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On Homaloidal Polynomials

Andrea Bruno

Let $\mathbb{P}^n$ be the projective space over a field k. If F is a homogeneous polynomial, we say that F is *homaloidal* if the polar map ∂F defined by the partial derivatives of F is a birational selfmap of $\mathbb{P}^n$. Although the problem of determining homaloidal polynomials has a classical flavor, the theme was only recently raised in an algebro-geometric context by Dolgachev [Do] following suggestions stemming from the theory of prehomogeneous varieties: the relative invariants of prehomogeneous spaces are, in fact, homaloidal polynomials [EKP; KiSa]. Dolgachev classifies square free homaloidal polynomials in $\mathbb{P}^2$ (see also [D]) and characterizes square free homaloidal polynomials in $\mathbb{P}^3$ that are products of four independent linear forms. Dolgachev also raises the following question: Is it true that a non–square free product of linear forms is homaloidal if and only if the product of its factors with multiplicity 1 is? This question has been given a positive answer in a specific case (see [KrS]) and in full generality (see [DP]) in a topological context. We will give an algebraic proof of the following result.

Theorem A. *Suppose that k is of characteristic* 0. *Let $L_0, \dots, L_r$ be linear forms and let $m_0, \dots, m_r$ be positive integers. Then $F = \prod_{i=0}^r L_i^{m_i}$ is a homaloidal polynomial if and only if* (i) $F_{\mathrm{red}} = \prod_{i=0}^r L_i$ *is homaloidal and* (ii) $r = n$ *and the linear forms $L_0, \dots, L_n$ are independent.* (Here the subscript "red" denotes "reduced".)

In particular, square free homaloidal polynomials that split as the product of linear forms all induce, up to a projectivity, standard Cremona transformations. The hypothesis on the characteristic of the ground field is essential because we need resolution of singularities. When this is possible, we obtain the following result.

Theorem B. *Assume that resolution of singularities holds in characteristic p and dimension n* (*e.g., if $n = 2$ or $n = 3$ and $p \geq 7$*). *Then $F = \prod_{i=0}^r L_i^{m_i}$ is a homaloidal polynomial if and only if* (i) *p does not divide m_i for each i and* (ii) $F_{\mathrm{red}} = \prod_{i=0}^r L_i$ *is homaloidal if and only if p does not divide m_i for each i and $r = n$ and the linear forms $L_0, \dots, L_n$ are independent.*

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1. Preliminaries

We start with the following classical definition.

DEFINITION 1. If $F \in H^0(\mathbb{P}^n, \mathcal{O}_{P^n}(d))$ is a homogeneous polynomial, then the *polar map defined by* F is the rational map

$$\mathbb{P}^n \overset{\partial F}{\dashrightarrow} \mathbb{P}^{n\vee}$$

defined by

$$\partial F(p) = \left[\frac{\partial F}{\partial X_0}(p), \dots, \frac{\partial F}{\partial X_n}(p)\right],$$

while the *polar system defined by* F is the linear system

$$|\partial F| := \left|\left\langle \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_n} \right\rangle\right| \subset |\mathcal{O}_{P^n}(d-1)|.$$

It follows from the definition that, if $Z_F \subset \mathbb{P}^n$ is the hypersurface defined by F, then the base locus of the polar map defined by F is the singular locus of Z_F. Moreover, since Z_F is a hypersurface, it follows that Z_F is not reduced if and only if F is not square free. If F is square free then the polar map ∂F is free of base divisors; if Z_F is smooth then the polar map ∂F is a morphism and the image of Z_F by ∂F is the dual variety $Z_F^\vee \subset \mathbb{P}^{n\vee}$ of Z_F.

If F is not square free, we write

$$F = \prod_{i=0}^{r} F_i^{m_i},$$

with $d = \sum_{i=0}^r m_i \deg(F_i)$. Then the divisorial components of the base locus of the polar system $|\partial F|$ are given by the hypersurface defined by the polynomial

$$F' = \prod_{i=0}^{r} F_i^{m_i - 1}.$$

We will indicate by F_{red} the polynomial F/F'.

The polar system defined by F is naturally split into a fixed and a moving part, as follows.

DEFINITION 2. The *moving part* of the polar system defined by a homogeneous polynomial F is the linear system $|M(\partial F)|$ obtained by removing all base components from the polar system $|\partial F|$. In particular, we have that

$$|M(\partial F)| \subset |\mathcal{O}_{\mathbb{P}^n}(d - 1 - \deg F')|.$$

Notice that Definition 2 makes perfect sense when F is square free, in which case F' is a constant.

DEFINITION 3. A *homaloidal polynomial* of degree d is a homogeneous polynomial $F \in H^0(\mathbb{P}^n, \mathcal{O}_{P^n}(d))$ such that the moving part $|M(\partial F)|$ of the polar system defined by F induces a birational map.

Well-known examples of homaloidal polynomials in $\mathbb{P}^n$ are those defining smooth quadrics. A remarkable result in [EKP] is the classification of homaloidal polynomials of degree $d = 3$: the irreducible ones define the secant varieties of the four Severi varieties and a classification seems at hand, at least in degree $d = 4$. Probably the best-known and most important example of a homaloidal polynomial is the polynomial $F = X_0 \cdots X_n$, which has degree $d = n + 1$ and whose associated polar map is a standard Cremona transformation. An infinite class of examples of homaloidal polynomials in characteristic 0 is given by the polynomials

$$F(m_0, \dots, m_n) := X_0^{m_0} \cdots X_n^{m_n},$$

with $m_i \geq 1$ for all $i = 0, \dots, n$, of arbitrarily large degree $d = \sum_{i=0}^n m_i$. The base locus of ∂F has a divisorial component defined by the polynomial $F' = \prod_{i=0}^n X_i^{m_i - 1}$. Once we remove it, we simply compute that

$$|M(\partial F)| = \left| \left\langle m_0 \prod_{i=1}^{n} X_i, \dots, m_j \prod_{i \neq j} X_i, \dots, m_n \prod_{i=0}^{n-1} X_i \right\rangle \right|.$$

Hence ∂F induces the same map as a composition of a (diagonal) projectivity and $\partial F_{\mathrm{red}} = \partial \prod_{i=0}^n X_i$, so that it is homaloidal. If the characteristic of k is p, then from our previous list we need to remove only the polynomials $F(m_0, \dots, m_n)$ in which some m_i is divisible by p.

We always assume that *resolution of indeterminacies is possible* over the ground field k. Under this hypothesis, to say that F is homaloidal is equivalent to saying that ∂F is dominant and that there exists a resolution of singularities

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\partial F"] & & \mathbb{P}^{n\vee} \end{tikzcd}$$

such that, if Y is a general member of $|M(\partial F)|$ and if $\bar{Y}$ denotes its strict transform on X, then

$$(\bar{Y})^n = 1,$$

because in fact $\bar{Y} \in |g^* \mathcal{O}_{\mathbb{P}^{n\vee}}(1)|$.

We next state an important property of homaloidal polynomials.

PROPOSITION 4. *If F is a homaloidal polynomial then Z_F is not a cone. In particular, if $F = \prod_{i=0}^r L_i^{m_i}$ is homaloidal then $\langle L_0, \dots, L_r \rangle = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, so that $r \geq n$.*

Proof. If Z_F is a cone, then the image of the polar map defined by F is contained in the linear space dual to the vertex of the cone of Z_F. If F is a product of linear forms, then Z_F is a cone if and only if $\langle L_0, \dots, L_r\rangle \neq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. $\square$

In fact (see [R]), in characteristic 0 even more is true: Z_F is a cone if and only if the image of the polar map associated to F lies in a hyperplane. This is false in characteristic p, as shown by the polynomial $F(p, 1, \dots, 1) = X_0^p X_1 \cdots X_n$.

2. Products of Linear Forms

In this section we will prove Theorems A and B. We will always assume that F is a homaloidal polynomial that splits as the product of linear forms. Suppose first that F is square free, so that ∂F is a linear system free of base components. We will fix a minimal resolution of singularities:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\partial F"] & & \mathbb{P}^{n\vee}. \end{tikzcd}$$

By definition, if Y is a general member of the polar system $|\partial F|$ and if $\bar{Y}$ is its strict transform on X, then $\bar{Y} \in |g^*\mathcal{O}_{\mathbb{P}^{n\vee}}(1)|$.

Let us write

$$F = \prod_{i=0}^{r} L_i.$$

Up to a projectivity we can assume that $L_0 = X_0$. We will denote by H_0 the hyperplane defined by $X_0 = 0$. Let us define the polynomial

$$G := \frac{F}{X_0} = \prod_{i=1}^{r} L_i.$$

With this choice, a basis of the polar system $|\partial F|$ defined by F is given by:

$$|\partial F| = \left|\left\langle G + X_0\frac{\partial G}{\partial X_0}, X_0\frac{\partial G}{\partial X_1}, \dots, X_0\frac{\partial G}{\partial X_n}\right\rangle\right|.$$

We first observe that if $D \subset X$ is the strict transform of the hyperplane H_0 then, looking at the equations for ∂F, the map g contracts D because ∂F contracts H_0 to its dual point $U_0 = [1 : 0 : \cdots : 0]$.

Now define $G_0 \in H^0(H_0, \mathcal{O}_{H_0}(d-1))$ as the restriction of G to H_0. Observe that G_0 does not need a priori to be reduced.

LEMMA 5. *The irreducible divisor $D \subset X$ is the unique divisor contracting to the point $U_0 \in \mathbb{P}^{n\vee}$, and the map $g\colon X \to \mathbb{P}^{n\vee}$ factors through the blowup $h'\colon Z \to \mathbb{P}^{n\vee}$ of $\mathbb{P}^{n\vee}$ at U_0.*

Proof. Suppose that $W \neq D$ is a divisor in X that is g-exceptional and such that $g(W) = g(D) = U_0$. By minimality of the resolution of the rational map ∂F, it follows that W is not f-exceptional and thus corresponds to a hypersurface $f(W) \subset \mathbb{P}^n$ that is distinct from H_0. Let J be an equation of $f(W)$. The equations of ∂F imply that J must divide $\frac{\partial G}{\partial X_i}$ for all $i \geq 1$, but G is reduced and so this is impossible. The irreducible divisor D then corresponds to the extraction of a valuation centered at $U_0 \in \mathbb{P}^{n\vee}$, and we must show that this valuation corresponds to the whole maximal ideal $\mathcal{M}_{U_0}$ of the point $U_0 \in \mathbb{P}^{n\vee}$. We have already remarked that the system $|\langle \frac{\partial G}{\partial X_1}, \dots, \frac{\partial G}{\partial X_n} \rangle|$ corresponds on X to the system $|g^*\mathcal{O}_{\mathbb{P}^{n\vee}}(1) - D|$ and that it is of codimension 1 in $|g^*\mathcal{O}_{\mathbb{P}^{n\vee}}(1)|$. Suppose that $g_*\mathcal{O}_X(-D) = \mathcal{M}'$ with $\sqrt{\mathcal{M}'} = \mathcal{M}_{U_0}$; then, since

$$g_* g^* \mathcal{O}_{\mathbb{P}^{n\vee}}(1) \otimes \mathcal{O}_X(-D) = \mathcal{M}' \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1),$$

we have that $H^0(\mathbb{P}^{n\vee}, \mathcal{M}' \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1)) = n$ and hence

$$\mathcal{M}_{U_0} = \mathcal{M}'.$$

Therefore, D is the strict transform of the exceptional divisor under the blowup of $\mathbb{P}^{n\vee}$ at U_0, $h' \colon Z \to \mathbb{P}^{n\vee}$, and the result follows. $\square$

Consider now the diagram of maps

$$\begin{tikzcd}
 & & X \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
H_0 \arrow[r, hook] & \mathbb{P}^n \arrow[rr, dashed, "\partial F"] & & \mathbb{P}^{n\vee} \arrow[r, dashed, "\pi"] & P,
\end{tikzcd}$$

where π is the projection from the point U_0 to the hyperplane P. We can use Lemma 5 to factorize the morphism g through the blowup Z of $\mathbb{P}^{n\vee}$ at U_0. This yields the diagram

$$\begin{tikzcd}
 & X \arrow[d, "h"] & \\
 & Z \arrow[dl, "h'"'] \arrow[dr, "t"] & \\
\mathbb{P}^{n\vee} \arrow[rr, dashed, "\pi"] & & P
\end{tikzcd}$$

with $g = h'h$. Recall that we denote by Y a general element in $|\partial F|$ and by D the strict transform of H_0 in X.

Lemma 6. *With notation as before, G_0 is square free and homaloidal on H_0.*

Proof. Since it is a composition of morphisms, the map th is a morphism. This implies that the linear system $(th)^*\mathcal{O}_P(1) \simeq \bar{Y} - D$ is base-point free. Moreover, we have $|\bar{Y} - D| = |(th)^*\mathcal{O}_P(1)| = |g^*\mathcal{O}_{\mathbb{P}^{n\vee}}(1) - D|$, so that $th = \pi g$ is given by the base-point free system associated to $|\partial G|$. The morphism m is then a resolution of singularities of the map $|\partial G_0|$, since $|(\bar{Y} - D)|_D|$ is the pullback on D of the system $|\partial G_0|$. Because m is a morphism, ∂G_0 is free of base components;

this means that G_0 is reduced. The map ∂G_0 is surjective because it is a composition of surjections. In order to show that G_0 is homaloidal on H_0, it suffices to show that

$$D \cdot ((th)^* \mathcal{O}_P(1) - D)^{n-1} = 1.$$

This follows from the fact that D is the strict transform of the exceptional divisor under the blowup $h' \colon Z \to \mathbb{P}^{n\vee}$. $\square$

We now consider homaloidal polynomials that are products of linear forms with at least a square factor. Quite surprisingly, *there is a priori no relation between ∂F and $\partial F_{\mathrm{red}}$*. Let us choose $r+1$ distinct linear forms $L_0, \ldots, L_r$ in $\mathbb{P}^n$ together with an identification $X_0 = L_0$ and consider the following polynomials, where H_0 is the hyperplane of equation $X_0 = 0$:

$$F = X_0^{m_0} \prod_{i=1}^{r} L_i^{m_i}, \quad F' = X_0^{m_0-1} \prod_{i=1}^{r} L_i^{m_i-1}, \quad F_{\mathrm{red}} = \frac{F}{F'};$$
$$G = \prod_{i=1}^{r} L_i^{m_i}, \quad G' = \prod_{i=1}^{r} L_i^{m_i-1}, \quad G_{\mathrm{red}} = \frac{G}{G'};$$
$$G_0 = G \cap H_0.$$

We may compute the moving parts of the polar systems defined by F and F_{red} as follows:

$$|M(\partial F)| = \left|\left\langle m_0 G_{\mathrm{red}} + X_0 \sum_{i=1}^{r} m_i \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \ldots, X_0 \sum_{i=0}^{r} m_i \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j \right\rangle\right|,$$

$$|M(\partial F_{\mathrm{red}})| = |\partial F_{\mathrm{red}}| = \left|\left\langle G_{\mathrm{red}} + X_0 \sum_{i=1}^{r} \frac{\partial L_i}{\partial X_0} \prod_{j \neq i,0} L_j, \ldots, X_0 \sum_{i=0}^{r} \frac{\partial L_i}{\partial X_n} \prod_{j \neq i,0} L_j \right\rangle\right|.$$

Consider now the following diagram of maps, where f and g induce a minimal resolution of the morphism induced by $|M(\partial F)|$:

$$\begin{tikzcd} & & X \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ H_0 \arrow[r, hook] & \mathbb{P}^n \arrow[rr, dashed, "M(\partial F)"] & & \mathbb{P}^{n\vee} \arrow[r, dashed, "\pi"] & P. \end{tikzcd}$$

We define D to be the strict transform of H_0 in X; we denote by Y a general member of $|M(\partial F)|$ and by $\bar{Y}$ its strict transform on X. It turns out that all the arguments used to prove Lemma 5 and Lemma 6 apply verbatim to prove the following lemma.

LEMMA 7. *With notation as before, the following statements hold.*

(1) *D is the strict transform on X of the exceptional divisor in the blowup $h' \colon Z \to \mathbb{P}^{n\vee}$ of $\mathbb{P}^{n\vee}$ at U_0.*

(2) *The restriction of the linear system $|(th)^*\mathcal{O}_P(1)|$ to D induces a morphism $m\colon D \to P$ that is a resolution of singularities of the polar map defined by G_0 on H_0; that is, $|(\bar{Y} - D)|_D| = |M(\partial G_0)|$.*
(3) *The polynomial G_0 is homaloidal in H_0.*

We are now able to prove Theorems A and B at once.

THEOREM 8. *Assume that resolution of singularities holds. Let $L_0, \dots, L_r$ be distinct linear forms, and let $F = \prod_{i=0}^r L^{m_i}$ with $m_i \geq 1$ for all $i = 0, \dots, r$. Then*:

(i) *F_{red} is homaloidal if and only if* (a) *$r = n$ and* (b) *the L_i are independent linear forms*; *and*
(ii) *F is homaloidal if and only if* (a) *no m_i is divisible by p and* (b) *$F_{\text{red}} = \prod_{i=0}^r L_i$ is homaloidal.*

Proof. The proof is by induction on n.

The starting point of the induction is the case $n = 1$, which is easy: if $F = \prod_{i=0}^r L_i^{m_i}$ is homaloidal then the base-point free system $|M(\partial F)|$ must be of degree 1, from which it follows easily that $r = 1$ and that L_0 and L_1 are in linear general position (they are distinct by hypothesis). Up to a projectivity, the converse has been proved by virtue of the examples following Definition 3. The same argument works a fortiori if F is square free.

Let us then move to $\mathbb{P}^n$ ($n > 1$) and consider first the case of a square free homaloidal polynomial $F = \prod_{i=0}^r L_i$. Setting $X_0 = L_0$, we apply Lemma 6 and so obtain a reduced homaloidal polynomial $G_0 = \prod_{i=1}^r L_{i,0}$ on H_0. By induction we have that $r = n$ and the $L_{i,0} = L_i \cap H_0$ are independent in H_0, from which it follows that $X_0, L_1, \dots, L_n$ are independent in $\mathbb{P}^n$.

Suppose now that $F = \prod_{i=0}^r L_i^{m_i}$ is nonreduced and homaloidal. We must prove that F_{red} is homaloidal and that no m_i is divisible by p (the converse is a consequence of the first part of this proof and the examples following Definition 3). If we are in characteristic p then we must ensure that there exists an index i for which m_i is not divisible by p; if this is not the case then F is a power of p, so that its polar map is identically zero. We may then assume that m_0 is not divisible by p. Putting $X_0 = L_0$ and applying Lemma 7, we get that $|M(\partial F)|$ induces on H_0 the homaloidal system defined by G_0. By induction and by the same argument as before, we obtain the thesis that $r = n$ and the linear forms $L_0, \dots, L_n$ are independent, so F_{red} is homaloidal, and that no exponent is divisible by p. $\square$

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Dipartimento di Matematica
Universita' degli Studi Roma Tre
Largo S. L: Murialdo, 1
00146 Roma
Italy

bruno@mat.uniroma3.it