is a conservative extension of q'' such that $\alpha_{\bar{q}}$ is greater than the height of υ_l and hence (16) holds. Since $\varphi_{q''}$ extends ψ and since

$$C^{N_i}_{\nu_i} \cap P = C^{N_i}_{\nu_i} \cap \alpha_{q''}$$

for all $i < k$, it follows that (17) holds. Since υ is consistent with q'', so is each σ_i $(i < k)$. This finishes the proof. □

We will now see that $\mathcal{Q}$ satisfies conditions that are sufficient to ensure it can be iterated while preserving cardinality and not introducing reals.

LEMMA 5.10. *The forcing $\mathcal{Q}$ is completely proper and satisfies the strong chain condition.*

Proof. First I will show that $\mathcal{Q}$ is completely proper. Let $M \to N_k$ $(k < 2)$ be a $\mathcal{Q}$-diagram and let $q \in \mathcal{Q}^M$. Fix an enumeration D_i $(i < \omega)$ of all dense open subsets of $\mathcal{Q}$ in $\hat{M}$ and an enumeration with infinite repetition (U_i, k_i) $(i < \omega)$ of all pairs (U, k) such that U is a subtree of a finite power of T, U is in $\hat{M}$, and $k < 2$. Using Lemma 5.9, construct a decreasing sequence $q(i)$ $(i < \omega)$ of conditions in $\mathcal{Q}^M$ that are below q and a sequence of $\sigma(i)$ $(i < \omega)$ such that, for all $j < \omega$:

(18) $q(j)$ is in D_j^M;
(19) if U_j is in $\mathcal{U}_{q(j)}$, then $\sigma(j)$ is in $U_j^{N_{k_j}}$ and of height $\delta = \omega_1^M$;
(20) $\{\sigma(i) \restriction \alpha_{q(j)} : i < j\}$ is a pairwise disjoint family;
(21) if $i < \omega$, then $\sigma(i)$ is consistent with $q(j)$; and
(22) if $i < j$, s is in $\sigma(i)$, and ξ is in $C_\delta^{N_{k_i}}$ with $\alpha_{q(i)} < \xi < \alpha_{q(j)}$, then we have $\varphi_{q(j)}(s \restriction \xi) = f_\delta^{N_{k_i}}(\xi)$.

Let G be the set of all p in $\mathcal{Q}^M$ such that there is an $i < \omega$ with $q(i) \leq p$. Clearly G is M-generic. Now suppose that $k < 2$ is given. To see that G is $\overrightarrow{MN_k}$-prebounded, it is sufficient to verify the hypotheses of Lemma 5.8. Toward this end, suppose that U is in $\bigcup_{p \in G} \mathcal{U}_p$. Pick an $i < \omega$ such that U is in $\mathcal{U}_{q(i)}$, $U_i = U$, and $k_i = k$. Set $\delta_0 = \alpha_{q(i)}$ and let p be any element of G. Pick a $j \geq i$ such that $q(j) \leq p$. By construction, $q(j)$ is consistent with σ and hence so is p. If s is in σ, $\delta_0 < \xi \leq \alpha_p \leq \alpha_{q(j)}$, and ξ is in $C_\delta^{N_k}$, then

$$\varphi_p(s \restriction \xi) = \varphi_{q(j)}(s \restriction \xi) = f_\delta^{N_k}(\xi).$$

By Lemma 5.8, I am now finished.

To see that $\mathcal{Q}$ has the strong chain condition, proceed as before with $N_0 = N_1 = N$ in order to construct $G \subseteq \mathcal{Q}^M$. Now let $N \to \tilde{N}$ be given such that G is in $\tilde{N}$ and let ε_i denote the arrows witnessing $M \to N_i$. Let $\bar{q}_i$ be the conservative lower bound of $\varepsilon_i G$ and observe that $\varphi_{\bar{q}_0} = \varphi_{\bar{q}_1}$; let $\varphi_{\bar{q}} = \varphi_{\bar{q}_0} = \varphi_{\bar{q}_1}$. Similarly, if U is in $\mathcal{U}_{\bar{q}_i}$ then there is a U' in $\mathcal{U}_{\bar{q}_{1-i}}$, which are equal when restricted to their elements of height less than δ; let $\mathcal{U}_{\bar{q}} = \mathcal{U}_{\bar{q}_0} \cup \mathcal{U}_{\bar{q}_1}$. It is easily verified that $\bar{q}$ is now the desired bound. □

LEMMA 5.11. *$\mathcal{Q}$ is totally $<\omega_1$-proper.*

Proof. Let $\gamma < \omega_1$ be given and fix an amenable $\mathcal{Q}$-diagram $M_\eta \to M_\zeta$ $(\eta < \zeta \leq \gamma)$. Following the methods of Lemma 5.10, we will construct a decreasing sequence q_ζ $(\zeta \leq \gamma)$ in $\mathcal{Q}$ by recursion in such a way that, if ζ is a limit, then q_ζ is the conservative lower bound for q_η $(\eta < \zeta)$.

For now, let us focus on the successor stages. Fix an enumeration D_i $(i < \omega)$ of all dense open subsets of $\mathcal{Q}$ in $\hat{M}_\gamma$ and an enumeration (U_i, ζ_i) $(i < \omega)$ with infinite repetition such that, for each i, U_i is a subtree of a finite power of T corresponding to an element of M_γ and $\zeta_i \leq \gamma$ is a limit ordinal. Also fix a well-order $\lhd$ of $H(\aleph_1^+)$ that is in M_0 and that well-orders γ in type ω. If $\zeta \leq \gamma$, let m_ζ be the number of $\eta \leq \gamma$ such that $\eta \lhd \zeta$.

Now suppose that q_ζ is given. Using Lemma 5.9, construct a decreasing sequence $q_\zeta(i)$ $(i < \omega)$ of conditions in $\mathcal{Q}^{M_{\zeta+1}}$ that are below q_ζ and a sequence of $\sigma_\zeta(i)$ $(i < \omega)$ such that the following conditions hold.

(23) $q_\zeta(i) = q_\zeta$ if $i < m_\zeta$.
(24) If $D_i^{M_{\zeta+1}}$ is in $M_{\zeta+1}$, then $q_\zeta(m_\zeta + i)$ is in $D_i^{M_{\zeta+1}}$.

(25) If U_i is in $\mathcal{U}_{q_\zeta(i)}$ and $\zeta \le \zeta_i$, then $\sigma_\zeta(i)$ is in $U_i^{M_{\zeta_i+1}}$ and of height $\nu_i = \omega_1^{M_{\zeta_i}}$.
(26) For all $k < \omega$, $\{\sigma_\zeta(i) \restriction \alpha_{q_\zeta(k)} : i < k\}$ is a pairwise disjoint family.
(27) For all i, j, $\sigma_\zeta(i)$ is consistent with $q_\zeta(j)$.
(28) If $i < k$, s is in $\sigma_\zeta(i)$, and ξ is in $C_{\nu_i}^{M_{\zeta_i}}$ with $\alpha_{q_\zeta(i)} < \xi < \alpha_{q_\zeta(k)}$, then we have $\varphi(s \restriction \xi) = f_{\nu_i}^{M_{\zeta_i}}(\xi)$.
(29) Given that the construction has been carried out for all $i < k$, it follows that $q_\zeta(k)$ and $\sigma_\zeta(k)$ are the $\lhd$-least objects satisfying conditions (23)–(28).

Arguing as in Lemma 5.10, the sequence $q_\zeta(i)$ $(i < \omega)$ generates an $M_{\zeta+1}$-generic filter. Note that $\langle q_\zeta(i) : i < \omega \rangle$ is in $M_{\zeta+1}$ by the recursion theorem (see [9, III.5.6]), so applying Lemma 5.8 shows that the conservative lower bound $q_{\zeta+1}$ of this sequence is in $M_{\zeta+1}$.

Now suppose that ζ is a limit ordinal. It is clear that $\{q_\eta : \eta < \zeta\}$ generates an M_ζ-generic filter. There are two obstacles to overcome: we need to show that this filter is in $M_{\zeta+1}$ and that it has a lower bound in $M_{\zeta+1}$. The first claim follows from the recursion theorem and the fact that, at each point of the recursion, we chose $\lhd$-minimal witnesses.

Now it remains to verify the hypothesis of Lemma 5.8. The key observation is as follows.

Claim 5.12. *If $m_{\zeta_0} < m_\eta$ whenever $\zeta_0 < \eta < \zeta$, then $\sigma_\eta(i) = \sigma_{\zeta_0}(i)$ for all $i < m_{\zeta_0}$.*

Proof. Suppose this is not the case and let (η, i) be the lexicographically least counterexample. Observe that q_η is consistent with $\sigma_{\zeta_0}(i)$ because q_η is the conservative lower bound for $q_{\eta'}(k)$ $(\eta' < \eta$ and $k < \omega)$. By the minimality of i, $\sigma_\eta(i') = \sigma_{\zeta_0}(i')$ for all $i' < i$ and so it is easily checked that $\sigma_{\zeta_0}(i)$ satisfies the conditions of the recursion (except possibly its $\lhd$-minimality). Notice, however, that if σ satisfies the conditions for $\sigma_\eta(i)$ then it does also for $\sigma_{\zeta_0}(i)$, since the conditions $q_\eta(k)$ $(k < \omega)$ are stronger than any of the conditions $q_{\zeta_0}(k)$ $(k < \omega)$. Since $\sigma_{\zeta_0}(i)$ was taken to be $\lhd$-minimal, it must be that $\sigma_\eta(i) = \sigma_{\zeta_0}(i)$. □

In order to verify the hypotheses of Lemma 5.8, let U be an element of $\bigcup_{\eta<\zeta} \mathcal{U}_{q_\eta}$. Let i be such that $U_i = U$ and $\zeta_i = \zeta$, and find a $\zeta_0 < \zeta$ such that $i < m_{\zeta_0}$ if $\zeta_0 \le \eta < \zeta$; then $m_{\zeta_0} < m_\eta$.

Now suppose that ξ is in $C_\zeta \setminus \alpha_{q_\zeta}$ and let s be in σ_i. Let η be the least ordinal such that $\xi < \alpha_{q_{\eta+1}}$, noting that then $\alpha_{q_\eta} < \xi$. By arrangement, $i < m_\eta$ and therefore $\alpha_{q_\eta(m_\eta-1)} \le \xi$, since $q_\eta = q_\eta(m_\eta - 1)$. Because there exists a $k < \omega$ such that $\xi < \alpha_{q_\eta(k)}$, (28) implies that

$$\varphi_{q_\eta(k)}(s \restriction \xi) = f_\zeta(\xi)$$

as desired. Lemma 5.8 now implies that q_η $(\eta < \zeta)$ has a unique conservative lower bound q_ζ in $M_{\zeta+1}$. □

We are now ready to finish the proof of Theorem 1.9. Let V be a given ground model. By doing a preliminary proper forcing if necessary, we may assume that V satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. In V, build a countable support iteration of length

ω_2 such that all of the iterands are forced to be of the form $\mathcal{Q}(\vec{f}; T)$ and, following [17] (or [9, VIII.6]), in such a way that, by the end of the iteration, $\mathcal{Q}(\vec{f}; T)$ has been forced with at some initial stage whenever $\vec{f}$ and T are appropriate elements of the final generic extension (this is possible by the ground model assumptions and Lemma 4.22). By Lemma 5.7, $\mathcal{Q}(\vec{f}; T)$ forces that the coloring $\vec{f}$ can be T-uniformized. Hence the final model satisfies (A). It follows from Lemmas 4.16 and 4.22 that the resulting iteration preserves cardinals and does not introduce reals. The latter consequence implies that the final extension satisfies CH.

6. Closing Remarks

I will finish this paper with some remarks and further consequences of the results so far. In Section 1 it was noted that, in hindsight, a more conventional approach to the main result should work. In particular, if we let $\mathcal{Q}(T)$ be the forcing that consists of the pairs $q = (X_q, \mathcal{U}_q)$, as in the definition of $\mathcal{Q}(T; \vec{f})$, then $\mathcal{Q}(T)$ introduces a subtree U of T into which T does not club-embed. Moreover, $\mathcal{Q}(T)$ can be iterated without adding reals. It seems likely that countable support iterations of forcings of this type preserve that U does not club-embed into T, though this probably involves a rather tedious argument if proved directly. This method would also not be as "portable" to future applications; see the further discussion at the end of this section.

Now consider the following theorem in relation to Theorem 1.7.

THEOREM 6.1. *Let C be a fixed Countryman line. The assertion that C is minimal can be made either true or false by proper forcing but cannot be changed by σ-closed forcing.*

In particular, $\diamondsuit^+$ does not imply that all minimal Aronszajn lines are Souslin. This partially addresses Baumgartner's question of whether his construction necessarily produces Souslin lines. The invariance of the minimality of C under σ-closed forcing is essentially due to Baumgartner [5]. Since MA$(\aleph_1)$ can always be forced by a proper (even countable chain condition) forcing [17], any Countryman line can be made minimal by proper forcing. On the other hand, Theorem 1.9 asserts that the conjunction of (A) and CH can be made true by proper forcing; by the subsequent theorems, this conjunction implies that C is not minimal.

We also have an example related to the following problem of Woodin. The reader is referred to [22] for undefined terminology.

QUESTION 6.2 [22]. Are there Π_2-sentences φ_1 and φ_2 such that

$$(H(\aleph_1^+), \in) \models \mathrm{CH} \wedge \varphi_1 \quad \text{and}$$
$$(H(\aleph_1^+), \in) \models \mathrm{CH} \wedge \varphi_2$$

are each Ω-consistent but whose conjunction Ω-implies the negation of CH?

This question is motivated by Woodin's celebrated result of [22] that the answer is negative if CH is replaced by ZFC. This offers an explanation of the observed phenomenon that every forcible Π_2-sentence about $H(\aleph_1^+)$ can be proved if one assumes a strong enough forcing axiom and, in particular, that all such sentences are mutually consistent. Whether the same can be said about the stronger theory ZFC + CH is the content of this question.

Let $\mathcal{L}^C$ be the expansion of the usual language $\mathcal{L}$ of set theory to include a predicate $\vec{C}$, and add an axiom asserting that $\vec{C}$ is a *C-sequence* of length ω_1—that is, C_α is a cofinal subset of α for every $\alpha < \omega_1$, and if $\gamma < \alpha$ then $C_\alpha \cap \gamma$ is finite. Such an extension of the language is not entirely contrived, since the analysis of minimal walks on ω_1 is based around a fixed C-sequence that is used to construct a number of 2-place "ϱ-functions". These have served as a unified approach to combinatorial constructions at this level (see [21]).

THEOREM 6.3. *There are two Π_2-sentences φ_1 and φ_2 in $\mathcal{L}$ and $\mathcal{L}^C$, respectively, such that*

$$(H(\aleph_1^+), \in) \models \mathrm{CH} \wedge \varphi_1 \quad \textit{and}$$
$$(H(\aleph_1^+), \in, \vec{C}) \models \mathrm{CH} \wedge \varphi_2$$

are each Ω-consistent but $\varphi_1 \wedge \varphi_2$ implies $2^{\aleph_0} = 2^{\aleph_1}$.

Theorem 6.1 also yields the following result, which is related to Steel's question on the Σ_2^2-completeness of $\diamondsuit$ and attendant assertions.

THEOREM 6.4. *There is a Σ_2-sentence ψ in $\mathcal{L}^C$ such that*

$$(H(\aleph_1^+), \in, \vec{C}) \models \psi$$

is Ω-independent although ψ is invariant under σ-closed forcing.

QUESTION 6.5. Is there a Σ_2-sentence ψ in the language of $(H(\aleph_1^+), \in)$ that is Ω-independent but invariant under σ-closed forcing?

Arguing as in [1, 2.3], Larson has noted that there exist T_x and subtrees $S_x \subseteq T_x$ indexed by 2^{ω_1} such that: (i) each $T_x \subseteq \omega^{<\omega_1}$ is coherent, is closed under finite changes, and consists of finite-to-one functions; and (ii) if $x \neq y$ and if ζ is the least ordinal such that $x(\zeta) \neq y(\zeta)$, then

$$(T_x)_\zeta = (T_y)_\zeta,$$
$$(S_x)_\zeta \cap (S_y)_\zeta = \emptyset.$$

This can be used with Lemma 2.9, Theorem 3.1, and [19, Thm. 3.4] to show that if $2^{\aleph_0} < 2^{\aleph_1}$ then there is a Countryman line that is not minimal. Hence it is not possible to remove the predicate in the previous examples by quantifying over all such C-sequences.

Finally, let us finish with the following question.

QUESTION 6.6. Is the forcing axiom for completely proper forcings (CPFA) consistent with CH relative to a large cardinal assumption?

The example of [15, XVIII.1] shows that some care needs to be taken in any approach to this question but does not suggest a negative answer. A positive answer would suggest a negative answer to Question 6.2.

While the conjunction of CPFA and CH is not known to be consistent, substantial fragments of CPFA are consistent with CH and there is a considerable body of literature surrounding this (see [2; 7; 15]). Observe that we have accomplished two tasks in this paper:

(1) the demonstration that the conjunction of CPFA and CH implies that ω_1 and $-\omega_1$ are the only minimal uncountable ordertypes; and
(2) that this conclusion requires only the fragment of CPFA that is known to be consistent with CH.

In recent joint work with Ishiu, Question 1.3 has been essentially reduced to Question 6.6.

THEOREM 6.7 [8]. *The conjunction of CPFA$^+$ and CH implies that there are no minimal non-σ-scattered ordertypes.*

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Department of Mathematics
Cornell University
555 Malott Hall
Ithaca, NY 14853-4201

justin@math.cornell.edu