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Rational Curves on Blowing-ups of Projective Spaces

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1. Introduction

Let $\mathrm{Mor}_\beta(\mathbb{P}^1, Y)$ denote the moduli space of morphisms f from a complex projective line $\mathbb{P}^1$ to a smooth complex projective variety Y such that $f_*[\mathbb{P}^1] = \beta$, where β is a given second homology class of Y. We study the irreducibility and the rational connectedness of the moduli space when Y is a successive blowing-up of a product of projective spaces with a suitable condition on β.

Before stating the Main Theorem proven in this paper, we introduce some notation. Let $X = \prod_{k=1}^m \mathbb{P}^{n_k}$, $X_0 = X$, and let $\pi_i : X_i \to X_{i-1}$ ($i = 1, \dots, r$) be a blowing-up of X_{i-1} along a smooth irreducible subvariety Z_i. Let $E_i^t \subset X_r$ be the total transform $(\pi_i \circ \dots \circ \pi_r)^{-1} Z_i$ of the exceptional divisor associated to Z_i, and let H_k be the divisor class coming from the hyperplane class of the kth projective space $\mathbb{P}^{n_k}$. Let $m_i = \#\{Z_j \mid j < i, (\pi_j \circ \dots \circ \pi_r)^{-1}(Z_j) \supset E_i^t\}$. So general points of Z_i are the (m_i)th infinitesimal points of X. Denote by $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ the open sublocus of $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ consisting of those f whose images do not lie on exceptional divisors: $f(\mathbb{P}^1) \not\subseteq E_i^t$ for all i.

Main Theorem. *Assume that* $\beta \cdot \bigl(\pi^* H_k - \sum_{i=1}^r (m_i + 1) E_i^t\bigr) \ge 0$ *for all k and that* $\beta \cdot E_i^t \ge 0$ *for all i, where* $\pi = \pi_1 \circ \dots \circ \pi_r$.

(1) *The moduli space* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *consists of free morphisms and is an irreducible smooth variety of expected dimension.*
(2) *If Z_i are rationally connected for all i, then a projective and birational model of* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *is also rationally connected.*
(3) *The moduli space* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ *is smooth, and* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *is dense in* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$, *if one of the following conditions hold*:
   (a) *all $\pi(E_i^t)$ are points in X*;
   (b) *all centers Z_i are convex (i.e., $H^1(\mathbb{P}^1, g^* T_{Z_i}) = 0$ for any morphism $g : \mathbb{P}^1 \to Z_i$), and $\pi(E_i^t)$ are disjoint to $\pi(E_j^t)$ for any $i \ne j$.*

Note that the irreducibility (respectively, the rational connectedness of a projective, birational model) of the morphism space $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ implies the irreducibility of the moduli space of rational curves C with numerical condition $[C] = \beta$.

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Our paper is motivated by the following two questions.

(1) If Y is rationally connected, is $\mathrm{Mor}_\beta(\mathbb{P}^1, Y)$ also irreducible and rationally connected? If this does not always hold, does it hold for special values of β?
(2) For surfaces fibered over $\mathbb{P}^1$ with genus-2 fibers (or, more generally, hyperelliptic fibers) and fixed numerical invariants, is the moduli space of such surfaces connected? If this does not always hold, does it hold for special values of the numerical invariants?

Several authors have studied the first question. The case of homogeneous spaces was treated by Kim and Pandharipande [7] and Thomsen [10]. The case of small, degree-d general hypersurfaces $Y = X_d \subset \mathbb{P}^n$ was handled by Harris, Roth, and Starr [4]. The case when Y is the moduli space of rank-2 stable vector bundles, with fixed determinant of degree 1 on a smooth projective curve of genus $g \geq 2$, was investigated by Castravet [1], who found all irreducible components and described the maximal rationally connected fibration of them.

Let $\bar{M}_{0,n}$ be the moduli space of stable n-pointed rational curve. As a corollary of the Main Theorem, the space $\mathrm{Mor}_\beta(\mathbb{P}^1, \bar{M}_{0,n})$ is connected for certain values of β because the space $\bar{M}_{0,n}$ is a successive blowing-up of $(\mathbb{P}^1)^{n-3}$ along smooth codimension-2 subvarieties [6] or a successive blowing-up of $\mathbb{P}^{n-3}$ [5]. This gives a step toward proving connectedness of the moduli space of hyperelliptic fibrations over $\mathbb{P}^1$ (presumably by replacing the hyperelliptic fibration by the fibration of quotient by the hyperelliptic involution as marked by the images of the Weierstrass points).

When Y is a blowing-up of a product of projective spaces along smooth closed (not necessarily irreducible) subvarieties, we prove a slightly stronger result: Theorem 1 in Section 2. In Section 3, we prove the Main Theorem. The key idea of both proofs is to express the moduli space as a fibration—a fiber consists of the morphisms f that pass through given points of $\coprod_i Z_i$ at given points of the domain $\mathbb{P}^1$—and then to show that the general fiber is rationally connected and has the expected dimension under the condition on β as in the Main Theorem (and also as in Theorem 1). When Y is a successive blowing-up, we will need to utilize jet spaces and jet conditions in order to show the rational connectedness of the general fiber. We shall also apply a result of Graber, Harris, and Starr [3].

Throughout the paper, we will employ the well-known results of the deformation theory of morphisms of curves as well as the established notation used in [8]. The complex number field $\mathbb{C}$ will be the base field.

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2. Blowing-ups along a Smooth Subvariety

2.1. Setup and a Morphism σ

Let $\pi : \tilde{X} \to X$ be the blowing-up of a smooth projective variety X along a smooth closed subvariety Z with the exceptional divisor E.

For a curve class $\beta \in H_2(\tilde{X}, \mathbb{Z})$, consider the evaluation morphism

$$\begin{aligned} \mathrm{ev}\colon \mathbb{P}^1 \times \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X}) &\to \tilde{X}, \\ (p, \tilde{f}) &\mapsto \tilde{f}(p). \end{aligned}$$

Assume that $\beta \cdot E_i \geq 0$ for all $i = 1, \ldots, r$, where the E_i are exceptional irreducible divisors over the irreducible components Z_i of $Z = \coprod_{i=1}^r Z_i$. In general, $\tilde{f}(\mathbb{P}^1) \subset E_i$ does not imply $\tilde{f}(\mathbb{P}^1) \cdot E_i < 0$, as the following example shows.

EXAMPLE 1. Consider the blowing-up $\tilde{X}$ of $X = \mathbb{P}^3$ along a curve $Z \cong \mathbb{P}^1$ with a normal bundle $N_{Z/X} = \mathcal{O}(1) \oplus \mathcal{O}(2)$. Then $E = \mathbb{P}(\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2))$. If C is a positive section (resp. the negative section) then, by the construction of the sections and the universal property of the projectivization E of $\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)$, we see that $C \cdot E = 1$ (resp. $C \cdot E = 2$).

This observation forces us to consider an open subvariety $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ of $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})$ consisting of $\tilde{f}$ such that $\tilde{f}(\mathbb{P}^1) \not\subseteq E$. Now the scheme-theoretic intersection $\Gamma_{\pi\circ\mathrm{ev}} \cap (\mathbb{P}^1 \times \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \times Z) \cong \mathrm{ev}^{-1}(E)$, where $\Gamma_{\pi\circ\mathrm{ev}}$ is the graph of the morphism $\pi \circ \mathrm{ev}$, can be regarded as a closed subscheme of $\mathbb{P}^1 \times \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \times Z$, which is proper and flat over $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$. Thus that intersection induces a natural morphism

$$\begin{aligned} \sigma : \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp &\to \textstyle\prod_{i=1}^r \mathrm{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i), \\ \tilde{f} &\mapsto (\Gamma_f \cap (\mathbb{P}^1 \times Z_1), \ldots, \Gamma_f \cap (\mathbb{P}^1 \times Z_r)), \end{aligned}$$

where $f := \pi \circ \tilde{f}$ and $e_i := \beta \cdot E_i$. Here $\mathrm{Hilb}^0\, Y$ of a variety Y is defined to be $\mathrm{Spec}\, \mathbb{C}$.

2.2. An Exact Sequence

The following lemma shows a sufficient condition for the generic smoothness of σ.

LEMMA 1. *Let $\pi : \tilde{X} \to X$ be as before and suppose that $\tilde{f}(\mathbb{P}^1) \not\subseteq E$.*

(1) *There is a natural injective morphism of sheaves*

$$\tilde{f}^*\pi^* T_X(-E) \to \tilde{f}^* T_{\tilde{X}}.$$

(2) *If $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ is transversal to E, then the injective morphism in part* (1) *induces a short exact sequence*

$$0 \to \tilde{f}^*\pi^* T_X(-E) \to \tilde{f}^* T_{\tilde{X}} \to (\mathrm{id}_{\mathbb{P}^1} \times f)^* T_{\mathbb{P}^1 \times Z} \to 0;$$

furthermore, the associated morphism

$$H^0(\mathbb{P}^1, \tilde{f}^* T_{\tilde{X}}) \to (\mathrm{id}_{\mathbb{P}^1} \times f)^* T_{\mathbb{P}^1 \times Z}$$

is the derivative of σ.

(3) *For every* $\tilde{f}$ *satisfying* $h^1(\mathbb{P}^1, \tilde{f}^* \pi^* T_X(-E)) = 0$, *the morphism* σ *is smooth at* $[\tilde{f}]$.

Proof. The first morphism is defined by the pull-back of the extension of the isomorphism $T_X \cong T_{\tilde{X}}$ away from E. To check for the existence of the extension, consider the blowing-up $\tilde{X} \to X$ locally as $(t, \mathbf{x}, \mathbf{y}) \mapsto (z_1 = t, \mathbf{z}_2 = t\mathbf{x}, \mathbf{z}_3 = \mathbf{y})$, where $t, \mathbf{x}, \mathbf{y}$ (resp. $z_1, \mathbf{z}_2, \mathbf{z}_3$) is a system of local parameters of $\tilde{X}$ (resp. X) and where the bold letters denote multivariables. Then the natural morphism of sheaves

$$\pi^* T_X(-E) \to T_{\tilde{X}}$$

defined by

$$\left(t\frac{\partial}{\partial z_1} \mapsto t\frac{\partial}{\partial t} - \sum x_i \frac{\partial}{\partial x_i}\right), \quad \left(t\frac{\partial}{\partial \mathbf{z}_2} \mapsto \frac{\partial}{\partial \mathbf{x}}\right), \quad \text{and} \quad \left(t\frac{\partial}{\partial \mathbf{z}_3} \mapsto t\frac{\partial}{\partial \mathbf{y}}\right)$$

is the extension. At p with $f(p) \in Z$, the second morphism is defined by

$$(f_*|_p)^{-1} \oplus \pi_*|_{\tilde{f}(p)} \colon T_{\tilde{X}}|_{\tilde{f}(p)} = f_* T_{\mathbb{P}^1}|_p \oplus T_E|_{\tilde{f}(p)} \to T_{\mathbb{P}^1}|_p \oplus T_Z|_{f(p)},$$

where $T_Y|_y$ denotes the tangent space of a variety Y at a point y. Now the rest of the proof is straightforward. □

REMARK 1. In fact, this proof shows that there is an exact sequence

$$0 \to (\mathrm{ev})^*(\pi^* T_X(-E)) \to \mathrm{ev}^* T_{\tilde{X}} \to (\mathrm{id}_{\mathbb{P}^1} \times \pi \circ \mathrm{ev})^* T_{\mathbb{P}^1 \times Z} \to 0$$

over $\mathbb{P}^1 \times \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ$, where $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ$ is the locus of all morphisms in $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})$ that are transversal to E.

REMARK 2. In Example 1, the one-to-one morphism whose image is the negative section is not free: the exact sequence

$$0 \to N_{C/E} = \mathcal{O}(-1) \to N_{C/\tilde{X}} \to N_{E/\tilde{X}}|_C = \mathcal{O}(2) \to 0$$

splits because C is a section. Therefore, Lemma 1(1) is not true in general without the condition $\tilde{f}(\mathbb{P}^1) \not\subseteq E$.

When $\tilde{f}(\mathbb{P}^1) \subseteq E$, instead of Lemma 1 we have the following lemma.

LEMMA 2. *Assume that* $k \leq e + 1$, $e = E \cdot \tilde{f}_*[\mathbb{P}^1] \geq 0$, *and* $\tilde{f}(\mathbb{P}^1) \subseteq E$. *Then, if* $H^1(\mathbb{P}^1, f^* T_X(-e - k)) = 0$ *and* $H^1(\mathbb{P}^1, f^* T_Z(-k)) = 0$, *we obtain* $H^1(\mathbb{P}^1, \tilde{f}^* T_{\tilde{X}}(-k)) = 0$.

Proof. (i) Note that $H^1(\mathbb{P}^1, f^* N_{Z/X}(-e - k)) = 0$ by $H^1(\mathbb{P}^1, f^* T_X(-e - k)) = 0$ and

$$0 \to T_Z \to T_X|_Z \to N_{Z/X} \to 0.$$

(ii) Observe that $H^1(\mathbb{P}^1, \tilde{f}^* T_\pi(-k)) = 0$ and $H^1(\mathbb{P}^1, \tilde{f}^* T_E(-k)) = 0$ by (i), $H^1(\mathbb{P}^1, f^* T_Z(-k)) = 0$, and the exact sequences

$$0 \to \mathcal{O} \to \pi^*(N_{Z/X}) \otimes \mathcal{O}_E(1) \to T_\pi \to 0,$$
$$0 \to T_\pi \to T_E \to \pi^* T_Z \to 0,$$

where T_π denotes the relative tangent bundle of π.

(iii) Finally, $H^1(\mathbb{P}^1, \tilde{f}^* T_{\tilde{X}}(-k)) = 0$ by

$$0 \to T_E \to T_{\tilde{X}}|_E \to \mathcal{O}_{\tilde{X}}(E)|_E \to 0.$$

Here we use the condition that $-k + e \geq -1$. □

2.3. *The Fiber of σ When $X = \mathbb{P}^n$*

Let $X = \mathbb{P}^n$ and set

$$c_H(\beta) = \beta \cdot (\pi^* H - E),$$

where H is the hyperplane class of $\mathbb{P}^n$. If $c_H(\beta) \geq -1$, then Lemma 1 implies the vanishing of the obstruction $H^1(\mathbb{P}^1, \tilde{f}^* T_{\tilde{X}}) = 0$ provided $\tilde{f}(\mathbb{P}^1) \nsubseteq E$, and so the space $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ is smooth. In addition, the general fiber of a morphism σ is smooth and has the expected dimension as a result of Lemma 1. Here the expected dimension of the fiber is, by definition,

$$\begin{aligned}&\text{expected}\dim \mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X}) - \dim \textstyle\prod_{i=1}^r \mathrm{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i)\\ &\qquad = \dim \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - ne = \dim \mathrm{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) - ne.\end{aligned}$$

In the next lemma we investigate the irreducibility of the fiber of the morphism σ for $X = \mathbb{P}^n$.

LEMMA 3. *Suppose that $\pi_*\beta \neq 0$ in $H_2(X, \mathbb{Z})$. Then the following statements hold.*

(1) *Every nonempty fiber of σ is isomorphic to an open subset of a projective space.*
(2) *If $c_H(\beta) \geq -1$, then σ is a smooth morphism at general points.*
(3) *If $c_H(\beta) \geq 0$, then the general fiber of σ is nonempty.*
(4) *If $c_H(\beta) \geq 0$ and $\dim Z_i = 0$ for all i, then σ is surjective.*

Proof. First note that $\mathrm{Mor}_{\pi_*\beta}(\mathbb{P}^1, X)$ contains $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ as a quasi-projective subvariety over which the scheme $\mathrm{ev}^{-1}(Z_i)$ has the relative Hilbert polynomial e_i for all $i = 1, \ldots, r$. We will describe a fiber of σ as a subscheme in

$$\mathrm{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1})),$$

where $d = (\pi_*\beta) \cdot H$. Let

$$P := \prod P_i \in \prod \mathrm{Hilb}^{e_i}\, \mathbb{P}^1 \times Z_i, \quad P_i = \textstyle\sum_a e_a^{(i)}(p_a^{(i)}, q^{(i,a)}),$$

and

$$\mathrm{Supp}(P_i) = \{(p_a^{(i)}, q^{(i,a)})\}_a,$$

where $p_a^{(i)} \neq p_{a'}^{(i')}$ if $(i,a) \neq (i',a')$, $q_0^{(i,a)} \neq 0$ for all (i,a), and $e_a^{(i)} = 1$ for all a if $\dim Z_{i'} \neq 0$ for some i'. Then $\sigma^{-1}(P)$ is a subvariety of $\mathbb{P}H^0(\mathbb{P}^1, K_P)$, where K_P is the kernel of the morphism of sheaves

$$\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1} \to \Big(\bigoplus\nolimits_{i,a} \mathcal{O}_{p_a^{(i)},\mathbb{P}^1}/m_{p_a^{(i)}}^{e_a^{(i)}}\Big) \otimes \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^n,$$

$$f \mapsto \sum\nolimits_{j=1}^n \Big(\bigoplus\nolimits_{i,a}[q_0^{(i,a)} f_j - q_j^{(i,a)} f_0]\Big) \otimes 1_j,$$

where $1_j = (\underbrace{0,\dots,1}_{j},0,\dots,0) \in \mathbb{C}^n$. Speaking more precisely: $\sigma^{-1}(P)_{\mathrm{red}}$ coincides with

$$\mathbb{P}H^0(\mathbb{P}^1, K_P) \cap \mathrm{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) \setminus \bigcup\nolimits_{P' \in \mathrm{Hilb}^{e+1}(\mathbb{P}^1 \times Z)\,:\, P' \supset P} \mathbb{P}H^0(\mathbb{P}^1, K_{P'}); \qquad (*)$$

here, if P' is not simple at (p,q) and $\dim Z_i \neq 0$ for some i, then $K_{P'}$ is defined as the kernel of

$$K_P \to m_p/m_p^2 \otimes \mathcal{O}_{\mathbb{P}^1}(d) \otimes N_{Z/X}|_q,$$

$$f \mapsto \sum\nolimits_j (q_0 f_j - q_j f_0) \otimes \Big[\frac{\partial}{\partial z_j}\Big],$$

where $N_{Z/X}|_q := T_X|_q / T_Z|_q$ (normal space) and where $\{z_j := x_j/x_0 : j = 1,\dots,n\}$ are the coordinates of $\mathbb{C}^n = \{x_0 \neq 0\} \subset \mathbb{P}^n$.

Since $\mathcal{O}_{\mathbb{P}^1}(d - \sum_{i,a} e_a^{(i)} p_a^{(i)}) \otimes \mathbb{C}^{n+1} \subset K_P \subset \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}$, it follows that the sheaf K_P is isomorphic to $\bigoplus_{j=0}^n \mathcal{O}_{\mathbb{P}^1}(k_j)$ for some k_j, where $d \geq k_j \geq d - \sum_{i=1}^r e_i$ for all j. Now, if $\dim Z_i = 0$ for all i and $k_j \geq -1$ for all j, then $\dim \mathbb{P}H^0(\mathbb{P}^1, K_P) = \dim \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - ne$. If $\dim Z_i > 0$ for some i and $k_j \geq 0$, then $\dim \mathbb{P}H^0(\mathbb{P}^1, K_{P'}) = \dim \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - n(e+1)$ for $P' \supset P$. These facts applied to $(*)$ complete the proof. □

REMARK 3. Note that Lemma 3 holds for a product $X = \prod_k \mathbb{P}^{n_k}$ of projective spaces if we let $c_H(\beta) := \min\{\beta \cdot (\pi^* H_k - E)\}_k$, where H_k is the hyperplane class of the kth component of the product space X.

2.4. Some Elementary Facts

The following results are standard facts.

PROPOSITION 1 (cf. [2; 8, Prop. II.3.7]). *Let X be a smooth variety and let Y be a subvariety. Then any free morphism $f : \mathbb{P}^1 \to X$ can be deformed to a morphism $f_\varepsilon : \mathbb{P}^1 \to X$ that is transversal to Y.*

LEMMA 4. *Let X and Y be varieties and assume that Y is irreducible. Let $f : X \to Y$ be a dominant morphism in any irreducible component of X. If the general fiber of f is irreducible, then X is irreducible.*

Proof. The proof is straightforward. □

2.5. A Consequence

Let X be a product $\prod_k \mathbb{P}^{n_k}$ of projective spaces $\mathbb{P}^{n_k}$, and let $\tilde{X}$ be a blowing-up of X along a smooth closed subvariety Z. Denote by E the exceptional divisor and denote by H_k the divisor class. We assume that $\pi_*\beta \neq 0$ and $e_i \geq 0$ for all i.

THEOREM 1

(1) *If* $\beta \cdot (\pi^* H_k - E) \geq -1$ *for all* k *and if* Z *consists of finite points, then* $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})$ *is an irreducible smooth variety.*
(2) *If* $\beta \cdot (\pi^* H_k - E) \geq 0$ *for all* k, *then* $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ *is a nonempty and irreducible smooth variety.*
(3) *If* $\beta \cdot (\pi^* H_k - E) \geq 0$ *for all* k *and if all centers* Z_i *are convex, then* $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})$ *is an irreducible smooth variety.*

Proof. We prove this for $X = \mathbb{P}^n$. The condition $\beta \cdot (\pi^* H - E) \geq -1$ implies that $H^1(\mathbb{P}^1, (f^* T_X)(-e)) = 0$ by the Euler sequence on $\mathbb{P}^n$. By Lemma 1(1) we have $H^1(\mathbb{P}^1, (\tilde{f}^* T_{\tilde{X}})) = 0$, which implies that every irreducible component of $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ is smooth with the expected dimension.

(1) The first assertion follows from Lemma 3 and Lemma 4.

(2) To prove the second assertion of the theorem, first note that every element in $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$ is a free morphism by Lemma 1(1). Therefore, by Proposition 1, it is enough to show the irreducibility of the sublocus $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ$ of morphisms that are transversal to E. By Lemma 2, the general fiber of σ restricted to any component of $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ$ has the expected dimension and is irreducible. Hence the morphism σ restricted to any component is dominant on the irreducible variety $\prod \mathrm{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i)$, so the proof now follows from Lemma 4. The moduli space is nonempty by Lemma 3.

(3) If a morphism $\tilde{f}$ lies on E, we can use Lemma 2 with $k = 0$ to deform it to an element in $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$: by Lemma 2 we have $H^1(\mathbb{P}^1, \tilde{f}^* T_{\tilde{X}}) = 0$, which implies that the moduli space $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})$ is smooth at $\tilde{f}$. By the exact sequence in part (iii) of the proof of Lemma 2, we see that there is a deformation of $\tilde{f}$ to an element in $\mathrm{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp$. Now assertion (2) completes the proof of assertion (3).

By Remark 3, the same proof as for the product of projective spaces works here. □

3. Successive Blowing-up Case

3.1. Setup

Let $X_0 = X$ be a smooth projective variety and let $\pi_i \colon X_i \to X_{i-1}$ $(i = 1, \ldots, r)$ be a blowing-up of X_{i-1} along a smooth irreducible subvariety Z_i. In general, the space X_r is a successive blowing-up of X. Let $E_i^t \subset X_r$ (resp. $E_i^s \subset X_r$) be the total (resp. strict) transform of the exceptional divisor associated to Z_i, and let $e_i = \beta \cdot E_i^s$. Denote by $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\circ$ the sublocus of $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ of the morphisms $\tilde{f}$ that are transversal to $E = \bigcup_{i=1}^r E_i^s$ and do not intersect with $E_i^s \cap E_j^s$ for $i \neq j$. Then, for $e_i \geq 0$, for all i we obtain a morphism

$$\sigma \colon \mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\circ \to \textstyle\prod_{i=1}^r \mathrm{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i),$$
$$\tilde{f} \mapsto \textstyle\prod_{i=1}^r (\Gamma_{\pi_i \circ \cdots \circ \pi_r \circ \tilde{f}} \cap \mathbb{P}^1 \times \big(Z_i \setminus \bigcup_{j>i} (\pi_i \circ \cdots \circ \pi_r(E_j^s)\big)$$

as the generalization of the σ in Section 2.1.

Inductive application of the exact sequence of Lemma 1 proves the following corollary.

COROLLARY 1. *Suppose that* $H^1(\mathbb{P}^1, f^*(T_X(-\sum E_i^t))) = 0$, *where* $f = \pi_1 \circ \cdots \circ \pi_r \circ \tilde{f}$. *Then the morphism* σ *is smooth at* $\tilde{f}$.

3.2. Jet Spaces

We want to show that the general fiber of σ is rationally connected under a suitable condition on β when $X = \mathbb{P}^n$ or their products. However, it is hard to analyze the fiber of σ directly as in Section 2.3. Our strategy is to introduce an auxiliary morphism τ by imposing further conditions on jets of the morphism $f: \mathbb{P}^1 \to X$. It turns out that the fiber of τ is simple to study. Since the jet conditions on f can be translated into the vanishing conditions on the blowing-up space, we will be able to express the general fiber of σ by the fibers of τ (more precisely, its product $\tau_{\mathbf{m}}$) that are rationally connected.

To introduce τ, let $J_q^k X = \mathrm{Mor}((\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^{k+1}), 0), (X, q))$ be the k-jet space of X at $q \in X$. Then the morphism $f: \mathbb{P}^1 \to X$ naturally assigns an element $[f]_p^k \in J_{f(p)}^k X$ for any $p \in \mathbb{P}^1$. Using this assignment, we define a morphism

$$\tau: ((\mathbb{P}^1)^l \setminus \Delta) \times \mathrm{Mor}_\beta(\mathbb{P}^1, X) \to (\mathbb{P}^1 \times J^k X)^l,$$

$$(\mathbf{p}, f) \mapsto (p_i, [f]_{p_i}^k)_{i=1,\dots,l},$$

where Δ is the big diagonal and $J^k X = \coprod_{p \in X} J_p^k X$.

LEMMA 5. *If* $H^1(\mathbb{P}^1, f^*T_X(-(k+1)l)) = 0$, *then* τ *is smooth at* $(\mathbf{p}, f)$.

Proof. The natural exact sequence

$$0 \to f^*T_X(-(k+1)l)) \to f^*T_X \to f^*T_X \otimes \left(\bigoplus_{i=1}^l \mathcal{O}_{p_i,\mathbb{P}^1}/m_{p_i}^{k+1}\right) \to 0$$

induces the map

$$H^0(\mathbb{P}^1, f^*T_X) \to H^0\left(\mathbb{P}^1, f^*T_X \otimes \left(\bigoplus_{i=1}^l \mathcal{O}_{p_i,\mathbb{P}^1}/m_{p_i}^{k+1}\right)\right),$$

which is the tangent map $T\tau_{(\prod p_i, f)}|_{0 \times H^0(\mathbb{P}^1, f^*T_X)}$. (Note that exactness holds because the p_i are pairwise distinct for all i.) Indeed,

$$T_{J^k X}|_{[f]_p^k} = H^0(\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon)^{k+1}, ([f]_p^k)^* T_X)) = H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_p/m_p^{k+1})$$

by base change. This induced morphism is surjective by assumption. □

3.3. A General Simple Fact

Let $\pi: \tilde{X} \to X$ be a blowing-up of a smooth variety X along a subvariety Z. Let $E_q := \pi^{-1}(q)$, where q is a smooth point of Z. Observe that there is a natural morphism $j_q^k: (J_q^k X)^\circ \to \bigcup_{w \in E_q} J_w^{k-1} \tilde{X}$ defined by lifting, where $(J_q^k X)^\circ$ consists of those $s \in J_q^k X$ that are, as k-jet arcs, transversal to Z at q.

LEMMA 6. *The morphism* j_q^k *is smooth, and every fiber is a rational variety.*

Proof. This is a local problem at q, so we may assume that $X = \mathbb{C}^n$, $q = 0$, $Z = \{0\} \times \mathbb{C}^l$, and $\pi(t, \mathbf{x}, \mathbf{y}) = (t, t\mathbf{x}, \mathbf{y})$. Consider a k-jet $s(t)$ at $t = 0$ such that $(ds_1/dt)|_{t=0} \neq 0$; then

$$\left(s_1(t), \frac{s_2(t)}{s_1(t)}, \ldots, \frac{s_{n-l}(t)}{s_1(t)}, s_{n-l+1}(t), \ldots, s_n(t)\right) \bmod t^k$$

is, by definition, $j^k(s(t))$. This shows that the morphism j^k is regular. It is now straightforward to check the smoothness of j^k_q and the rationality of the fiber. □

3.4. The Morphism $\tau_{\mathbf{m}}$ and Its Fibers When $X = \prod_{j=1}^m \mathbb{P}^{n_j}$

We define a morphism

$$\tau_{\mathbf{m}} : ((\mathbb{P}^1)^{\sum e_i} \setminus \Delta) \times \mathrm{Mor}_\beta(\mathbb{P}^1, X) \to \textstyle\prod_{i=1}^r (\mathbb{P}^1 \times J^{m_i} X)^{e_i}$$

that is similar to the τ described in Section 3.2. Here the m_i are nonnegative integers and $\mathbf{m} = (m_1, \ldots, m_r)$.

Lemma 7. *If the target space X is a product of projective spaces, $\prod_{j=1}^m \mathbb{P}^{n_j}$, then the fibers of $\tau_{\mathbf{m}}$ (with their induced reduced scheme structure) are rational varieties.*

Proof. We will prove that when $X = \mathbb{P}^n$ and $l = 1$, every fiber of τ defined in Section 3.2 is a linear subvariety of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \otimes \mathbb{C}^{n+1})$; the general case then follows in a straightforward fashion. It is easy to check that, if $p \times f$ and $p \times g$ are in a same fiber of τ, then $p \times (\mu f + \lambda g := (\mu f_0 + \lambda g_0, \ldots, \mu f_n + \lambda g_n))$ is in the same fiber for all but a finite number of $(\mu, \lambda) \in \mathbb{P}^1$. □

3.5. A Proof of the Main Theorem and Example

Let $m_i = \#\{Z_j \mid j < i, (\pi_j \circ \cdots \circ \pi_r)^{-1}(Z_j) \supset E_i^t\}$. Then general points of Z_i are the (m_i)th infinitesimal points of X. For each i, we re-index Z_j so that $\pi^{-1}(Z_{i_k}) \supset E_i$, where $i_1 < \cdots < i_{m_i}$.

Proof of parts (1) *and* (2) *of the Main Theorem.* By Lemma 1(1) and the assumption on β, the space $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ is smooth and has the expected dimension, and its elements are free morphisms. Hence $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\circ$ is open dense in $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ by Proposition 1. Also note that σ is a smooth morphism by Corollary 1. Therefore, by Lemma 4 and Theorem 2, to verify parts (1) and (2) it is enough to show that the general fiber of σ is rationally connected. Let $P = \sum (p_i, q_i) \in \prod \mathrm{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i)$ be such that all points in P are simple. Then we obtain the inclusion

$$\pi_\beta \circ \sigma^{-1}(P) \subset \mathrm{pr}_2 \circ (\tau_{\mathbf{m}})^{-1}((p_i \times (j^1_{\pi_{i_{m_i}}(q_i)} \circ \cdots \circ j^{m_i}_{\pi_{i_1} \circ \cdots \circ \pi_{i_{m_i}}(q_i)})^{-1}(q_i))_{i=1}^{\sum e_j}), \quad (**)$$

where $\pi_\beta : \mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\circ \subset \mathrm{Mor}_{\pi_* \beta}(\mathbb{P}^1, X)$ is the natural embedding and pr_2 is the projection to the second factor. In $(**)$ the right-hand side (RHS) includes morphisms $\tilde{f}$ with $\deg \tilde{f}^{-1}(E) \geq \beta \cdot E$. This is reason why both sides may not coincide.

By Lemmas 5 and 7, $\tau_{\mathbf{m}}$ is a dominant morphism with a rationally connected general fiber; hence the RHS of $(**)$ is also rationally connected by virtue of Theorem 2. Since the LHS is open subset of the RHS, we conclude that the LHS is also rationally connected.

Proof of part (3) *of the Main Theorem.* (a) If f lies on E_i^s for some i, then $e_i < 0$: take a hypersurface Y of X such that Y contains the image of f; then the strict transform D of Y under π does not contain the image of $\tilde{f}$. Hence $0 = Y \cdot f_*[\mathbb{P}^1] = \pi^*Y \cdot \tilde{f}_*[\mathbb{P}^1] = (D + \sum a_i E_i^s) \cdot \tilde{f}_*[\mathbb{P}^1]$ shows that $E_i^s \cdot \tilde{f}_*[\mathbb{P}^1]$ is negative for some i.

(b) This is part (3) of Theorem 1.

THEOREM 2 [3]. *Let $f : X \to Y$ be a dominant morphism between irreducible varieties X and Y. If Y and the general fiber of f is rationally connected, then X is rationally connected.*

EXAMPLE 2. Let Y be a quadratic line complex in $\mathbb{P}^5$, which is a complete intersection of two smooth quadrics in $\mathbb{P}^5$. Then Y is isomorphic to the moduli space of isomorphism classes of a stable rank-2 vector bundle on a curve of genus $g = 2$ with fixed determinant of degree 1 [9]. Let $\tilde{X}$ be the blowing-up of Y along a line and let $\tilde{Q}$ be the inverse image of the line. Then $\tilde{X}$ is a blowing-up of $\mathbb{P}^3$ along a smooth quintic curve C. Let E_C be the inverse image of the curve C:

$$\begin{tikzcd} & E_C \subset \tilde{X} \supset \tilde{Q}' \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ C \subset \mathbb{P}^3 & & Y \supset \text{line.} \end{tikzcd}$$

Then $\pi(\tilde{Q})$ is a quadric surface Q containing C. Thus $\tilde{Q} = 2H - E_C$, where H is the proper transform of a hyperplane class in $\mathbb{P}^3$. Castravet [1] has shown that there are at least two components (a nice one and an almost nice one) of $\mathrm{Mor}_d(\mathbb{P}^1, Y)$ with the expected dimension. The almost nice component consists of morphisms $\mathbb{P}^1 \to Y$ that are d-to-one onto lines in Y, where $0 < d \in \mathbb{Z} \cong H_2(Y, \mathbb{Z})$ with respect to the ample generator of $\operatorname{Pic} Y$. These two components of $\mathrm{Mor}_d(\mathbb{P}^1, Y)$ are birational to two components of $\mathrm{Mor}_{(d,e)}(\mathbb{P}^1, \tilde{X})$ with $e = 2d$, where $(d, e) \in \mathbb{Z} \times \mathbb{Z} \cong H_2(\tilde{X}, \mathbb{Z})$ with respect to $\pi^*(H)$ and E_C. In this case we observe that, at every point in the corresponding component of the almost nice component, σ is not smooth.

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