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Sharp Bounds for Eigenvalues of Triangles

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1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let T be a triangle in a plane and suppose T has area A and perimeter L. Then the first eigenvalue λ_T of the Dirichlet Laplacian on T satisfies*

$$\frac{\pi^2 L^2}{16 A^2} < \lambda_T \leq \frac{\pi^2 L^2}{9 A^2}. \tag{1.1}$$

The constants 9 and 16 are optimal, and equality in the upper bound holds only for the equilateral triangle.

The lower bound was proved in a more general context in [3]. In Section 6 we show that for "tall" isosceles triangles there is an asymptotic equality in the lower bound. Hence it is impossible to decrease the constant 16.

The upper bound was recently stated as a conjecture in [2], and numerical evidence for its validity is given in [1]. Bounds of this form but with different constants have been the subject of many papers in the literature. The eigenvalue of any doubly connected domain is bounded above by the same fraction but with the constant 4; see [7] and remarks in [5]. There is also a sharper upper bound due to Freitas [2] that is not of this form, but it seems that in the worst case ("tall" isosceles triangle) it gives the constant 6 and in the best (equilateral) 9. Observe that the constant 9 cannot be improved because equilateral triangles give equality in the upper bound of Theorem 1.1.

It is worth mentioning that the same theorem can be equivalently stated in terms of the inradius $R = 2A/L$ of the triangle.

Theorem 1.2. *Let T be a triangle in a plane and let T have inradius R. Then the first eigenvalue λ_T of the Dirichlet Laplacian on T satisfies*

$$\frac{\pi^2}{4} < \lambda_T R^2 \leq \frac{4\pi^2}{9}. \tag{1.2}$$

The equality in the upper bound holds only for the equilateral triangle.

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The spectral properties of a Dirichlet Laplacian on an arbitrary planar domain are important both in physics and in mathematics. Unfortunately, it is almost impossible to find the exact spectrum even for some simple classes of domains. Except for rectangles, balls, and annuli, not much can be said in general. In the case of triangles, the full spectrum is known only for equilateral and right triangles with smallest angles $\pi/4$ or $\pi/6$ (for more information about these see [4; 6]). For all other triangles, the best we can do is give bounds for the eigenvalues, such as those described here.

Even though Theorem 1.1 gives sharp bounds in the sense that the constants are the best possible given the form of the bound, there is certainly room for improvement. In fact, sharper lower bounds are already known (see [2]). One of these bounds is good for both equilateral and "tall" triangles; it gives the constant 9 for the first and 16 for the second. An upper bound that is good in both cases is still unknown.

By comparing our numerical results with the numerical studies contained in [1, Sec. 5.1], we conjecture as follows.

CONJECTURE 1.3. *Let T be a triangle in a plane and let T have area A and perimeter L. Then the first eigenvalue λ_T of the Dirichlet Laplacian on T satisfies*

$$\frac{\pi^2 L^2}{16A^2} + \frac{7\sqrt{3}\pi^2}{12A} \leq \lambda_T \leq \frac{\pi^2 L^2}{12A^2} + \frac{\sqrt{3}\pi^2}{3A}. \tag{1.3}$$

Here both bounds are of the form

$$E_3(L, A, \theta) = \frac{4\pi^2}{\sqrt{3}A} + \theta \frac{L^2 - 12\sqrt{3}A}{A^2}$$

considered in [1].

The lower bound with $\theta = \pi^2/16$ is the best bound we can expect given this particular form. Indeed, this is the only bound that is sharper than the lower bound of Theorem 1.1 and that might hold for "tall" triangles. The upper bound from our main result is also of this form but with $\theta = \pi^2/9$. Hence the conjectured upper bound is sharper ($\theta = \pi^2/12$), and it is best in the sense that the bound with $\theta = \pi^2/13$ is not valid. However, since only the constant 16 can give a good upper bound for "tall" triangles, it is not possible to find a bound of the form E_3 that is good for both equilateral and "tall" triangles.

Our proof of the upper bound from Theorem 1.1 contains two main parts. The first deals with "almost equilateral" triangles—that is, with triangles for which the longest side is comparable to the shortest side. For these our strategy is to find a suitable test function ψ. That is, we try to find a function that is 0 on the boundary of the triangle T and than apply the Rayleigh quotient to derive the upper bound for λ_T. Thus,

$$\lambda_T \leq \frac{\int_T |\nabla \psi|^2}{\int_T \psi^2}. \tag{1.4}$$

This part of the proof is contained in Sections 2–5. Section 2 includes also some preliminary results.

The second part of the proof, contained in Section 6, deals with "tall" triangles. These can be approximated by circular sections for which the eigenvalues can be found explicitly.

2. Eigenfunctions and Notation

An arbitrary triangle T' can be rotated and rescaled to obtain a triangle T with vertices $(0,0)$, $(1,0)$, and (a,b). This—together with the fact that the bound in the main theorem is invariant under translations, rotations, and scaling—allows us to restrict our attention to the triangles with such vertices. We can also assume that the side contained in the x-axis is the shortest. Hence we have

$$a^2+b^2 \geq 1 \quad \text{and} \quad a \leq 1/2$$

for our triangles. We will denote the length of the other two sides by M and N, with N denoting the longest.

We start with the first eigenfunction of an equilateral triangle, proceeding as in [2]. Such a function is given by

$$f(x,y) = \sin\left(\frac{4\pi y}{\sqrt{3}}\right) - \sin\left[2\pi\left(x+\frac{y}{\sqrt{3}}\right)\right] + \sin\left[2\pi\left(x-\frac{y}{\sqrt{3}}\right)\right]. \tag{2.1}$$

We can compose f with a linear transformation to obtain a function ϕ that is equal to 0 on the boundary of T. Namely, consider

$$\begin{aligned}\phi(x,y) &= f\left(x-\frac{a-1/2}{b}y, \frac{\sqrt{3}}{2b}y\right)\\ &= \sin\left(\frac{2\pi y}{b}\right) - \sin\left[2\pi\left(x+\frac{(1-a)y}{b}\right)\right] + \sin\left[2\pi\left(x-\frac{ay}{b}\right)\right].\end{aligned} \tag{2.2}$$

This function was used in [2] to obtain the upper bound from the Rayleigh quotient. Since the function f is the first eigenfunction of the Dirichlet Laplacian on an equilateral triangle and since its eigenvalue yields equality in the main bound, it is reasonable to expect that taking any linear transformation can only decrease the constant 9 in Theorem 1.1.

Hence we want to find another eigenfunction of some other triangle. We will use the eigenfunctions of the equilateral triangle to find a test function for the right triangle with angles $\pi/3$ and $\pi/6$. In [4] the author constructs two families of eigenfunctions of the equilateral triangle. The antisymmetric mode has the property that it is 0 on the altitude. Thus, such a function is also the eigenfunction for the right triangle. We can then take the antisymmetric eigenfunction corresponding to the smallest eigenvalue as our test function. A calculation then leads to

$$\begin{aligned}g(x,y) = \sin(\sqrt{3}\pi y)\sin\left(\frac{\pi x}{3}\right)&\\ + \sin\left(\frac{\sqrt{3}\pi y}{3}\right)\sin\left(\frac{5\pi x}{3}\right)&\\ + \sin\left(\frac{2\sqrt{3}\pi y}{3}\right)\sin\left(\frac{4\pi x}{3}\right)&.\end{aligned} \tag{2.3}$$

This function, as can be easily checked, is the eigenfunction of the Dirichlet Laplacian on the triangle with vertices $(0,0)$, $(1,0)$, and $(0,\sqrt{3})$. The corresponding eigenvalue gives a better bound than the one in Theorem 1.1: the constant is about 9.6. Therefore, a linear transformation of this function should give a correct bound at least for the neighborhood of the point $(0,\sqrt{3})$. Applying a suitable linear transformation yields the second test function

$$
\begin{aligned}
\varphi_1(x,y) &= g\left(x-\frac{ay}{b},\frac{\sqrt{3}y}{b}\right)\\
&= \sin\left(\frac{3\pi y}{b}\right)\sin\left[\frac{\pi}{3}\left(x-\frac{ay}{b}\right)\right]\\
&\quad+\sin\left(\frac{\pi y}{b}\right)\sin\left[\frac{5\pi}{3}\left(x-\frac{ay}{b}\right)\right]\\
&\quad+\sin\left(\frac{2\pi y}{b}\right)\sin\left[\frac{4\pi}{3}\left(x-\frac{ay}{b}\right)\right].
\end{aligned}
\tag{2.4}
$$

We can similarly obtain the last two test functions. One will be a linear transformation of the eigenfunction of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,\sqrt{3})$; the other will be a linear transformation of the eigenfunction of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1/\sqrt{3})$. Thus:

$$
\begin{aligned}
\varphi_2(x,y) &= \sin\left(\frac{3\pi y}{b}\right)\sin\left[\frac{\pi}{3}\left(1-x+\frac{(a-1)y}{b}\right)\right]\\
&\quad+\sin\left(\frac{\pi y}{b}\right)\sin\left[\frac{5\pi}{3}\left(1-x+\frac{(a-1)y}{b}\right)\right]\\
&\quad+\sin\left(\frac{2\pi y}{b}\right)\sin\left[\frac{4\pi}{3}\left(1-x+\frac{(a-1)y}{b}\right)\right];
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\varphi_3(x,y) &= \sin\left(\frac{5\pi y}{3b}\right)\sin\left[\pi\left(x-\frac{ay}{b}\right)\right]\\
&\quad+\sin\left(\frac{4\pi y}{3b}\right)\sin\left[2\pi\left(x-\frac{ay}{b}\right)\right]\\
&\quad+\sin\left(\frac{\pi y}{3b}\right)\sin\left[3\pi\left(x-\frac{ay}{b}\right)\right].
\end{aligned}
\tag{2.6}
$$

Now we can take a linear combination of these test functions. That is, consider

$$\psi(x,y)=\alpha\varphi_1(x,y)+\beta\varphi_2(x,y)+\gamma\varphi_3(x,y)+\varepsilon\phi(x,y). \tag{2.7}$$

We can calculate the Rayleigh quotient for this function. Optimizing over all possible values of α, β, γ, and ε will give us an appropriate bound for the first eigenvalue. To prove Theorem 1.1, we must check that

$$\lambda_T \le \frac{\int_T |\nabla\psi|^2}{\int_T \psi^2} \le \frac{\pi^2 L^2}{9A^2} \tag{2.8}$$

for some α, β, γ, and ε (possibly depending on T) and also that the last inequality becomes an equality only for the equilateral triangle.

The last inequality of (2.8) is equivalent to

$$9A^2 \int_T |\nabla\psi|^2 \le \pi^2 L^2 \int_T \psi^2. \tag{2.9}$$

Because the function ψ is given explicitly and is a trigonometric function, it is possible to find the exact values for these integrals. Yet because the calculations are cumbersome, we will use Mathematica for the long ones. We emphasize, however, that all the calculations are done symbolically.

By our assumptions we have $L = 1 + \sqrt{a^2+b^2} + \sqrt{(a-1)^2+b^2}$ and $A = b/2$. Now running Mathematica shows that, in order to prove (2.9), we must find $\alpha, \beta, \gamma, \varepsilon$ such that the following inequality is valid:

$$\begin{aligned}
0 \ge{} & 8041366333 \\
& \times \{(-1594323 - 1792090a + 531441(a^2+b^2) + 201600(3+a^2+b^2)\pi^2)\alpha^2 \\
& \quad + (-2854972 + 729208a + 531441(a^2+b^2) + 201600(3+(a-1)^2+b^2)\pi^2)\beta^2 \\
& \quad + (531441 - 1792090a - 1594323(a^2+b^2) + 201600(1+3a^2+3b^2)\pi^2)\gamma^2\} \\
& + 5558192409369600(1 - a + (a^2+b^2))\pi^2\varepsilon^2 \\
& + 67672797192 \\
& \times \{(729\sqrt{3}(454 - 128a + 339(a^2+b^2)) + 24640(4 - 8a + 9(a^2+b^2))\pi)\gamma\varepsilon \\
& \quad + (729\sqrt{3}(665 - 780a + 454(a^2+b^2)) + 24640(5 + 4(a^2+b^2))\pi)\beta\varepsilon \\
& \quad + (729\sqrt{3}(339 - 128a + 454(a^2+b^2)) + 24640(9 - 8a + 4(a^2+b^2))\pi)\alpha\varepsilon\} \\
& + (1990033124626008a + 2553294638054160\sqrt{3}(3 - 2a + 3(a^2+b^2))\pi)\alpha\gamma \\
& + 1151172000(35341051 - 26756686a + 32479596(a^2+b^2))\beta\gamma \\
& + 189\{819452341268271 - 73323642839420a + 73323642839420(a^2+b^2) \\
& \qquad - 79935610875120\sqrt{3}\pi + 24336134222400\sqrt{3}(-a + a^2 + b^2)\pi\}\alpha\beta \\
& - 9\big(1 + \sqrt{(-1+a)^2+b^2} + \sqrt{a^2+b^2}\big)^2 \\
& \times \{444001222376712\sqrt{3}(\alpha+\gamma)\varepsilon \\
& \quad - 1629547920\pi(\sqrt{3}\alpha(4251\beta - 99484\gamma) - 113696(\alpha+\beta+\gamma)\varepsilon) \\
& \quad + 51464744531200\pi^2(\alpha^2+\beta^2+\gamma^2+2\varepsilon^2) \\
& \quad + 3\beta(346474423262177\alpha + 85272(3297684500\gamma + 1735627257\sqrt{3}\varepsilon))\}.
\end{aligned} \tag{2.10}$$

We will actually show that this inequality is strict for the triangles that are not equilateral.

Expression (2.10) clearly shows that it would be most difficult to do the calculations by hand. Notice that this expression depends only on b^2 and a and that its "building blocks" are exactly equal to the length of the sides of the triangle T.

Hence we make the substitution $M^2 = a^2 + b^2$ and $N^2 = (a-1)^2 + b^2$. The result is a polynomial of degree 2 in M and N, where $N \geq M \geq 1$. For further simplification (and to improve our chances of finding the appropriate coefficients) we divide all triangles into four classes:

1. $N \geq 2$ and $M \leq 15$;
2. $1 \leq N \leq 2$ and $(N+1)/2 \leq M \leq 2$;
3. $1 \leq N \leq 2$ and $1 \leq M \leq (N+1)/2$;
4. $M \geq 15$.

Each class will be handled separately in the sections that follow. The method used to handle the last case will be totally different than the previous ones.

3. Class 1: $N \geq 2$ and $M \leq 15$

Let us take $\varepsilon = \beta = 0$, $\alpha = 1$, and $\gamma = -1/6$. Then (2.10) simplifies to

$$\begin{aligned} 0 &< P(M,N) \\ &= -90851035780 - 16374894040M^2 + 33929984593N^2 \\ &\quad - 272432160\sqrt{3}(10 - 8M + 10M^2 - 8N - 8MN + 3N^2)\pi \\ &\quad + 28828800(689 - 148M + 199M^2 - 148N - 148MN - 74N^2)\pi^2. \end{aligned} \tag{3.1}$$

To prove this inequality we first find all the critical points of the right side and later check the values on the boundary. Both $\partial_M P$ and $\partial_N P$ are linear with respect to M and N, so we have exactly one critical point with $N \approx -42.2$. Hence it is enough to check this inequality on the boundary.

The boundary conditions are given by $M = N$, $M = 15$, $N = 2$, and $M = N - 1$. For each of these conditions, P is a quadratic equation and we need only check (i) that the roots are outside of the bounds for M or N and (ii) that the inequality is true at the endpoints. Thus:

- $P(M,M) = 0$ for $M \approx 1.6$ and $M \approx 15.15$, $P(2,2) < 0$ and $P(15,15) < 0$;
- $P(15,N) = 0$ for $N \approx 14.97$ and $N \approx 42.5$, $P(15,15) < 0$ and $P(15,16) < 0$;
- $P(M,2) = 0$ for $M \approx 0.96$ and $M \approx 2.61$, $P(1,2) < 0$ and $P(2,2) < 0$;
- $P(N-1,N) = 0$ for $N \approx 1.97$ and $N \approx 20.56$, $P(1,2) < 0$ and $P(15,16) < 0$.

This shows that the desired inequality is true on the boundary and therefore holds everywhere.

4. Class 2: $1 \leq N \leq 2$ and $(N+1)/2 \leq M \leq 2$

In this section and the next we must deal with cases for which an equilateral triangle ($N = M = 1$) is one of the possible triangles. We take $\varepsilon = 1$, since only the eigenfunction of the equilateral triangle can give the constant 9 in Theorem 1.1. We also require that all the other coefficients vanish near the equilateral triangle. Let us take $\gamma = 0$ and $\alpha = \beta$; then we just have to choose the common value for α and β. The calculations are already so complicated that we cannot afford to pick a complicated coefficient, and hence we take $\alpha = (N + M - 2)/2$. This choice has one additional advantage: here we are working with the eigenfunctions of one equilateral triangle and two right triangles with shortest side $(0,0)-(1,0)$; hence we have a symmetry about $a = 1/2$ or (in terms of M and N) about $M = N$. It is therefore natural to introduce the rotated coordinates $U = (M+N)/2 - 1$ and $V = (N-M)/2$. Observe that $\alpha = \beta = U$. This also moves the equilateral triangle to the origin.

After these transformations are applied, the inequality (2.10) becomes

$$
\begin{aligned}
0 &\geq P(U,V)\\
&= U^2(3293385188722144 - 451048860827136\sqrt{3} - 952832984463360\pi\\
&\qquad - 874782993324240\sqrt{3}\pi + 463182700780800\pi^2 - 4817666363010084U\\
&\qquad + 916192998555120\sqrt{3}U + 710514087323040\sqrt{3}\pi U\\
&\qquad - 330844786272000\pi^2 U - 1072431834636645U^2\\
&\qquad + 346350633108480\sqrt{3}\pi U^2 - 33084478627200\pi^2 U^2)\\
&\quad + 9V^2(44112638169600\pi^2 + 355514206276944\sqrt{3}U + 105870331607040\pi U\\
&\qquad\quad + 177818984344461U^2 + 36504201333600\sqrt{3}\pi U^2\\
&\qquad\quad + 25732372265600\pi^2 U^2).
\end{aligned}
\tag{4.1}
$$

This is a polynomial of degree 4 in U and of degree 2 in V. Hence we expect it is possible to solve $\partial_V P(U,V) = 0$. (In fact, $\partial_V P$ is equal to V multiplied by an irreducible quadratic polynomial in U.) We therefore have exactly one solution: $V = 0$, or $N = M$. But this is a boundary of the region, so we need only check the boundary values.

In this case the boundary conditions are $M = N$, $M = (N+1)/2$, and $N = 2$. Once we change variables to U and V, these conditions become $V = 0$, $U = 3V$, and $U + V = 1$, respectively. Each time we obtain a polynomial of degree 4, so we now proceed as in the previous section.

- $P(U,0) = 0$ for $U = 0$ (double root), $U \approx 5.65$ and $U \approx -0.24$; $P(0,0) = 0$ and $P(1,0) < 0$.
- $P(3V,V) = 0$ for $V = 0$ (double root), $V \approx 0.55$ and $V \approx -0.04$; $P(0,0) = 0$ and $P(3/4,1/4) < 0$.
- $P(1-V,V) = 0$ for $V \approx -0.52$ and $V \approx 0.29$ (2 complex roots); $P(1,0) < 0$ and $P(3/4,1/4) < 0$.

Therefore, the inequality is true on the boundary and hence also inside of the region. And because we are dealing with polynomials, this inequality is strict in all the points except for $U = 0$, $V = 0$ (equilateral triangle).

5. Class 3: $1 \leq N \leq 2$ and $1 \leq M \leq (N+1)/2$

Here we take $\varepsilon = 1$, $\beta = 0$, and $\alpha = \gamma = (N+M-2)/\sqrt{2}$. Even though the symmetry described in the previous section does not exist here, we still use the same rotated coordinates $U = (M+N)/2 - 1$ and $V = (N-M)/2$. This time the inequality (2.10) becomes

$$
\begin{aligned}
0 &\geq P(U,V) \\
&= 32133332U^2(-1898955433 - 549628092\sqrt{6} + 103783680\sqrt{2}\pi \\
&\qquad - 22702680\sqrt{3}\pi + 345945600\pi^2 - 1063944882U \\
&\qquad + 222614730\sqrt{6}U + 259459200\sqrt{2}\pi U - 136216080\sqrt{3}\pi U \\
&\qquad + 115315200\pi^2 U - 531972441U^2 + 113513400\sqrt{3}\pi U^2 \\
&\qquad + 172972800\pi^2 U^2) \\
&\quad - 64266664(824442138\sqrt{6} - 155675520\sqrt{2}\pi - 2201993543U \\
&\qquad + 158918760\sqrt{3}\pi U + 403603200\pi^2 U)(U + U^2)V \\
&\quad + 3759599844(1478400\pi^2 + 10405746\sqrt{6}U + 5765760\sqrt{2}\pi U \\
&\qquad - 4546773U^2 + 4074840\sqrt{3}\pi U^2 + 3449600\pi^2 U^2)V^2. \qquad (5.1)
\end{aligned}
$$

Note that this is still a polynomial of degree 2 in V, so we may proceed as in the previous section. Unfortunately, this time the only solution of $\partial_V P = 0$ is V a rational function of U with an irreducible denominator of degree 2. Hence, plugging this into $\partial_U P = 0$ yields a rational equation with squared irreducible polynomial of degree 2 in the denominator—an equation that is equivalent to the numerator being 0. Fortunately, the numerator is a solvable polynomial of degree 7 with four imaginary roots and three real roots (0, ≈ -0.18, and ≈ 1.8).

Here we have the following bounds: $U = 3V$ (equivalent to $M = (N+1)/2$), $U + V = 1$ ($N = 2$), and $U = V$ ($M = 1$). Hence this triangle has vertices $(0,0)$, $(3/4, 1/4)$, and $(1/2, 1/2)$, so neither critical point is inside of this region. This leaves only the boundary values to check, which (as before) means we must find the roots of certain polynomials of degree 4 as well as values at the endpoints.

- $P(V,V) = 0$ for $V = 0$ (double root), $V \approx -0.27$ and $V \approx 0.64$; $P(0,0) = 0$ and $P(1/2, 1/2) < 0$.
- $P(3V,V) = 0$ for $V = 0$ (double root), $V \approx -0.06$ and $V \approx 0.51$; $P(0,0) = 0$ and $P(3/4, 1/4) < 0$.
- $P(U, 1-U) = 0$ for $U \approx 0.48$ and $U \approx 0.79$ (2 complex roots); $P(1/2, 1/2) < 0$ and $P(3/4, 1/4) < 0$.

Therefore, the inequality (5.1) holds. Note also that, as in the previous section, the inequality is strict for any triangle other than equilateral.

6. Class 4: $M \geq 15$

For this class our method is different than for the previous classes. Because we are dealing with the triangles for which two sides are long and almost equal, we will estimate the eigenvalue by the eigenvalue of a circular sector contained in the triangle T.

We shall use γ to denote the angle between the sides of length N and M. First we take the isosceles triangle with angle γ between the sides of length M. We can certainly put this triangle inside the triangle T. Since the shortest side of this isosceles triangle has length no greater than 1, the altitude h satisfies $h \geq \sqrt{M^2 - 1/4}$.

We now use $S(\alpha, r)$ to denote a circular sector with angle α and radius r. It is known (see [8]) that the first eigenvalue of the sector $S(\alpha, r)$ is $j^2_{\pi/\alpha} r^{-2}$, where j_ν is the first zero of the Bessel function $J_\nu(x)$ of order ν.

It is clear that we can put a sector $S(\gamma, h)$ inside the triangle T. Hence, by domain monotonicity we have

$$\lambda_T \leq \lambda_{S(\gamma,h)} = \frac{j^2_{\pi/\gamma}}{h^2}. \tag{6.1}$$

We need to prove that

$$\frac{j^2_{\pi/\gamma}}{h^2} < \frac{\pi^2 L_T^2}{9A_T^2}. \tag{6.2}$$

We have $L_T = 1+M+N \geq 2N$ and $A_T = \sin(\gamma)NM/2 \leq \gamma NM/2$. Therefore, it is enough to show that

$$\frac{9j^2_{\pi/\gamma}(\gamma NM/2)^2}{(M^2 - 1/4)^2(2N)^2} < 1 \tag{6.3}$$

or that

$$\frac{9j^2_{\pi/\gamma}\gamma^2 M^2}{16\pi^2(M^2 - 1/4)^2} < 1. \tag{6.4}$$

To find the bound for j_ν, we will use the estimate obtained in [9]:

$$j_\nu \leq \nu - \frac{a_1}{2^{1/3}}\nu^{1/3} + \frac{3a_1^2 2^{1/3}}{20}\nu^{-1/3}, \tag{6.5}$$

where $a_1 \approx -2.338$ is the first negative zero of the Airy function. Hence

$$\frac{j_\nu}{\nu} \leq 1 + 2\nu^{-2/3} + 2\nu^{-4/3} \tag{6.6}$$

and so

$$\frac{9j^2_{\pi/\gamma}\gamma^2 M^2}{16\pi^2(M^2 - 1/4)^2} \leq \left(1 + 2\left(\frac{\gamma}{\pi}\right)^{2/3} + 2\left(\frac{\gamma}{\pi}\right)^{4/3}\right)^2 \frac{9M^2}{16(M^2 - 1/4)}. \tag{6.7}$$

Inequality (6.7) is increasing with γ, as can be easily verified by differentiating. Given M, the angle γ is maximized for the isosceles triangle; hence $\gamma \leq 2\sin^{-1}(1/2M)$. In order to arrive at (6.4), it is enough to show that

$$\left(1 + 2\left(\frac{2\sin^{-1}(1/2M)}{\pi}\right)^{2/3} + 2\left(\frac{2\sin^{-1}(1/2M)}{\pi}\right)^{4/3}\right)^2 \frac{9M^2}{16(M^2 - 1/4)} < 1. \tag{6.8}$$

It is easy to check that the function on the left side is decreasing with M and that, for $M = 15$, the inequality is true. Hence (6.8) holds for any triangle with $M \geq 15$.

Note also that if $M \to \infty$ then the whole expression tends to $9/16$. This shows that the constant 16 in the lower bound in Theorem 1.1 is optimal.

7. Script in Mathematica

Here we give the script written in Mathematica to handle all the cumbersome calculations of Sections 2–5. It is important to note that all the calculations are done symbolically; the exact values of the polynomials' roots are converted to numerical form only at the end.

```
(* Section 2 *)
(* isosceles triangle with vertices (0,0), (1,0) and (Sqrt[3],0) *)
g[x_,y_]=Sin [Sqrt[3]\[Pi] y]Sin[\[Pi] x/3] +  \
        Sin[\[Pi] y/Sqrt[3]]Sin[5\[Pi] x/3] +  \
        Sin[2\[Pi] y/Sqrt[3]]Sin[4\[Pi] x/3];
(* other right triangles *)
g2[x_,y_]=g[1-x,y];
g3[x_,y_]=g[Sqrt[3]y,Sqrt[3]x];
(* test functions obtained from right triangles *)
\[CurlyPhi]1=g[x-(a y /b),Sqrt[3]y/b];
\[CurlyPhi]2=g2[x-((a-1) y /b),Sqrt[3]y/b];
\[CurlyPhi]3=g3[x-(a y /b),y/(Sqrt[3]b)];
(* equilateral triangle after linear transformation *)
\[Phi]:=Sin[2\[Pi]y/b]-Sin[2\[Pi](x+(1-a)y/b)]+Sin[2\[Pi](x-a y/b)];
(* final test function *)
\[Psi]=\[Alpha] \[CurlyPhi]1 + \[Beta] \[CurlyPhi]2 +  \
        \[Gamma] \[CurlyPhi]3 + \[Epsilon] \[Phi];
grad=Simplify[Integrate[D[\[Psi],x]^2+D[\[Psi],y]^2,{y,0,b},  \
        {x,a y/b, (a-1) y/b+1}]];
int=Simplify[ Integrate[\[Psi]^2,{y,0,b},{x,a y/b , (a-1)y/b +1}]];
(* we have to prove that this is <= 0 *)
in=9b^2grad-4\[Pi]^2(1+Sqrt[a^2+b^2]+Sqrt[(a-1)^2+b^2])^2int;
(* change from (a, b) to (M, N) and cancel b *)
in2=Simplify[in/b /. b^2 -> M^2 - a^2 /. a -> (M^2 - N^2 + 1)/2,  \
        (N > 0) && (M > 0)];
(* inequality (2.9) *)
Simplify[308788467187200in/b]

(* Section 3 *)
W=in2/. \[Epsilon] -> 0 /. \[Gamma] -> -1/6 /. \[Beta] -> 0 /.   \
        \[Alpha] -> 1;
(* Inequality (3.1) *)
Apart[1383782400W]
(* Critical point *)
Reduce[(D[W, M] == 0) && (D[W, N] == 0), {M, N}] // N
(* Boundary : roots and endpoints *)
Reduce[W == 0 /. N -> 2] // N
Reduce[W == 0 /. M -> N - 1] // N
```

```
Reduce[W == 0 /. M -> N] // N
Reduce[W == 0 /. M -> 15] // N
W /. M -> {1, 2} /. N -> 2 // N
W /. M -> 15 /. N -> {15, 16} // N

(* Section 4 *)
W=in2/. \[Epsilon] -> 1 /. \[Gamma] -> 0 /. \[Beta] -> \[Alpha] /.\
        \[Alpha] -> (N + M - 2)/2;
pol = W/. M -> U - V /. N -> U + V /. U -> U + 1;
(* inequality (4.1) *)
Apart[22056319084800pol, V]
(* Critical point *)
Reduce[D[pol, V] == 0, V] // N
(* Boundary : roots and endpoints *)
Reduce[pol == 0 /. V -> 0] // N
Reduce[pol == 0 /. U -> 1 - V] // N
Reduce[pol == 0 /. U -> 3V] // N
pol /. V -> 0 /. U -> {0, 1} // N
pol /. V -> 1/4 /. U -> 3/4 // N

(* Section 5 *)
W=in2/. \[Epsilon] -> 1 /. \[Beta] -> 0 /. \[Gamma] -> \[Alpha] /.\
        \[Alpha] -> (M + N - 2)/Sqrt[2];
pol = W /. M -> U - V /. N -> U + V /. U -> U + 1;
(* inequality (5.1) *)
Apart[9609600pol, V]
(* Critical points *)
Vs = Solve[D[pol, V] == 0, V];
Reduce[D[pol, V] == 0, V, Reals]
(* denominator with complex roots only *)
Reduce[Denominator[Together[D[pol, U] /. Vs]] == 0] // N
(* polynomial of degree 7 in U *)
Reduce[Numerator[Together[
      D[pol, U] /. Vs]] == 0] // N
(* Boundary : roots and endpoints *)
Reduce[pol == 0 /. U -> 3V] // N
Reduce[pol == 0 /. U -> V] // N
Reduce[pol == 0 /. V -> 1 - U] // N
pol /. U -> 1 - V /. V -> {1/4, 1/2} // N
pol /. U -> 0 /. V -> 0 // N
```

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