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Monads and Regularity of Vector Bundles on Projective Varieties

L. Costa & R. M. Miró-Roig

1. Introduction

In the seventies, Horrocks showed that every vector bundle E on $\mathbb{P}^2$ and $\mathbb{P}^3$ admits a certain "double-ended resolution" by line bundles that he called *monads.* Monads appear in a wide variety of contexts within algebraic geometry, and they are now one of the most important tools that permit us to construct vector bundles with prescribed invariants (e.g., rank, determinant, Chern classes) and to study them with the methods of linear algebra. A large number of vector bundles on $\mathbb{P}^n$ are the cohomology of monads, and most of them can be constructed as cohomologies of quasi-linear monads (see Definition 2.9). Indeed, any vector bundle on $\mathbb{P}^3$ is the cohomology of a quasi-linear monad, and any instanton bundle on $\mathbb{P}^{2l+1}$ is also the cohomology of a quasi-linear monad. On the other hand, in the mid-sixties Mumford introduced the concept of regularity for a coherent sheaf F on a projective space $\mathbb{P}^n$. Since then, Castelnuovo–Mumford regularity has become a fundamental invariant—in both commutative algebra and algebraic geometry—for measuring the complexity of a sheaf. In [4; 5] we generalized the notion of Castelnuovo–Mumford regularity for coherent sheaves on projective spaces and also introduced a notion of regularity for coherent sheaves on other projective varieties (as multiprojective spaces, Grassmanians, and hyperquadrics $Q_n \subset \mathbb{P}^{n+1}$) that is equivalent to the Castelnuovo–Mumford regularity when the variety is a projective space $\mathbb{P}^n$.

The main goal of this paper is to bound the regularity of vector bundles E on $\mathbb{P}^n$ and Q_n defined as the cohomology of quasi-linear monads. As a by-product, we obtain a bound for the regularity of any vector bundle on $\mathbb{P}^3$, the regularity of any instanton bundle on $\mathbb{P}^{2l+1}$, and the regularity of any instanton bundle on a hyperquadric $Q_{2l+1} \subset \mathbb{P}^{2l+2}$.

The paper is organized as follows. In Section 2, we briefly recall the notion of regularity for coherent sheaves on projective varieties and summarize its main formal properties. We also recall the use of monads for constructing vector bundles and introduce the notion of a quasi-linear monad. Sections 3 and 4 are the heart of the paper. In Section 3, we give effective bounds for the regularity of any coherent sheaf on $\mathbb{P}^n$ that is the cohomology of a quasi-linear monad (see Theorem 3.2). In particular, we bound the regularity of any vector bundle on $\mathbb{P}^3$ as well as the

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regularity of any instanton bundle on $\mathbb{P}^{2n+1}$; at the end of the section, we prove that the bounds obtained so far are sharp. In Section 4, we bound the regularity of any coherent sheaf on a hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ that is the cohomology of a quasi-linear monad. We also give a bound for the regularity of any instanton bundle on Q_n, and by means of an example we will see that the bound is sharp (for the definition and existence of such bundles see [6]).

Notation. Throughout this paper, X will be a smooth projective variety defined over the complex numbers $\mathbb{C}$, and we denote by $\mathcal{D} = \mathcal{D}^b(\mathcal{O}_X\text{-mod})$ the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_X$-modules. Notice that $\mathcal{D}$ is an abelian linear triangulated category. We identify, as usual, any coherent sheaf F on X to the object $(0 \to F \to 0) \in \mathcal{D}$ concentrated in degree 0, and we will not distinguish between a vector bundle and its locally free sheaf of sections.

2. Regularity of Coherent Sheaves on Projective Varieties

Here we recall, for coherent sheaves on projective varieties, the definition of regularity with respect to an n-block collection; we then gather the main formal properties that it verifies. More information on the subject can be found in [4; 5].

Let X be a smooth projective variety of dimension n. Let $\sigma = (\mathcal{E}_0, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\beta_i}^i)$, be an n-block collection of coherent sheaves on X that generates $\mathcal{D}$, and let $\mathcal{H}_\sigma := \{\mathcal{E}_i\}_{i\in\mathbb{Z}}$ be the helix associated to σ. For any $j \in \mathbb{Z}$, denote by

$$\sigma_j^\vee := (\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_n), \quad \mathcal{H}_i = (H_1^i, \dots, H_{\alpha_i}^i),$$

the right dual n-block collection of $\sigma_j = (\mathcal{E}_j, \dots, \mathcal{E}_{j+n})$ (see [5] for the precise definitions). That is, $\sigma_j^\vee$ is the n-block collection that is uniquely determined by $\mathcal{H}_0 = \mathcal{E}_{j+n}$ and the orthogonality conditions

$$\operatorname{Hom}^p(H_j^i, E_l^k) = 0 \quad \text{for any } \ p \geq 0 \tag{2.1}$$

except for

$$\operatorname{Ext}^k(H_i^k, E_i^{m-k}) = \mathbb{C}. \tag{2.2}$$

In [5], the goal was to extend the notion of Castelnuovo–Mumford regularity for coherent sheaves on a projective space to coherent sheaves on an n-dimensional smooth projective variety with an n-block collection of coherent sheaves on X that generates $\mathcal{D}$. With notation as before, let us now recall the precise definition.

Definition 2.1. Let X be a smooth projective variety of dimension n with an n-block collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_i = (E_1^i, \dots, E_{\alpha_i}^i)$, of coherent sheaves on X that generates $\mathcal{D}$. Let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i\in\mathbb{Z}}$ be the helix of blocks associated to σ, and let F be a coherent $\mathcal{O}_X$-module. We say that F is *m-regular with respect to σ* if, for $q > 0$, we have

$$\begin{cases} \bigoplus_{s=1}^{\alpha_{-m+p}} \operatorname{Ext}^q(H_s^{-p}, F) = 0 & \text{for } -n \leq p \leq -1, \\ \bigoplus_{s=1}^{\alpha_{-m}} \operatorname{Ext}^q(E_s^{-m}, F) = 0 & \text{for } p = 0, \end{cases}$$

where $(\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_n)$, $\mathcal{H}_i = (H_1^i, \dots, H_{\alpha_i}^i)$, is the right dual n-block collection of $\sigma_{-m-n} = (\mathcal{E}_{-m-n}, \dots, \mathcal{E}_{-m})$. We define the *regularity of F with respect to σ*,

$\mathrm{Reg}_\sigma(F)$, as the least integer m such that F is m-regular with respect to σ. We set $\mathrm{Reg}_\sigma(F) = -\infty$ if there is no such integer.

REMARK 2.2. Roughly speaking, F is m-regular with respect to σ if the spectral sequence ${}^{-n-m}E_1^{pq} = 0$ given in [5, Thm. 3.10], situated in the square $-n \leq p \leq 0$ and $0 \leq q \leq n$, collapses at E_2. This fact motivates our definition of regularity and justifies the use of that notation.

EXAMPLE 2.3. Consider the n-block collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$ on $\mathbb{P}^n$ and the associated helix $\mathcal{H}_\sigma = \{\mathcal{O}_{\mathbb{P}^n}(i)\}_{i \in \mathbb{Z}}$. The right dual n-block collection of an n-block collection $\sigma_i = (\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(i+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(i+n))$ is

$$\sigma_i^\vee = (\mathcal{O}_{\mathbb{P}^n}(i+n), T_{\mathbb{P}^n}(i+n-1), \ldots, \textstyle\bigwedge^j T_{\mathbb{P}^n}(i+n-j), \ldots, \bigwedge^n T_{\mathbb{P}^n}(i)).$$

Hence, for any coherent sheaf F on $\mathbb{P}^n$, our definition reduces to: F is m-regular with respect to σ if $\mathrm{Ext}^q\big(\bigwedge^{-p} T(-m+p), F\big) = H^q(\mathbb{P}^n, \Omega^{-p}(m-p) \otimes F) = 0$ for all $q > 0$ and all p, where $-n \leq p \leq 0$.

In [13, Lec. 14], Mumford defined the notion of regularity for a coherent sheaf over a projective space. A coherent sheaf F on $\mathbb{P}^n$ is said to be m-regular in the sense of Castelnuovo–Mumford if $H^i(\mathbb{P}^n, F(m-i)) = 0$ for $i > 0$. The Castelnuovo–Mumford regularity of F, $\mathrm{Reg}^{\mathrm{CM}}(F)$, is the least integer m such that F is m-regular. It is important to remark that, for coherent sheaves on $\mathbb{P}^n$, the σ-regularity in the sense of Definition 2.1 and the Castelnuovo–Mumford regularity agree. Indeed, we can make the following statement.

PROPOSITION 2.4. *A coherent sheaf F on $\mathbb{P}^n$ is m-regular in the sense of Castelnuovo–Mumford if and only if it is m-regular with respect to the n-block collection $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$. Therefore,*

$$\mathrm{Reg}_\sigma(F) = \mathrm{Reg}^{\mathrm{CM}}(F).$$

Proof. See [4, Prop. 4.6; 5, Prop. 4.11]. □

To emphasize that the new notion of regularity generalizes the original definition of Castelnuovo–Mumford regularity, in [5] we proved that its basic formal properties remain true in this new setting. More precisely, by [5, Prop. 4.14 and Prop. 4.15] we have the following result.

PROPOSITION 2.5. *Let X be a smooth projective variety of dimension n with an n-block collection of coherent sheaves $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n)$, $\mathcal{E}_j = (E_1^j, \ldots, E_{\alpha_j}^j)$, that generates $\mathcal{D}$. Let F and G be coherent $\mathcal{O}_X$-modules.*

(a) *If F is m-regular with respect to σ, then the canonical map*

$$\textstyle\bigoplus_{s=1}^{\alpha_{-m}} \mathrm{Hom}(E_s^{-m}, F) \otimes E_s^{-m} \twoheadrightarrow F$$

is surjective and F is k-regular with respect to σ for any $k \geq m$.

(b) *If $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules, then*

$$\mathrm{Reg}_\sigma(F_2) \leq \max\{\mathrm{Reg}_\sigma(F_1), \mathrm{Reg}_\sigma(F_3)\}.$$

(c) $\mathrm{Reg}_\sigma(F \oplus G) = \max\{\mathrm{Reg}_\sigma(F), \mathrm{Reg}_\sigma(G)\}$.

The following technical result, which establishes the regularity with respect to $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ of any coherent sheaf $E_t^i \in \mathcal{E}_i$, will prove to be useful.

PROPOSITION 2.6. *Let X be a smooth projective variety of dimension n with an n-block collection $\sigma = (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$, $\mathcal{E}_j = (E_1^j, \dots, E_{\alpha_j}^j)$, that generates $\mathcal{D}$, and let $\mathcal{H}_\sigma = \{\mathcal{E}_i\}_{i\in\mathbb{Z}}$ be the associated helix. Then, for any $i \in \mathbb{Z}$ and any $E_t^i \in \mathcal{E}_i$, $\mathrm{Reg}_\sigma(E_t^i) = -i$.*

Proof. See [5, Prop. 4.9]. □

We will end this section by recalling the definition of and some basic facts about monads.

DEFINITION 2.7. Let X be a smooth projective variety. A *monad* on X is a complex of vector bundles

$$M_\bullet\colon 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

that is exact at A and at C. The sheaf $E := \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha)$ is called the *cohomology sheaf* of the monad $M_\bullet$.

REMARK 2.8. Clearly, the cohomology sheaf E of a monad $M_\bullet$ is always a coherent sheaf, but more can be said in particular cases. In fact: E is torsion free if and only if the localized maps α_x are injective away from a subset $Y \subset X$ of codimension 2; E is reflexive if and only if the localized maps α_x are injective away from a subset $Y \subset X$ of codimension 3; and E is locally free if and only if the localized maps α_x are injective for all $x \in X$.

A monad $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ has a *display,* which is a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r] \arrow[d, "\beta"'] & Q \arrow[r] \arrow[d] & 0; \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

here $K := \mathrm{Ker}(\beta)$ and $Q := \mathrm{Coker}(\alpha)$. From the display one easily deduces that, if a coherent sheaf E on X is the cohomology sheaf of a monad $M_\bullet$, then

(i) $\mathrm{rk}(E) = \mathrm{rk}(B) - \mathrm{rk}(A) - \mathrm{rk}(C)$ and
(ii) $c_t(E) = c_t(B)c_t(A)^{-1}c_t(C)^{-1}$.

DEFINITION 2.9. A monad on X is called *quasi-linear* if it has the form

$$0 \longrightarrow \bigoplus_{j=1}^{r} \mathcal{L}'_j \xrightarrow{\alpha} \bigoplus_{i=1}^{s} \mathcal{L}_i \xrightarrow{\beta} \bigoplus_{k=1}^{t} \mathcal{L}''_k \longrightarrow 0,$$

where $\mathcal{L}_i, \mathcal{L}'_j, \mathcal{L}''_k$ are line bundles on X.

Quasi-linear monads appear often in the literature. For instance, Horrocks proved that any vector bundle on $\mathbb{P}^3$ is the cohomology of a quasi-linear monad, and Okonek–Spindler proved that any instanton bundle on $\mathbb{P}^{2l+1}$ is the cohomology of a quasi-linear monad (see Section 3 for the precise statements). The goal of the sequel here is to bound the regularity of the cohomology bundle of a quasi-linear monad on a projective space $\mathbb{P}^n$ and on a hyperquadric Q_n.

3. Regularity of Sheaves on $\mathbb{P}^n$

In [11] Horrocks proved that any vector bundle E on $\mathbb{P}^3$ is the cohomology of a quasi-linear monad

$$0 \longrightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \bigoplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

The goal of this section is to give effective bounds for the Castelnuovo–Mumford regularity of any rank-r vector bundle E on $\mathbb{P}^n$ that is the cohomology of a quasi-linear monad

$$0 \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow \bigoplus_{k=1}^{r+s+t} \mathcal{O}_{\mathbb{P}^n}(b_k) \longrightarrow \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(c_j) \longrightarrow 0$$

in terms of the integers a_i, b_k, and c_j. In particular, we will bound the Castelnuovo–Mumford regularity of any vector bundle on $\mathbb{P}^3$ and the Castelnuovo–Mumford regularity of any mathematical instanton bundle on $\mathbb{P}^{2n+1}$, since mathematical instanton bundles are the cohomology of a quasi-linear monad of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}^{2n+2k} \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}(1)^k \longrightarrow 0.$$

REMARK 3.1. According to Proposition 2.4, all the bounds of the Castelnuovo–Mumford regularity of a coherent sheaf E on $\mathbb{P}^n$ are valid for the regularity of E with respect to $\sigma = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$ and vice versa.

Let us now prove the main result of this section.

THEOREM 3.2. *Let E be a rank-r vector bundle on $\mathbb{P}^n$ that is the cohomology of a quasi-linear monad*

$$0 \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(a_i) \xrightarrow{\alpha} \bigoplus_{k=1}^{r+s+t} \mathcal{O}_{\mathbb{P}^n}(b_k) \xrightarrow{\beta} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(c_j) \longrightarrow 0,$$

with $a_1 \le \cdots \le a_s$, $b_1 \le \cdots \le b_{r+s+t}$, and $c_1 \le \cdots \le c_t$. Let $c := \sum_{j=1}^{t} c_j$. Then E is m-regular for any integer m such that

$$m \ge \max\{(n-1)c_t - (b_1 + \cdots + b_{t+n}) - n + c + 1, -a_1, -b_1\}.$$

Proof. Consider the short exact sequences

$$\begin{gathered} 0 \longrightarrow K \longrightarrow \bigoplus_{k=1}^{r+s+t} \mathcal{O}_{\mathbb{P}^n}(b_k) \longrightarrow \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(c_j) \longrightarrow 0, \\ 0 \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow K \longrightarrow E \longrightarrow 0 \end{gathered} \tag{3.1}$$

associated to the monad

$$0 \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(a_i) \xrightarrow{\alpha} \bigoplus_{k=1}^{r+s+t} \mathcal{O}_{\mathbb{P}^n}(b_k) \xrightarrow{\beta} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(c_j) \longrightarrow 0,$$

where $K := \mathrm{Ker}(\beta)$. Using the cohomological exact sequences associated to them, it can be easily seen that, for any $p \geq \max\{-b_1 - n, -a_1 - n\}$,

$$H^i(E(p)) = H^i(K(p)) = 0 \quad \text{for } 2 \leq i \leq n;$$

also, for any $p \in \mathbb{Z}$ we have $H^1(E(p)) = H^1(K(p))$. To see for which p the equality $H^1(K(p)) = 0$ holds, consider the Buchsbaum–Rim complex associated to

$$\mathcal{F} := \bigoplus_{k=1}^{r+s+t} \mathcal{O}_{\mathbb{P}^n}(b_k) \xrightarrow{\beta} \mathcal{G} = \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(c_j) \longrightarrow 0,$$

that is, the complex

$$\begin{aligned} 0 &\longrightarrow S^{r+s-1}\mathcal{G}^* \otimes \bigwedge\nolimits^{r+s+t} \mathcal{F} \longrightarrow S^{r+s-2}\mathcal{G}^* \otimes \bigwedge\nolimits^{r+s+t-1} \mathcal{F} \longrightarrow \cdots \\ &\longrightarrow S^2\mathcal{G}^* \otimes \bigwedge\nolimits^{t+3} \mathcal{F} \longrightarrow \mathcal{G}^* \otimes \bigwedge\nolimits^{t+2} \mathcal{F} \longrightarrow \bigwedge\nolimits^{t+1} \mathcal{F} \\ &\longrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow 0. \end{aligned}$$

We cut this complex into short exact sequences as follows:

$$\begin{gathered} 0 \longrightarrow K \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow 0, \\ 0 \longrightarrow K_2 \longrightarrow \bigwedge\nolimits^{t+1} \mathcal{F} \longrightarrow K \otimes \mathcal{O}_{\mathbb{P}^n}(c) \longrightarrow 0, \\ 0 \longrightarrow K_3 \longrightarrow \mathcal{G}^* \otimes \bigwedge\nolimits^{t+2} \mathcal{F} \longrightarrow K_2 \longrightarrow 0, \\ \vdots \\ 0 \longrightarrow K_n \longrightarrow S^{n-2}\mathcal{G}^* \otimes \bigwedge\nolimits^{t+n-1} \mathcal{F} \longrightarrow K_{n-1} \longrightarrow 0, \\ 0 \longrightarrow K_{m+1} \longrightarrow S^{n-1}\mathcal{G}^* \otimes \bigwedge\nolimits^{t+n} \mathcal{F} \longrightarrow K_n \longrightarrow 0. \end{gathered}$$

We then consider the cohomological exact sequence associated to these short exact sequences tensored by $\mathcal{O}_{\mathbb{P}^n}(p - c)$:

$$\begin{aligned} \cdots &\longrightarrow H^1\big(\bigwedge\nolimits^{t+1} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)\big) \longrightarrow H^1(K \otimes \mathcal{O}_{\mathbb{P}^n}(p)) \\ &\longrightarrow H^2(K_2 \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)) \longrightarrow H^2\big(\bigwedge\nolimits^{t+1} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)\big) \longrightarrow \cdots \\ &\vdots \\ \cdots &\longrightarrow H^q\big(S^{q-1}\mathcal{G}^* \otimes \bigwedge\nolimits^{t+q} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)\big) \\ &\longrightarrow H^q(K_q \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)) \longrightarrow H^{q+1}(K_{q+1} \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)) \\ &\longrightarrow H^{q+1}\big(S^{q-1}\mathcal{G}^* \otimes \bigwedge\nolimits^{t+q} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p - c)\big) \longrightarrow \cdots. \end{aligned}$$

Observe that, for any $q > 0$,

$$S^{q-1}\mathcal{G}^* \otimes \bigwedge^{t+q} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p-c) = \bigoplus_l \mathcal{O}_{\mathbb{P}^n}(d_l),$$

where $d_l = (b_{k_1} + \cdots + b_{k_{t+q}}) - (c_{j_1} + \cdots + c_{j_{q-1}}) + p - c$ with $j_1 \le \cdots \le j_{q-1}$ and $k_1 < \cdots < k_{t+q}$. Hence, for any $p \ge (n-1)c_t - (b_1 + \cdots + b_{t+n}) - n + c$,

$$\begin{aligned} h^1(K(p)) &= h^2(K_2(p-c)) = \cdots = h^n(K_n(p-c)) \\ &\le h^n\big(S^{n-1}\mathcal{G}^* \otimes \bigwedge^{t+n} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(p-c)\big) = 0. \end{aligned}$$

Therefore, for any $p \ge (n-1)c_t - (b_1 + \cdots + b_{t+n}) - n + c$, we have $H^1(E(p)) = 0$ and hence, for any $i > 0$ and $m \ge \max\{(n-1)c_t - (b_1 + \cdots + b_{t+n}) - n + c + 1, -a_1, -b_1\}$,

$$H^i(E(m-i)) = 0;$$

that is, E is m-regular. □

As a by-product we obtain an immediate proof of a result concerning the regularity of mathematical instanton bundles on $\mathbb{P}^{2n+1}$. (See [7] for $n = 1$ and [2] for $n \ge 2$; see also [10] for better bounds for general instanton bundles on $\mathbb{P}^3$.)

Corollary 3.3. *If E is a mathematical instanton bundle on $\mathbb{P}^{2n+1}$ with quantum number k, then E is k-regular.*

Proof. It was proved by Okonek and Spindler [14] that any mathematical instanton bundle on $\mathbb{P}^{2n+1}$ is the cohomology of a quasi-linear monad of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}^{2n+2k} \longrightarrow \mathcal{O}_{\mathbb{P}^{2n+1}}(1)^k \longrightarrow 0.$$

Hence, the result follows from Theorem 3.2. □

Remark 3.4. Arguing as in Theorem 3.2, we can see that any Schwarzenberger type bundle is 0-regular—that is, any vector bundle E on $\mathbb{P}^n$ arising from an exact sequence of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^c \longrightarrow \mathcal{O}_{\mathbb{P}^n}^d \longrightarrow E \longrightarrow 0$$

is 0-regular.

We would like to know how far is the bound given in Theorem 3.2 from being sharp. We will end this section by showing cases where the bounds obtained in Theorem 3.2 are sharp.

Example 3.5. (1) Let a, b, c be integers such that $b \ne c(n+1)$. Any vector bundle E on $\mathbb{P}^n$ with $H^0(\mathbb{P}^n, E) = 0$ and that is the cohomology bundle of a quasi-linear monad of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

has $H^1(E) \ne 0$ (see [8] for the existence of such bundles). Indeed, using the display associated to the monad together with the fact that $H^0(\mathbb{P}^n, E) = 0$, we easily deduce that $h^1(\mathbb{P}^n, E) = c(n+1) - b \ne 0$. So, any vector bundle E on $\mathbb{P}^n$ with $H^0(\mathbb{P}^n, E) = 0$ and that is the cohomology of a quasi-linear monad of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^2 \longrightarrow 0$$

is 2-regular but is not 1-regular. Hence, the bound in Theorem 3.2 is sharp. In particular, the bound of Theorem 3.2 is sharp for any mathematical instanton bundle E on $\mathbb{P}^{2l+1}$ with quantum number $k = 2$. Indeed, any mathematical instanton bundle E on $\mathbb{P}^{2l+1}$ with quantum number $k = 2$ is the cohomology bundle of a quasi-linear monad of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}(-1)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}^{2l+4} \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}(1)^2 \longrightarrow 0,$$

and the condition $H^0(\mathbb{P}^{2l+1}, E) = 0$ follows from [1, Lemma 3.4] if $l \geq 2$ and from the stability of E if $l = 1$.

(2) By [7, Prop. IV.1], the bound given in Theorem 3.2 is sharp for certain mathematical instanton bundles on $\mathbb{P}^3$ with quantum number $k \geq 1$. In fact, any mathematical instanton bundle E on $\mathbb{P}^3$ associated to $k+1$ lines on a smooth quadric verifies $h^1(E(k-2)) \neq 0$ and is the cohomology bundle of a quasi-linear monad of the type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^3}^{2k+2} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^k \longrightarrow 0.$$

Therefore, E is k-regular but is not $(k-1)$-regular, and the bound given in Theorem 3.2 is sharp. More generally, by [2, Thm. 3.6] it follows that, for any special symplectic instanton bundle E on $\mathbb{P}^{2l+1}$ with quantum number k and for any t with $-1 \leq t \leq k-2$, we have $H^1(E(t)) \neq 0$ and that E is the cohomology bundle of a quasi-linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}^{2l+2k} \longrightarrow \mathcal{O}_{\mathbb{P}^{2l+1}}(1)^k \longrightarrow 0.$$

Hence, for any special symplectic instanton bundle E on $\mathbb{P}^{2l+1}$ with quantum number k, the bound given in Theorem 3.2 is sharp.

4. Regularity of Sheaves on Hyperquadrics

In this section we will restrict our attention to coherent sheaves over $Q_n \subset \mathbb{P}^{n+1}$. We will bound the regularity of vector bundles that are the cohomology of quasi-linear monads, and in particular we will bound the regularity of mathematical instanton bundles on Q_{2l+1}.

Let $n \in \mathbb{Z}$ be an integer and let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. In [12], Kapranov defined the locally free sheaves ψ_i $(i \geq 0)$ on Q_n and considered the Spinor bundles on Q_n, Σ if n is odd and Σ_1, Σ_2 if n is even, to construct a resolution of the diagonal $\Delta \subset Q_n \times Q_n$ and to describe the bounded derived category $D^b(\mathcal{O}_{Q_n}\text{-mod})$. In particular, he found that $\bar{\sigma} = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$ where

$$\mathcal{F}_j = \mathcal{O}_{Q_n}(-n+j) \quad \text{for } 1 \leq j \leq n,$$

$$\mathcal{F}_0 = \begin{cases} (\Sigma_1(-n), \Sigma_2(-n)) & \text{if } n \text{ is even,} \\ (\Sigma(-n)) & \text{if } n \text{ is odd} \end{cases}$$

is an n-block collection of locally free sheaves on Q_n that generates $\mathcal{D}$ [12, Prop. 4.9]. Dualizing each bundle of this collection and reversing the order, we get that $\sigma_0 := (\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n)$ where

$$\mathcal{E}_j = \mathcal{O}_{Q_n}(j) \quad \text{for } 0 \le j \le n-1,$$
$$\mathcal{E}_n = \begin{cases} (\Sigma_1(n-1), \Sigma_2(n-1)) & \text{if } n \text{ is even},\\ (\Sigma(n-1)) & \text{if } n \text{ is odd} \end{cases}$$

is also an n-block collection of locally free sheaves on Q_n that generates $\mathcal{D}$.

In order to define the right dual n-block collection $\sigma_j^\vee$ of any subcollection

$$\sigma_j := (\mathcal{E}_j, \mathcal{E}_{j+1}, \dots, \mathcal{E}_{j+n})$$

of the helix $\mathcal{H}_{\sigma_0} = \{\mathcal{E}_i\}_{i\in\mathbb{Z}}$ associated to σ_0, we shall give an elementary description of the locally free sheaves ψ_i and their basic properties (for more details the reader can refer to [12]). With notation as before, we will collect the cohomological properties of the Spinor bundles we need later.

LEMMA 4.1. *Let $n = 2l + e \in \mathbb{Z}$ $(e = 0, 1)$ be an integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface, and let j be an integer with $1 \le j \le n$.*

(i) *For any i such that $0 < i < n$ and for all $t \in \mathbb{Z}$,*

$$H^i(Q_{2l+1}, \Sigma(t)) = H^i(Q_{2l}, \Sigma_1(t)) = H^i(Q_{2l}, \Sigma_2(t)) = 0.$$

(ii) *For all $t < 0$,*

$$H^0(Q_{2l+1}, \Sigma(t)) = H^0(Q_{2l}, \Sigma_1(t)) = H^0(Q_{2l}, \Sigma_2(t)) = 0$$

and

$$h^0(Q_{2l+1}, \Sigma) = h^0(Q_{2l}, \Sigma_1) = h^0(Q_{2l}, \Sigma_2) = 2^{\lceil (n+1)/2 \rceil}.$$

(iii) $\operatorname{Ext}^i(\Sigma(j), \Sigma) = \begin{cases} 0 & \text{if } i \ne j,\\ \mathbb{C} & \text{if } i = j. \end{cases}$

(iv) $\operatorname{Ext}^i(\Sigma_2(j), \Sigma_1) = \operatorname{Ext}^i(\Sigma_1(j), \Sigma_2) = \begin{cases} 0 & \text{if } i \ne j \text{ or } j \equiv 0 \bmod 2,\\ \mathbb{C} & \text{if } i = j \text{ and } j \equiv 1 \bmod 2. \end{cases}$

(v) $\operatorname{Ext}^i(\Sigma_1(j), \Sigma_1) = \operatorname{Ext}^i(\Sigma_2(j), \Sigma_2) = \begin{cases} 0 & \text{if } i \ne j \text{ or } j \equiv 1 \bmod 2,\\ \mathbb{C} & \text{if } i = j \text{ and } j \equiv 0 \bmod 2. \end{cases}$

Proof. The assertions (i) and (ii) follow from [15, Thm. 2.3]. The assertion (iii) follows by induction on j, using (i) and (ii), the fact that Σ is an exceptional vector bundle, and the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}(\mathcal{O}_{Q_n}(j)^{2(n+1)/2}, \Sigma) \longrightarrow \operatorname{Ext}^{i-1}(\Sigma(j-1), \Sigma)$$
$$\longrightarrow \operatorname{Ext}^i(\Sigma(j), \Sigma) \longrightarrow \operatorname{Ext}^i(\mathcal{O}_{Q_n}(j)^{2(n+1)/2}, \Sigma) \longrightarrow \cdots$$

obtained by applying the functor $\operatorname{Hom}(\cdot, \Sigma)$ to the exact sequence

$$0 \longrightarrow \Sigma(j-1) \longrightarrow \mathcal{O}_{Q_n}(j)^{2(n+1)/2} \longrightarrow \Sigma(j) \longrightarrow 0.$$

The assertions (iv) and (v) follow by induction on j, using (i) and (ii), the fact that (Σ_1, Σ_2) is an exceptional pair of vector bundles, and (for $1 \le m, k \le 2$) the long exact sequences

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}(\mathcal{O}_{Q_n}(j)^{2^l}, \Sigma_k) \longrightarrow \operatorname{Ext}^{i-1}(\Sigma_m(j-1), \Sigma_k)$$
$$\longrightarrow \operatorname{Ext}^i(\Sigma_k(j), \Sigma_k) \longrightarrow \operatorname{Ext}^i(\mathcal{O}_{Q_n}(j)^{2^l}, \Sigma_k) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}(\mathcal{O}_{Q_n}(j)^{2^l}, \Sigma_m) \longrightarrow \operatorname{Ext}^{i-1}(\Sigma_m(j-1), \Sigma_m)$$
$$\longrightarrow \operatorname{Ext}^{i}(\Sigma_k(j), \Sigma_m) \longrightarrow \operatorname{Ext}^{i}(\mathcal{O}_{Q_n}(j)^{2^l}, \Sigma_m) \longrightarrow \cdots$$

obtained by applying the functors $\operatorname{Hom}(\cdot, \Sigma_k)$, $1 \le k \le 2$, to the exact sequences

$$0 \longrightarrow \Sigma_1(j-1) \longrightarrow \mathcal{O}_{Q_n}(j)^{2^l} \longrightarrow \Sigma_2(j) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma_2(j-1) \longrightarrow \mathcal{O}_{Q_n}(j)^{2^l} \longrightarrow \Sigma_1(j) \longrightarrow 0. \qquad \square$$

From now on, we set $\Omega^j := \Omega^j_{\mathbb{P}^{n+1}}$ and define ψ_j inductively as

$$\psi_0 := \mathcal{O}_{Q_n}, \qquad \psi_1 := \Omega^1(1)|_{Q_n}.$$

For all $j \ge 2$, we may define the locally free sheaf ψ_j as the unique nonsplitting extension

$$0 \longrightarrow \Omega^j(j)|_{Q_n} \longrightarrow \psi_j \longrightarrow \psi_{j-2} \longrightarrow 0$$

(note that $\operatorname{Ext}^1(\psi_{j-2}, \Omega^j(j)|_{Q_n}) = \mathbb{C}$). In particular, $\psi_{j+2} = \psi_j$ for $j \ge n$ and $\psi_n = \Sigma(-1)^{2\lceil (n+1)/2 \rceil}$.

The locally free shaves ψ_j have the following cohomological properties.

LEMMA 4.2. *Let $n = 2l + e \in \mathbb{Z}$ $(e = 0, 1)$ be an integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface, and let j be an integer with $1 \le j \le n$. Then, for p, i with $1 \le p, i \le n-1$, the following statements hold.*

(i) *If $i \equiv 1 \bmod 2$ and $p \ne 1, \dots, i$ then, for any $l \in \mathbb{Z}$,*

$$H^p(\psi_i(l)) = 0.$$

(ii) *If $i \equiv 1 \bmod 2$ and $p \in \{1, \dots, i\}$, then*

$$H^p(\psi_i(l)) = \begin{cases} 0 & \text{if } l \ne -p \text{ and } p \equiv 1 \bmod 2, \\ 0 & \text{if } l \ne -p+1 \text{ and } p \equiv 0 \bmod 2. \end{cases}$$

(iii) *If $i \equiv 0 \bmod 2$ and $p \ne 1, \dots, i$ then, for any $l \in \mathbb{Z}$,*

$$H^p(\psi_i(l)) = 0.$$

(iv) *If $i \equiv 0 \bmod 2$ and $p \in \{1, \dots, i\}$, then*

$$H^p(\psi_i(l)) = \begin{cases} 0 & \text{if } l \ne -p \text{ and } p \equiv 0 \bmod 2, \\ 0 & \text{if } l \ne -p+1 \text{ and } p \equiv 1 \bmod 2. \end{cases}$$

For $p = 0, n$ and $1 \le i \le n-1$:

(v) $H^0(\psi_i(l)) = 0$ *for any $l < 0$;*
(vi) $H^n(\psi_i(l)) = 0$ *for any $l > -k$.*

Proof. First we consider the cohomological exact sequence

$$\cdots \longrightarrow H^p(\Omega^i(i+l)) \longrightarrow H^p(\Omega^i(i+l)|_{Q_n})$$
$$\longrightarrow H^{p+1}(\Omega^i(i+l-2)) \longrightarrow \cdots \qquad (4.1)$$

associated to the exact sequence

$$0 \longrightarrow \Omega^j(j-2) \longrightarrow \Omega^j(j) \longrightarrow \Omega^j(j)|_{Q_n} \longrightarrow 0.$$

Using Bott's formulas, from (4.1) we deduce that, for any $l \in \mathbb{Z}$,

$$\begin{aligned} H^p(\Omega^j(l)|_{Q_n}) &= 0 \quad \text{if } p \neq j \text{ or } j-1, \\ H^{i-1}(\Omega^i(i+l)|_{Q_n}) &= 0 \quad \text{if } l \neq -i+2, \\ H^i(\Omega^i(i+l)|_{Q_n}) &= 0 \quad \text{if } l \neq -i, \\ H^n(\Omega^i(i+l)|_{Q_n}) &= 0 \quad \text{if } l > -n, \\ H^0(\Omega^i(i+l)|_{Q_n}) &= 0 \quad \text{if } l < 0. \end{aligned} \tag{4.2}$$

Assume that $i \equiv 1 \bmod 2$ (we leave the case $i \equiv 0 \bmod 2$ to the reader). If $i = 1$ then, by definition, $\psi_1(l) = \Omega^1(1+l)|_{Q_n}$. Hence the result follows from (4.2). For $i \geq 3$, consider the cohomological exact sequence

$$\cdots \longrightarrow H^p(\Omega^i(i+l)|_{Q_n}) \longrightarrow H^p(\psi_i(l)) \longrightarrow H^p(\psi_{i-2}(l)) \longrightarrow \cdots \tag{4.3}$$

associated to the exact sequence

$$0 \longrightarrow \Omega^i(i+l)|_{Q_n} \longrightarrow \psi_i(l) \longrightarrow \psi_{i-2}(l) \longrightarrow 0.$$

The result follows if we argue by induction on i, using (4.2) and the exact sequence (4.3). $\square$

REMARK 4.3. According to Lemma 4.2, for any $1 \leq p \leq n-1$ and $k \in \mathbb{Z} \setminus \{0, -2, \ldots, -j\}$, if $j \equiv 0 \bmod 2$ and $k \in \mathbb{Z} \setminus \{-1, -3, \ldots, -j\}$ if $j \equiv 1 \bmod 2$, then

$$H^p(Q_n, \psi_j(k)) = 0.$$

PROPOSITION 4.4. *Let $n \in \mathbb{Z}$ be an integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface, and let $\mathcal{H}_{\sigma_0} = \{\mathcal{E}_i\}_{i \in \mathbb{Z}}$ be the helix associated to*

$$\sigma_0 = (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \ldots, \mathcal{O}_{Q_n}(n-1), \mathcal{E}_n),$$

$$\mathcal{E}_n = \begin{cases} (\Sigma_1(n-1), \Sigma_2(n-1)) & \textit{if } n \textit{ is even}, \\ (\Sigma(n-1)) & \textit{if } n \textit{ is odd}. \end{cases}$$

Then the following statements hold.

(i) *The right dual base of σ_0 is*

$$(\mathcal{H}_0, \psi_{n-1}(n), \psi_{n-2}(n), \ldots, \psi_1(n), \psi_0(n)),$$

$$\mathcal{H}_0 = \begin{cases} (\Sigma_1(n-1), \Sigma_2(n-1)) & \textit{if } n \textit{ is even}, \\ (\Sigma(n-1)) & \textit{if } n \textit{ is odd}. \end{cases}$$

(ii) *For any j $(1 \leq j \leq n)$, the right dual base of the n-block collection*

$$\begin{aligned} \sigma_j = (&\mathcal{O}_{Q_n}(j), \ldots, \mathcal{O}_{Q_n}(n-1), \mathcal{E}_{n-j}, \\ &\mathcal{O}_{Q_n}(n), \mathcal{O}_{Q_n}(n+1), \ldots, \mathcal{O}_{Q_n}(n+j-1)), \end{aligned}$$

$$\mathcal{E}_{n-j} = \begin{cases} (\Sigma_1(n-1), \Sigma_2(n-1)) & \textit{if } n \textit{ is even}, \\ (\Sigma(n-1)) & \textit{if } n \textit{ is odd} \end{cases}$$

is

$$(\mathcal{O}_{Q_n}(n+j-1), \psi_1^*(n+j-1), \dots, \psi_{j-1}^*(n+j-1),\\ \mathcal{H}_j, \psi_{n-j-1}(n+j), \dots, \psi_0(n+j)),$$

$$\mathcal{H}_j = \begin{cases} (\Sigma_1(n+j-1), \Sigma_2(n+j-1)) & \text{if } n \text{ is even,} \\ (\Sigma(n+j-1)) & \text{if } n \text{ is odd.} \end{cases}$$

(iii) *For any $\lambda \in \mathbb{Z}$, the right dual base of the n-block collection*

$$\sigma_{\lambda(n+1)} = (\mathcal{O}_{Q_n}(\lambda n), \mathcal{O}_{Q_n}(1+\lambda n), \dots, \mathcal{O}_{Q_n}(n-1+\lambda n), \mathcal{E}_{\lambda(n+1)+n}),$$

$$\mathcal{E}_{\lambda(n+1)+n} = \begin{cases} (\Sigma_1(n-1+\lambda n), \Sigma_2(n-1+\lambda n)) & \text{if } n \text{ is even,} \\ (\Sigma(n-1+\lambda n)) & \text{if } n \text{ is odd} \end{cases}$$

is

$$(\mathcal{H}_0, \psi_{n-1}((\lambda+1)n), \psi_{n-2}((\lambda+1)n), \dots, \psi_1((\lambda+1)n), \psi_0((\lambda+1)n)),$$

$$\mathcal{H}_0 = \begin{cases} (\Sigma_1(n-1+\lambda n), \Sigma_2(n-1+\lambda n)) & \text{if } n \text{ is even,} \\ (\Sigma(n-1+\lambda n)) & \text{if } n \text{ is odd.} \end{cases}$$

(iv) *For any j $(1 \le j \le n)$ and any $\lambda \in \mathbb{Z}$, the right dual base of the n-block collection*

$$\sigma_{j+\lambda(n+1)} = (\mathcal{O}_{Q_n}(j+\lambda n), \dots, \mathcal{O}_{Q_n}(n-1+\lambda n), \mathcal{E}_{\lambda(n+1)+n},\\ \mathcal{O}_{Q_n}((\lambda+1)n), \dots, \mathcal{O}_{Q_n}(j-2+(\lambda+1)n),\\ \mathcal{O}_{Q_n}(j-1+(\lambda+1)n)),$$

$$\mathcal{E}_{\lambda(n+1)+n} = \begin{cases} (\Sigma_1(n-1+\lambda n), \Sigma_2(n-1+\lambda n)) & \text{if } n \text{ is even,} \\ (\Sigma(n-1+\lambda n)) & \text{if } n \text{ is odd} \end{cases}$$

is

$$(\mathcal{O}_{Q_n}((\lambda+1)n+j-1), \psi_1^*((\lambda+1)n+j-1), \dots,\\ \psi_{j-1}^*((\lambda+1)n+j-1), \mathcal{H}_j, \psi_{n-j-1}((\lambda+1)n+j), \dots,\\ \psi_0((\lambda+1)n+j)),$$

$$\mathcal{H}_j = \begin{cases} (\Sigma_1((\lambda+1)n+j-1), \Sigma_2((\lambda+1)n+j-1)) & \text{if } n \text{ is even and } j \equiv 0 \bmod 2, \\ (\Sigma_2((\lambda+1)n+j-1), \Sigma_1((\lambda+1)n+j-1)) & \text{if } n \text{ is even and } j \equiv 1 \bmod 2, \\ (\Sigma((\lambda+1)n+j-1)) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The case of n odd is proved in [4, Prop. 4.2], so let us assume that n is even. Since the proof is similar in all the cases, we will prove (i) and leave the remaining cases to the reader. The orthogonality conditions (2.1) and (2.2) uniquely determine right dual basis, so the collection

$$(H_1^0, H_2^0, H_1^1, H_1^2, \dots, H_1^n)\\ = (\Sigma_1(n-1), \Sigma_2(n-1), \psi_{n-1}(n), \psi_{n-2}(n), \dots, \psi_1(n), \psi_0(n))$$

is the right dual base of

$$(E_1^0, E_1^1, E_1^2, \dots, E_1^n, E_2^n) = (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \dots, \mathcal{O}_{Q_n}(n-1), \Sigma_1(n-1), \Sigma_2(n-1))$$

if and only if, for any i, j, k, l,

$$\mathrm{Ext}^p(H_j^i, E_l^k) = 0 \quad \text{for} \;\; p \geq 0$$

unless $\mathrm{Ext}^k(H_i^k, E_i^{n-k}) = \mathbb{C}$—that is, if and only if, for any $p \geq 0$, the following conditions are satisfied.

(a) $\mathrm{Ext}^k(\psi_{n-k}(n), \mathcal{O}_{Q_n}(n-k)) = \mathbb{C}$ for $1 \leq k \leq n$.
(b) $\mathrm{Hom}(\Sigma_1(n-1), \Sigma_1(n-1)) = \mathbb{C}$.
(c) $\mathrm{Hom}(\Sigma_2(n-1), \Sigma_2(n-1)) = \mathbb{C}$.
(d) $\mathrm{Ext}^p(\Sigma_2(n-1), \Sigma_1(n-1)) = 0$.
(e) $\mathrm{Ext}^p(\Sigma_1(n-1), \Sigma_2(n-1)) = 0$.
(f) $\mathrm{Ext}^p(\Sigma_1(n-1), \mathcal{O}(k)) = 0$ for $0 \leq k \leq n-1$.
(g) $\mathrm{Ext}^p(\Sigma_2(n-1), \mathcal{O}(k)) = 0$ for $0 \leq k \leq n-1$.
(h) $\mathrm{Ext}^p(\psi_{n-j}(n), \mathcal{O}_{Q_n}(k)) = 0$ for $k \neq n-j$ or $p \neq j$ $(1 \leq j \leq n)$.
(i) $\mathrm{Ext}^p(\psi_{n-j}(n), \Sigma_1(n-1)) = 0$ for $1 \leq j \leq n$.
(j) $\mathrm{Ext}^p(\psi_{n-j}(n), \Sigma_2(n-1)) = 0$ for $1 \leq j \leq n$.

Conditions (a)–(h) follow from Serre's duality, Lemma 4.1, and the fact that (Σ_1, Σ_2) is a strongly exceptional pair of vector bundles. On the other hand, by [12, 4.8 and Prop. 4.11] we have that

$$(\mathcal{O}, \psi_1^*, \psi_2^*, \dots, \psi_{n-1}^*, \Sigma_1^*(1), \Sigma_2^*(1))$$

is an exceptional collection. Thus, for any $p \geq 0$, $1 \leq j \leq n$, and $i = 1, 2$,

$$\mathrm{Ext}^p(\psi_j, \Sigma_i(-1)) = 0,$$

and this proves (i) and (j). □

According to Proposition 2.5(c) and Proposition 2.6, if $a_1 \leq \cdots \leq a_r$, $b_1 \leq \cdots \leq b_s$, and $c_1 \leq \cdots \leq c_t$ then

$$\begin{aligned}\max\Big\{\mathrm{Reg}_{\sigma_0}\Big(\textstyle\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i)\Big), \mathrm{Reg}_{\sigma_0}\Big(\bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l)\Big), \mathrm{Reg}_{\sigma_0}\Big(\bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k)\Big)\Big\}& \\ = \max\{-(a_1+l_1), -(b_1+u_1), -(c_1+v_1)\},&\end{aligned}$$

where $a_1 = l_1 n + r_1$, $b_1 = u_1 n + s_1$, and $c_1 = v_1 n + k_1$ for some integers $0 \leq r_1, s_1, k_1 \leq n-1$. Keeping this notation, we can state the main result of this section as follows.

Theorem 4.5. *Let $n \in \mathbb{Z}$ be an integer, let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface, and let E be the cohomology of a monad*

$$0 \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i) \xrightarrow{\alpha} \bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l) \xrightarrow{\beta} \bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k) \longrightarrow 0$$

with $a_1 \leq \cdots \leq a_r$, $b_1 \leq \cdots \leq b_s$, $c_1 \leq \cdots \leq c_t$, and $\gamma = c_1 + \cdots + c_t$. Let

$$m = -\alpha(n+1) - j - n \geq \max\{-(a_1+l_1), -(b_1+u_1), -(c_1+v_1)\}$$

for some integers α and j $(1 \leq j \leq n)$, and assume that

$$\alpha n + (n-2)c_t - (b_1 + \cdots + b_{t+n}) + \gamma + j - 1 + n < 0.$$

Then E is m-regular with respect to the n-block collection

$$\sigma_0 = (\mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \dots, \mathcal{O}_{Q_n}(n-1), \mathcal{E}_n),$$
$$\mathcal{E}_n = \begin{cases} (\Sigma_1(n-1), \Sigma_2(n-1)) & \textit{if } n \textit{ is even,} \\ (\Sigma(n-1)) & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Denote by $\sigma^\vee_{-m-n} = (\mathcal{H}_0, \dots, \mathcal{H}_n)$, $\mathcal{H}_i = (H^i_1, \dots, H^i_{\alpha_i})$, the right dual n-block collection of $\sigma_{-m-n} = (\mathcal{E}_{-m-n}, \dots, \mathcal{E}_{-m})$. We need to show that, for any i and h with $0 \le i \le n$ and $1 \le h \le \alpha_i$,

$$H^q(E \otimes H^{i*}_h) = 0, \quad q > 0.$$

Toward this end, consider the short exact sequences

$$\begin{gathered} 0 \longrightarrow K \longrightarrow \bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l) \longrightarrow \bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k) \longrightarrow 0, \\ 0 \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i) \longrightarrow K \longrightarrow E \longrightarrow 0, \end{gathered} \tag{4.4}$$

where $K := \operatorname{Ker}(\beta)$. By assumption,

$$\begin{aligned} m \ge \max\Big\{&\operatorname{Reg}_{\sigma_0}\Big(\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i)\Big), \\ &\operatorname{Reg}_{\sigma_0}\Big(\bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l)\Big), \operatorname{Reg}_{\sigma_0}\Big(\bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k)\Big)\Big\}. \end{aligned}$$

As a result, according to Proposition 2.5(a), $\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i)$, $\bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l)$, and $\bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k)$ are m-regular with respect to σ_0. Hence, for any $q > 0$ and any i, h with $0 \le i \le n$ and $1 \le h \le \alpha_i$,

$$\begin{aligned} H^q\Big(\Big(\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i)\Big) \otimes H^{i*}_h\Big) &= H^q\Big(\Big(\bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l)\Big) \otimes H^{i*}_h\Big) \\ &= H^q\Big(\Big(\bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k)\Big) \otimes H^{i*}_h\Big) = 0. \end{aligned}$$

Considering the cohomological exact sequences

$$\begin{aligned} \cdots \longrightarrow H^q\Big(\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i) \otimes H^{i*}_h\Big) &\longrightarrow H^q(K \otimes H^{i*}_h) \\ &\longrightarrow H^q(E \otimes H^{i*}_h) \longrightarrow H^{q+1}\Big(\bigoplus_{i=1}^r \mathcal{O}_{Q_n}(a_i) \otimes H^{i*}_h\Big) \end{aligned}$$

and

$$\begin{aligned} \cdots \longrightarrow H^{q-1}\Big(\bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k) \otimes H^{i*}_h\Big) &\longrightarrow H^q(K \otimes H^{i*}_h) \\ &\longrightarrow H^q\Big(\bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l) \otimes H^{i*}_h\Big) \longrightarrow \cdots \end{aligned}$$

associated to (4.4), we obtain

$$\begin{gathered} H^q(E \otimes H^{i*}_h) = H^q(K \otimes H^{i*}_h), \quad q > 0, \\ H^q(K \otimes H^{i*}_h) = 0, \quad q \ge 2. \end{gathered}$$

Therefore, we now need only show that, for any i and h with $0 \le i \le n$ and $1 \le h \le \alpha_i$,

$$H^1(E \otimes H^{i*}_h) = H^1(K \otimes H^{i*}_h) = 0. \tag{4.5}$$

So consider the Buchsbaum–Rim complex associated to

$$\mathcal{F} := \bigoplus_{l=1}^s \mathcal{O}_{Q_n}(b_l) \longrightarrow \mathcal{G} := \bigoplus_{k=1}^t \mathcal{O}_{Q_n}(c_k) \longrightarrow 0;$$

that is,

$$\begin{aligned}0 \longrightarrow S^{s-t-1}\mathcal{G}^* \otimes \bigwedge^{s} \mathcal{F} \to S^{s-t-2}\mathcal{G}^* \otimes \bigwedge^{s-1} \mathcal{F} \longrightarrow \cdots\\ \longrightarrow S^2\mathcal{G}^* \otimes \bigwedge^{t+3} \mathcal{F} \longrightarrow \mathcal{G}^* \otimes \bigwedge^{t+2} \mathcal{F} \longrightarrow \bigwedge^{t+1} \mathcal{F}\\ \longrightarrow \mathcal{F} \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow \mathcal{G} \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow 0.\end{aligned}$$

We first cut the complex into short exact sequences:

$$\begin{gathered}0 \longrightarrow K \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow \mathcal{F} \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow \mathcal{G} \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow 0,\\ 0 \longrightarrow K_2 \longrightarrow \bigwedge^{t+1} \mathcal{F} \longrightarrow K \otimes \mathcal{O}_{Q_n}(\gamma) \longrightarrow 0,\\ 0 \longrightarrow K_3 \longrightarrow \mathcal{G}^* \otimes \bigwedge^{t+2} \mathcal{F} \longrightarrow K_2 \longrightarrow 0,\\ \vdots\\ 0 \longrightarrow K_n \longrightarrow S^{n-2}\mathcal{G}^* \otimes \bigwedge^{t+n-1} \mathcal{F} \longrightarrow K_{n-1} \longrightarrow 0,\\ 0 \longrightarrow K_{m+1} \longrightarrow S^{n-1}\mathcal{G}^* \otimes \bigwedge^{t+n} \mathcal{F} \longrightarrow K_n \longrightarrow 0.\end{gathered}$$

Then we consider the cohomological exact sequence associated to these short exact sequences tensored by $H_h^{i*} \otimes \mathcal{O}_{Q_n}(-\gamma)$:

$$\begin{aligned}\cdots \longrightarrow H^q\big(S^{q-1}\mathcal{G}^* \otimes \bigwedge^{t+q} \mathcal{F} \otimes H_h^{i*} \otimes \mathcal{O}_{Q_n}(-\gamma)\big)\\ \longrightarrow H^q(K_q \otimes H_h^{i*} \otimes \mathcal{O}_{Q_n}(-\gamma)) \longrightarrow H^{q+1}(K_{q+1} \otimes H_h^{i*} \otimes \mathcal{O}_{Q_n}(-\gamma))\\ \longrightarrow H^{q+1}\big(S^{q-1}\mathcal{G}^* \otimes \bigwedge^{t+q} \mathcal{F} \otimes H_h^{i*} \otimes \mathcal{O}_{Q_n}(-\gamma)\big) \longrightarrow \cdots .\end{aligned}$$

Observe that, for any $q > 0$,

$$S^{q-1}\mathcal{G}^* \otimes \bigwedge^{t+q} \mathcal{F} \otimes \mathcal{O}_{Q_n}(-\gamma) = \bigoplus\nolimits_l \mathcal{O}_{Q_n}(d_l),$$

where $d_l = (b_{j_1} + \cdots + b_{j_{t+q}}) - (c_{k_1} + \cdots + c_{k_{q-1}}) - \gamma$ with $k_1 \le \cdots \le k_{q-1}$ and $j_1 < \cdots < j_{t+q}$.

Hereafter we distinguish two cases according to the equivalence of $-m-n$ modulo $n+1$.

Case 1: Assume that $-m-n \equiv 0 \bmod (n+1)$; that is, $-m-n = \alpha(n+1)$. In this case, by Proposition 4.4(c) we have

$$\begin{aligned}&(\mathcal{H}_0, \ldots, \mathcal{H}_n)\\ &\quad = (\mathcal{H}_0, \psi_{n-1}((\alpha+1)n), \psi_{n-2}((\alpha+1)n), \ldots, \psi_1((\alpha+1)n), \psi_0((\alpha+1)n)),\end{aligned}$$

$$\mathcal{H}_0 = \begin{cases} (\Sigma_1(n-1+\alpha n), \Sigma_2(n-1+\alpha n)) & \text{if } n \text{ is even},\\ (\Sigma(n-1+\alpha n)) & \text{if } n \text{ is odd},\end{cases}$$

and (by assumption) $\alpha n + (n-2)c_t - (b_1 + \cdots + b_{t+n}) + \gamma - 1 + n < 0$. Hence, by Lemma 4.1 and Remark 4.3, the following statements hold.

- For any $1 \le q \le n-1$,

$$\begin{aligned}&H^q(\mathcal{O}_{Q_n}(d_l) \otimes \mathcal{H}_0^*)\\ &\quad = \begin{cases} H^{n-q}(\Sigma_1(\alpha n - 1 - d_l)) = H^{n-q}(\Sigma_2(\alpha n - 1 - d_l)) = 0 & \text{if } n \text{ is even},\\ H^{n-q}(\Sigma(\alpha n - 1 - d_l)) = 0 & \text{if } n \text{ is odd}.\end{cases}\end{aligned}$$

- For any $1 \le q \le n-1$ and i with $1 \le i \le n$,

$$H^q(\mathcal{O}_{Q_n}(d_l) \otimes H_h^{i*}) = H^{n-q}(\psi_{n-i}(\alpha n - d_l)) = 0.$$

Therefore,

$$H^q\big(S^{q-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+q+1}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = H^{q+1}\big(S^{q-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+q+1}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = 0$$

and hence

$$h^1(K \otimes H_h^{i*}) = h^2(K_2 \otimes H_h^{i*} \otimes \det(\mathcal{G})^*) = \cdots = h^n(K_n \otimes H_h^{i*} \otimes \det(\mathcal{G})^*) \leq h^n\big(S^{n-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+n}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = 0.$$

It then follows that $H^1(E \otimes H_h^{i*}) = H^1(K \otimes H_h^{i*}) = 0$, which finishes the proof in this case.

Case 2: Assume that $-m-n \equiv j \bmod (n+1)$ with $1 \leq j \leq n$; that is, assume $-m-n = \alpha(n+1)+j$ for some $1 \leq j \leq n$. In this case, by Proposition 4.4(d) we have

$$(\mathcal{H}_0, \dots, \mathcal{H}_n) = (\mathcal{O}_{Q_n}((\alpha+1)n+j-1), \psi_1^*((\alpha+1)n+j-1), \dots, \psi_{j-1}^*((\alpha+1)n+j-1), \mathcal{H}_j, \psi_{n-j-1}((\alpha+1)n+j), \dots, \psi_0((\alpha+1)n+j)),$$

$$\mathcal{H}_j = \begin{cases} (\Sigma_1((\alpha+1)n+j-1), \Sigma_2((\alpha+1)n+j-1) & \text{if } n \text{ is even,} \\ (\Sigma((\alpha+1)n+j-1) & \text{if } n \text{ is odd,} \end{cases}$$

and (by assumption) $\alpha n + (n-2)c_t - (b_1 + \cdots + b_{t+n}) + \gamma + j - 1 + n < 0$. Hence, by Lemma 4.1 and Remark 4.3, the following statements hold.

- For any $1 \leq q \leq n-1$ and $i = 0$,
$$H^q(\mathcal{O}_{Q_n}(d_l) \otimes H_h^{i*}) = H^q(\mathcal{O}_{Q_n}(d_l - (\alpha+1)n - j + 1)) = 0.$$
- For any $1 \leq q \leq n-1$ and i with $1 \leq i \leq j-1$,
$$H^q(\mathcal{O}_{Q_n}(d_l) \otimes H_h^{i*}) = H^q(\psi_i(d_l - (\alpha+1)n - j + 1)) = 0.$$
- For any $1 \leq q \leq n-1$ and $i = j$,
$$H^q(\mathcal{O}_{Q_n}(d_l) \otimes H_h^{i*}) = \begin{cases} H^{n-q}(\Sigma_1((\alpha+1)n+j-1-d_l)) = 0 & \text{if } n \text{ is even,} \\ H^{n-q}(\Sigma_2((\alpha+1)n+j-1-d_l)) = 0 & \text{if } n \text{ is even;} \\ H^{n-q}(\Sigma((\alpha+1)n+j-1-d_l)) = 0 & \text{if } n \text{ is odd.} \end{cases}$$
- For any $1 \leq q \leq n-1$ and i with $j+1 \leq i \leq n$,
$$H^q(\mathcal{O}_{Q_n}(d_l) \otimes H_h^{i*}) = H^{n-q}(\psi_{n-i}(\alpha n + j - d_l)) = 0.$$

Therefore,

$$H^q\big(S^{q-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+q+1}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = H^{q+1}\big(S^{q-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+q+1}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = 0$$

and hence

$$h^1(K \otimes H_h^{i*}) = h^2(K_2 \otimes H_h^{i*} \otimes \det(\mathcal{G})^*) = \cdots = h^n(K_n \otimes H_h^{i*} \otimes \det(\mathcal{G})^*) \leq h^n\big(S^{n-1}\mathcal{G}^* \otimes \textstyle\bigwedge^{t+n}\mathcal{F} \otimes H_h^{i*} \otimes \det(\mathcal{G})^*\big) = 0.$$

As a result, $H^1(E \otimes H_h^{i*}) = H^1(K \otimes H_h^{i*}) = 0$, which finishes the proof. $\square$

In [6] we introduced the notion of an instanton bundle on hyperquadrics $Q_n \subset \mathbb{P}^{n+1}$ (see [6, Prop. 4.7] for the existence of such bundles). Extending the ideas used in Theorem 4.5, we obtain a lower bound for the regularity of instanton bundles on hyperquadrics. The bound that we obtain in the next theorem is better than any we could obtain as a consequence of Theorem 4.5 because, with linear monads, we have better control of all the cohomology groups involved. By means of an example we will see that the bound obtained so far is sharp.

THEOREM 4.6. *Let E be a mathematical instanton bundle on Q_n $(n = 2l+1)$ with quantum number k, and let $m = -\alpha(n+1) - n - j \geq 2$ be an integer for some $\alpha, j \in \mathbb{Z}$ with $0 \leq j \leq n$. Assume that $m + \alpha \geq k$. Then E is m-regular with respect to the n-block collection*

$$\sigma_0 = (\mathcal{O}_{Q_{2l+1}}, \mathcal{O}_{Q_{2l+1}}(1), \dots, \mathcal{O}_{Q_{2l+1}}(2l), \Sigma(2l)).$$

Proof. Any mathematical instanton bundle E on Q_n is the cohomology bundle of a quasi-linear monad

$$0 \longrightarrow \mathcal{O}_{Q_{2l+1}}(-1)^k \xrightarrow{\alpha} \mathcal{O}_{Q_{2l+1}}^{2k+2l} \xrightarrow{\beta} \mathcal{O}_{Q_{2l+1}}(1)^k \longrightarrow 0. \tag{4.6}$$

Denote by $\sigma^\vee_{-m-n} = (H_0, \dots, H_n)$, the right dual n-block collection of $\sigma_{-m-n} = (\mathcal{E}_{-m-n}, \dots, \mathcal{E}_{-m})$. We must show that, for any i with $0 \leq i \leq n$,

$$H^q(E \otimes H_i^*) = 0, \quad q > 0.$$

Toward this end, consider the short exact sequences

$$\begin{gathered} 0 \longrightarrow K \longrightarrow \mathcal{O}_{Q_n}^{2k+2l} \longrightarrow \mathcal{O}_{Q_n}(1)^k \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_{Q_n}(-1)^k \longrightarrow K \longrightarrow E \longrightarrow 0 \end{gathered} \tag{4.7}$$

associated to the monad (4.6), where $K := \operatorname{Ker}(\beta)$. Because $m \geq 2$ (by assumption), it follows from Proposition 2.5(a) that $\mathcal{O}_{Q_n}(-1)^k$, $\mathcal{O}_{Q_n}^{2k+2l}$, and $\mathcal{O}_{Q_n}(1)^k$ are m-regular with respect to σ_0. Using the cohomological exact sequences

$$\begin{aligned} \cdots \longrightarrow H^q(\mathcal{O}_{Q_n}(-1)^k \otimes H_i^*) &\longrightarrow H^q(K \otimes H_i^*) \\ &\longrightarrow H^q(E \otimes H_i^*) \longrightarrow H^{q+1}(\mathcal{O}_{Q_n}(-1)^k \otimes H_i^*) \end{aligned}$$

and

$$\cdots \longrightarrow H^{q-1}(\mathcal{O}_{Q_n}(1)^k \otimes H_i^*) \longrightarrow H^q(K \otimes H_i^*) \longrightarrow H^q(\mathcal{O}_{Q_n}^{2k+2l} \otimes H_i^*) \longrightarrow \cdots$$

associated to (4.7), we obtain

$$\begin{aligned} H^q(E \otimes H_i^*) &= H^q(K \otimes H_i^*), \quad q > 0, \\ H^q(K \otimes H_i^*) &= 0, \quad q \geq 2. \end{aligned}$$

Hence, it remains only to show that, for any i with $0 \leq i \leq n$,

$$H^1(E \otimes H_i^*) = H^1(K \otimes H_i^*) = 0.$$

So consider the Buchsbaum–Rim complex associated to

$$\mathcal{O}_{Q_n}^{2k+2l} \longrightarrow \mathcal{O}_{Q_n}(1)^k \longrightarrow 0;$$

that is,

$$\begin{aligned}
&S^{k+2l-1}\mathcal{O}_{Q_n}(-1)^k \otimes \bigwedge^{2k+2l} \mathcal{O}_{Q_n}^{2k+2l} \\
&\quad\longrightarrow S^{k+2l-2}\mathcal{O}_{Q_n}(-1)^k \otimes \bigwedge^{2k+2l-1} \mathcal{O}_{Q_n}^{2k+2l} \longrightarrow \cdots \\
&\quad\longrightarrow S^{2}\mathcal{O}_{Q_n}(-1)^k \otimes \bigwedge^{k+3} \mathcal{O}_{Q_n}^{2k+2l} \longrightarrow \mathcal{O}_{Q_n}(-1)^k \otimes \bigwedge^{k+2} \mathcal{O}_{Q_n}^{2k+2l} \\
&\quad\longrightarrow \bigwedge^{k+1} \mathcal{O}_{Q_n}^{2k+2l} \longrightarrow \mathcal{O}_{Q_n}^{2k+2l} \otimes \mathcal{O}_{Q_n}(k) \longrightarrow \mathcal{O}_{Q_n}(1)^k \otimes \mathcal{O}_{Q_n}(k) \longrightarrow 0.
\end{aligned}$$

Cutting the complex into short exact sequences and twisting by $H_i^*(-k)$ then yields, for suitable integers $\lambda, \lambda_1, \dots, \lambda_n$ and ε,

$$0 \longrightarrow K \otimes H_i^* \longrightarrow H_i^{*\lambda} \longrightarrow H_i^{*\varepsilon}(1) \longrightarrow 0,$$

$$0 \longrightarrow K_2 \otimes H_i^*(-k) \longrightarrow H_i^*(-k)^{\lambda_1} \longrightarrow K \otimes H_i^* \longrightarrow 0,$$

$$0 \longrightarrow K_3 \otimes H_i^*(-k) \longrightarrow H_i^*(-k-1)^{\lambda_2} \longrightarrow K_2 \otimes H_i^*(-k) \longrightarrow 0,$$

$$\vdots$$

$$0 \longrightarrow K_n \otimes H_i^*(-k) \longrightarrow H_i^*(-k-n+2)^{\lambda_{n-1}} \longrightarrow K_{n-1} \otimes H_i^*(-k) \longrightarrow 0,$$

$$H_i^*(-k-n+1)^{\lambda_n} \longrightarrow K_n \otimes H_i^*(-k) \longrightarrow 0.$$

Hence, to prove $H^1(K \otimes H_i^*) = 0$ it suffices to show that, for any q with $1 \le q \le n-1$,

$$\begin{aligned}
H^q(H_i^*(-k-q+1)) &= 0,\\
H^{q+1}(H_i^*(-k-q+1)) &= 0, \qquad (4.8)\\
H^n(H_i^*(-k-n+1)) &= 0.
\end{aligned}$$

Assume that $-m-n = \alpha(n+1)+j$ for some $1 \le j \le n$. In this case, by Proposition 4.4(iv) we have

$$\begin{aligned}
(H_0, \dots, H_n) = (&\mathcal{O}_{Q_n}((\alpha+1)n+j-1), \psi_1^*((\alpha+1)n+j-1), \dots,\\
&\psi_{j-1}^*((\alpha+1)n+j-1), \Sigma((\alpha+1)n+j-1),\\
&\psi_{n-j-1}((\alpha+1)n+j), \dots, \psi_0((\alpha+1)n+j)).
\end{aligned}$$

By assumption, $m+\alpha \ge k$; that is, $\alpha n+n+j+k \le 0$. In particular,

$$\alpha n+n+j+k \ne 1,2,3.$$

Hence, by Lemma 4.1 and Lemma 4.2, the following statements hold.

- For any $1 \le q \le n-1$,

$$\begin{aligned}
H^q(\mathcal{O}_{Q_n}(-(\alpha+1)n-j+1-k-q+1)) &= 0,\\
H^{q+1}(\mathcal{O}_{Q_n}(-(\alpha+1)n-j+1-k-q+1)) &= 0,\\
H^n(\mathcal{O}_{Q_n}(-(\alpha+1)n-j+1-k-n+1)) &= 0.
\end{aligned}$$

- For any $1 \le q \le n-1$ and i with $1 \le i \le j-1$,

$$\begin{aligned}
H^q(\psi_i(-(\alpha+1)n-j+1-k-q+1)) &= 0,\\
H^{q+1}(\psi_i(-(\alpha+1)n-j+1-k-q+1)) &= 0,\\
H^n(\psi_i(-(\alpha+1)n-j+1-k-n+1)) &= 0.
\end{aligned}$$

- For any $1 \leq q \leq n-1$, since $\Sigma^* \cong \Sigma(-1)$ it follows that

$$H^q(\Sigma^*(-(\alpha+1)n - j + 1 - k - q + 1)) = 0,$$
$$H^{q+1}(\Sigma^*(-(\alpha+1)n - j + 1 - k - q + 1)) = 0,$$
$$H^n(\Sigma^*(-(\alpha+1)n - j + 1 - k - n + 1)) = 0.$$

- For any $1 \leq q \leq n-1$ and i with $j+1 \leq i \leq n$,

$$H^q(\psi^*_{n-i}(-(\alpha+1)n - j - k - q + 1)) = 0,$$
$$H^{q+1}(\psi^*_{n-i}(-(\alpha+1)n - j - k - q + 1)) = 0,$$
$$H^n(\psi^*_{n-i}(-(\alpha+1)n - j - k - n + 1)) = 0.$$

Therefore, conditions (4.8) are satisfied and thus

$$H^1(E \otimes H_i^*) = H^1(K \otimes H_i^*) = 0.$$

The case $j = 0$ follows in exactly the same way and so is left to the reader. □

It would be nice to know how far are the bounds given in Theorem 4.5 and Theorem 4.6 from being sharp. In the next example we will see that the bound of Theorem 4.6 is indeed sharp.

EXAMPLE 4.7. Let a, b, c, d, e, f, g be homogenous coordinates in $\mathbb{P}^6$, let $Q_5 \subset \mathbb{P}^6$ be the hyperquadric defined by the equation $a^2+b^2+c^2+d^2+e^2+f^2+g^2 = 0$, and consider the matrix with linear entries

$$A = \begin{pmatrix} a & b & c & 0 & 0 & d & e & f & 0 & 0 \\ 0 & a & b & c & 0 & 0 & d & e & f & 0 \\ 0 & 0 & a & b & c & 0 & 0 & d & e & f \end{pmatrix}.$$

Let $\alpha\colon \mathcal{O}_{Q_5}(-1)^3 \longrightarrow \mathcal{O}_{Q_5}^{10}$ be the morphism associated to the 3×10 matrix A and let $\beta\colon \mathcal{O}_{Q_5}^{10} \longrightarrow \mathcal{O}_{Q_5}(1)^3$ be the morphism associated to the 10×3 matrix with linear entries $B = A^t$; for this we transpose with respect to the standard symplectic form

$$G := \begin{pmatrix} 0 & -1_4 \\ 1_4 & 0 \end{pmatrix}.$$

Since the localized maps α_x are injective for all $x \in Q_5$, the cohomology sheaf of the monad

$$0 \longrightarrow \mathcal{O}_{Q_5}(-1)^3 \xrightarrow{\alpha} \mathcal{O}_{Q_5}^{10} \xrightarrow{\beta} \mathcal{O}_{Q_5}(1)^3 \longrightarrow 0$$

is a rank-4 vector bundle E on Q_5. By Theorem 4.6, E is 5-regular with respect to $\sigma_0 = (\mathcal{O}_{Q_5}, \mathcal{O}_{Q_5}(1), \ldots, \mathcal{O}_{Q_5}(4), \Sigma(4))$. On the other hand, E is by definition 4-regular with respect to σ_0 if and only if, for any $q > 0$,

$$H^q(E(3)) = H^q(E \otimes \psi_1(3)) = H^q(E \otimes \psi_2(3)) = 0,$$
$$H^q(E \otimes \Sigma^*(3)) = H^q(E \otimes \psi_1^*(2)) = H^q(E(2)) = 0.$$

Using Macaulay [9], we can compute the cohomology of E; in particular, we obtain

$$H^1(E(2)) \cong \mathbb{C}^{10}.$$

Hence E is not 4-regular with respect to σ_0. Therefore, for this instanton bundle, the bound given in Theorem 4.6 is sharp.

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L. Costa
Facultat de Matemàtiques
Departament d'Algebra i Geometria
Gran Via de les Corts Catalanes 585
08007 Barcelona
Spain

costa@ub.edu

R. M. Miró-Roig
Facultat de Matemàtiques
Departament d'Algebra i Geometria
Gran Via de les Corts Catalanes 585
08007 Barcelona
Spain

miro@ub.edu