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Changes of Variables in ELSV-type Formulas

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1. Introduction

In [5], Goulden, Jackson, and Vakil formulated a conjecture relating certain Hurwitz numbers (enumerating ramified coverings of the sphere) to the intersection theory on a conjectural Picard variety $\mathrm{Pic}_{g,n}$. This variety, of complex dimension $4g-3+n$, is supposedly endowed with a natural morphism to the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The fiber over a point $x \in \mathcal{M}_{g,n}$ lying in the open part of the moduli space is equal to the Jacobian of the corresponding smooth curve C_x. The variety $\mathrm{Pic}_{g,n}$ is also supposed to carry a universal curve $\mathcal{C}_{g,n}$ with n disjoint sections $s_1, \dots, s_n$. Denote by $\mathcal{L}_i$ the pull-back under s_i of the cotangent line bundle to the fiber of $\mathcal{C}_{g,n}$. Then we obtain n tautological 2-cohomology classes $\psi_i = c_1(\mathcal{L}_i)$ on $\mathrm{Pic}_{g,n}$.

We shall use the formula from [5] to study the intersection numbers of the classes ψ_i on $\mathrm{Pic}_{g,n}$ (if it is ever to be constructed). In particular, we prove a Witten–Kontsevich-type theorem relating the intersection theory and integrable hierarchies. These equations, together with the string and dilation equations, allow us to compute all the intersection numbers under consideration.

Independently of the conjecture of [5], our results can be interpreted as meaningful statements about Hurwitz numbers. Our methods are close to those of Kazarian and Lando in [7] and make use of Hurwitz numbers. We also extend the results of [7] to include the Hodge integrals over the moduli spaces involving one λ-class.

1.1. The Conjecture

Fix n positive integers $b_1, \dots, b_n$. Let $d = \sum b_i$ be their sum.

Definition 1.1. The number of degree-d ramified coverings of the sphere by a genus-g surface possessing

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- a unique preimage of 0,
- n numbered preimages of ∞ with multiplicities $b_1, \dots, b_n$, and
- $2g - 1 + n$ fixed simple branch points

is called a *Hurwitz number* and is denoted by $h_{g;b_1,\dots,b_n}$.

CONJECTURE 1.2 [5]. *There exist a compactification of the Picard variety over the moduli space $\mathcal{M}_{g,n}$ by a smooth $(4g-3+n)$-dimensional orbifold $\mathrm{Pic}_{g,n}$, natural cohomology classes $\Lambda_2, \dots, \Lambda_{2g}$ on $\mathrm{Pic}_{g,n}$ of (complex) degrees $2, \dots, 2g$, and an extension of the tautological classes $\psi_1, \dots, \psi_n$ such that*

$$h_{g;b_1,\dots,b_n} = (2g-1+n)!\, d \int_{\mathrm{Pic}_{g,n}} \frac{1 - \Lambda_2 + \cdots \pm \Lambda_{2g}}{(1-b_1\psi_1)\cdots(1-b_n\psi_n)}.$$

Assuming that the conjecture is true we can define

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\mathrm{Pic}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{1}$$

By convention, this bracket vanishes unless $\sum d_i = 4g - 3 + n$. We also introduce the following generating series for the intersection numbers of the ψ-classes on $\mathrm{Pic}_{g,n}$:

$$F(t_0, t_1, \dots) = \sum_n \frac{1}{n!} \sum_{d_1,\dots,d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}, \tag{2}$$

and we denote by

$$U = \frac{\partial^2 F}{\partial^2 t_0} \tag{3}$$

its second partial derivative.

While Conjecture 1.2 remains open, the situation should be seen in the following way. The Hurwitz numbers turn out to have the unexpected property of being polynomial in variables b_i (first conjectured in [4] and proved in [5]). The coefficients of these polynomials are denoted by $\langle \tau_{d_1} \cdots \tau_{d_n} \Lambda_{2k} \rangle$. (We restrict ourselves to the case $k = 0$ with $\Lambda_0 = 1$.) The conjectured relation of these coefficients with geometry is a strong motivation to study them. Our goal is to find out as much as we can about the values of the bracket in the combinatorial framework, waiting for their geometrical meaning to be clarified.

This study was actually initiated in [5]. In particular, the authors proved that the values of the bracket satisfy the following string and dilation equations:

$$\frac{\partial F}{\partial t_0} = \sum_{d \geq 1} t_d \frac{\partial F}{\partial t_{d-1}} + \frac{t_0^2}{2}, \tag{4}$$

$$\frac{\partial F}{\partial t_1} = \frac{1}{2} \sum_{d \geq 0} (d+1) t_d \frac{\partial F}{\partial t_d} - \frac{1}{2} F. \tag{5}$$

By abuse of language we will usually speak of the coefficients of F as intersection numbers, implicitly assuming the conjecture to be true.

1.2. Results

We will soon see that F is related to the following generating function for the Hurwitz numbers:

$$H(\beta, p_1, p_2, \dots) = \sum_{g,n} \frac{1}{n!} \frac{\beta^{2g-1+n}}{(2g-1+n)!} \sum_{b_1,\dots,b_n} \frac{h_{g;b_1,\dots,b_n}}{d} p_{b_1} \cdots p_{b_n}. \tag{6}$$

Here, as before, $d = \sum b_i$ is the degree of the coverings and $2g - 1 + n$ is the number of simple branch points.

Denote by L_p the differential operator

$$L_p = \sum b p_b \frac{\partial}{\partial p_b}.$$

Its action on H consists in multiplying each term by its total degree d.

THEOREM 1. *The series $L_p^2 H$ is a τ-function of the Kadomtsev–Petviashvili* (*or KP*) *hierarchy in variables p_i; that is, it satisfies the full set of bilinear Hirota equations. In addition, $L_p^2 H$ satisfies the dispersionless limit of the KP equations.*

The proof of this theorem follows in an almost standard way from the general theory of integrable systems. We will discuss it in Section 3.

THEOREM 2. *The series U is a τ-function for the KP hierarchy in variables $T_i = t_{i-1}/(i-1)!$; that is, it satisfies the full set of bilinear Hirota equations in these variables. In addition, it satisfies the dispersionless limit of the KP equations in the same variables.*

Theorem 2 follows from Theorem 1—but far from trivially, in spite of their apparent similarity.

EXAMPLE 1.3. The string and dilation equations allow one to compute all the values of the bracket in $g = 0, 1, 2$ knowing only the following values, which can be obtained using Theorem 2:

$$g = 0\colon \quad \langle \tau_0^3 \rangle = 1;$$

$$g = 1\colon \quad \langle \tau_2 \rangle = \frac{1}{24};$$

$$g = 2\colon \quad \langle \tau_6 \rangle = \frac{1}{1920},\ \langle \tau_2 \tau_5 \rangle = \frac{19}{5760},\ \langle \tau_3 \tau_4 \rangle = \frac{11}{1920},\ \langle \tau_2^2 \tau_4 \rangle = \frac{37}{1440},$$
$$\langle \tau_2 \tau_3^2 \rangle = \frac{5}{144},\ \langle \tau_2^3 \tau_3 \rangle = \frac{5}{24},\ \langle \tau_2^5 \rangle = \frac{25}{16}.$$

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2. Intersection Numbers and Hurwitz Numbers

Here we establish a link between the generating series H (for Hurwitz numbers) and F (for intersection numbers of the ψ-classes on $\mathrm{Pic}_{g,n}$).

We introduce the following linear triangular change of variables:

$$p_b = \sum_{d=b-1}^{\infty} \beta^{-(d+1)/2} \frac{(-1)^{d-b+1}}{(d-b+1)!\,(b-1)!} t_d. \tag{7}$$

Thus,

$$\begin{aligned} p_1 &= \beta^{-1/2} t_0 - \beta^{-1} t_1 + \tfrac{1}{2}\beta^{-3/2} t_2 - \cdots, \\ p_2 &= \qquad\qquad \beta^{-1} t_1 - \quad \beta^{-3/2} t_2 - \cdots, \\ p_3 &= \qquad\qquad\qquad\qquad \tfrac{1}{2}\beta^{-3/2} t_2 - \cdots. \end{aligned}$$

Let us separate the generating series H into two parts. The *unstable* part, corresponding to the cases $g = 0$, $n = 1, 2$, is given by

$$H_{\mathrm{unst}}(\beta, p_1, p_2, \dots) = \sum_{b=1}^{\infty} \frac{p_b}{b^2} + \frac{\beta}{2} \sum_{b_1, b_2 = 1}^{\infty} \frac{p_{b_1} p_{b_2}}{b_1 + b_2}.$$

The *stable* part is given by $H_{\mathrm{st}} = H - H_{\mathrm{unst}}$.

The change of variables was designed to make the following proposition work.

PROPOSITION 2.1. *The change of variables* (7) *transforms the series H_{st} into a series of the form $\sqrt{\beta} F + O(\beta)$.*

Proof. First let $\beta = 1$. It is readily seen that, for any $d \geq 0$,

$$\sum_{b=1}^{d+1} \frac{(-1)^{d-b+1}}{(d-b+1)!\,(b-1)!} \cdot \frac{1}{1 - b\psi} = \psi^d + O(\psi^{d+1}) \tag{8}$$

as a power series in ψ. Using Conjecture 1.2, it follows that for any $d_1, \dots, d_n$ we have

$$\sum_{\substack{b_1, \dots, b_n \\ 1 \leq b_i \leq d_i + 1}} \frac{(-1)^{d-b+1}}{(d-b+1)!\,(b-1)!} \frac{h_{g; b_1, \dots, b_n}}{(2g-1+n)!\,d}$$
$$= \int_{\mathrm{Pic}_{g,n}} (1 - \Lambda_2 + \cdots \pm \Lambda_{2g}) \prod_{i=1}^{n} (\psi_i^{d_i} + O(\psi_i^{d_i+1})).$$

Now *assume* that $\sum d_i = \dim(\mathrm{Pic}_{g,n}) = 4g - 3 + n$. Then each factor on the right-hand side contributes to the integral only through its lowest-order term. Therefore, the right-hand side is equal to

$$\int_{\mathrm{Pic}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

which is, up to a combinatorial factor, precisely a coefficient of F. The purpose of introducing the parameter β in the change of variables (7) is to isolate such terms from the others. Indeed, we claim that the power of β in a term obtained by the change of variables equals

$$\frac{\dim(\mathrm{Pic}_{g,n}) - \sum d_i + 1}{2}.$$

To check this, recall that the power of β in a term of H equals $2g-1+n$ by definition of H. After subtracting $(d+1)/2$ for each variable t_d, we obtain

$$2g - 1 + \frac{n}{2} - \sum \frac{d_i}{2} = \frac{4g - 2 + n - \sum d_i}{2} = \frac{\dim(\mathrm{Pic}_{g,n}) - \sum d_i + 1}{2}$$

as claimed. Applying the change of variables to H, we obtain a series with only positive (half-integer) powers of β, and the lowest-order terms in β form the series $\sqrt{\beta}F$. ■

The transformation of the partial derivatives corresponding to (7) is obtained by computing the inverse matrix, which is given by

$$\frac{\partial}{\partial p_b} = \sum_{d=0}^{b-1} \beta^{(d+1)/2} \frac{(b-1)!}{(b-d-1)!} \frac{\partial}{\partial t_d}. \tag{9}$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial p_1} &= \beta^{1/2} \frac{\partial}{\partial t_0}, \\
\frac{\partial}{\partial p_2} &= \beta^{1/2} \frac{\partial}{\partial t_0} + \beta \frac{\partial}{\partial t_1}, \\
\frac{\partial}{\partial p_3} &= \beta^{1/2} \frac{\partial}{\partial t_0} + 2\beta \frac{\partial}{\partial t_1} + 2\beta^{3/2} \frac{\partial}{\partial t_2}.
\end{aligned}$$

PROPOSITION 2.2. *The change of variables* (7) *induces the following transformations*:

$$\begin{aligned}
L_p &\longrightarrow L_t = \sum_{d \geq 0} (d+1) t_d \frac{\partial}{\partial t_d} + \frac{1}{\sqrt{\beta}} \sum_{d \geq 1} t_d \frac{\partial}{\partial t_{d-1}}; \\
L_p^2 H_{\mathrm{unst}} &\longrightarrow \frac{1}{\sqrt{\beta}} t_0 (t_1 + 1) + t_0^2.
\end{aligned}$$

Both claims of the proposition are obtained by simple computations.

The concinnity of this result is striking. Indeed, both transforms could have contained arbitrarily large negative powers of β, but they happen to cancel out in both cases. Furthermore, $L_p^2 H_{\mathrm{unst}}$ is an infinite series, but after the change of variables it has become a polynomial with only three terms. Most important of all, the coefficients L_{-1} and L_0 of $\beta^{-1/2}$ and β^0 (respectively) in the operator L_t are precisely the string and dilation operators from equations (4) and (5). This leads to the following corollaries.

COROLLARY 2.3. *We have*

$$L_t F = F + 2\frac{\partial F}{\partial t_1} + \frac{1}{\sqrt{\beta}}\left(\frac{\partial F}{\partial t_0} - \frac{t_0^2}{2}\right),$$

$$L_t \frac{\partial F}{\partial t_0} = 2\frac{\partial^2 F}{\partial t_0 \partial t_1} + \frac{1}{\sqrt{\beta}}\left(\frac{\partial^2 F}{\partial t_0^2} - t_0\right).$$

COROLLARY 2.4. *The change of variables* (7) *induces the following transformations of generating series*:

$$L_p^2 H_{\mathrm{st}} \longrightarrow \frac{1}{\sqrt{\beta}}[U - t_0(t_1+1)] + O_\beta(1),$$

$$L_p^2 H \longrightarrow \frac{1}{\sqrt{\beta}} U + O_\beta(1).$$

Here $O_\beta(1)$ is a series containing only nonnegative powers of β.

Proof. The first result follows from Corollary 2.3; the second one is obtained after a (yet another!) cancellation of the term $t_0(t_1+1)$ with the contribution of $L_p^2 H_{\mathrm{unst}}$. ∎

3. Hirota Equations and KP Hierarchy

In this section we recall some necessary facts about the Hirota and the KP hierarchies and then use them to prove Theorem 1. We start with a brief introduction to these hierarchies (see e.g. [6] for more details).

The *semi-infinite wedge space* W is the vector space of formal (possibly infinite) linear combinations of infinite wedge products of the form

$$z^{k_1} \wedge z^{k_2} \wedge \cdots,$$

with $k_i \in \mathbb{Z}$, $k_i = i$, starting from some i. Consider a sequence $\varphi_1, \varphi_2, \ldots$ of Laurent series $\varphi_i \in \mathbb{C}[z^{-1}, z]$ such that $\varphi_i = z^i +$ (lower-order terms) starting from some i. Then $\varphi_1 \wedge \varphi_2 \wedge \cdots$ is an element of W. The elements that can be represented in this way are called *decomposable*. One way to check whether an element is decomposable is to verify that it satisfies the Plücker equations.

Now we will assign an element of W to any power series in variables $p_1, p_2, \ldots$. The series will turn out to be a solution of the Hirota hierarchy if and only if the corresponding element of W is decomposable.

To a Young diagram μ with d squares we assign the *Schur polynomial* s_μ in variables $p_1, p_2, \ldots$ defined by

$$s_\mu = \frac{1}{d!}\sum_{\sigma \in S_d} \chi_\mu(\sigma) p_\sigma.$$

Here S_d is the symmetric group, σ is a permutation, $\chi_\mu(\sigma)$ is the character of σ in the irreducible representation assigned to μ, and $p_\sigma = p_{l_1} \cdots p_{l_k}$, where $l_1, \ldots, l_k$ are the lengths of cycles of σ. The Schur polynomials s_μ with $\operatorname{area}(\mu) = d$ form a basis of the space of quasihomogeneous polynomials of weight d (the weight of p_i being equal to i).

Consider a power series τ in variables p_i. Decomposing τ in the basis of Schur polynomials, we can uniquely assign to it a (possibly infinite) linear combination of Young diagrams μ (of all areas). Now, in this linear combination we replace each Young diagram $\mu = (\mu_1, \mu_2, \dots)$ by the wedge product

$$z^{1-\mu_1} \wedge z^{2-\mu_2} \wedge \cdots,$$

where $(\mu_1, \mu_2, \dots)$ are the lengths of the columns of μ in decreasing order with an infinite number of zeroes added at the end. We have thus obtained an element $w_\tau \in W$.

The bilinear Plücker equations on the coordinates of w_τ happen to combine into bilinear differential equations on τ, called the Hirota equations. Thus, as we said, w_τ is decomposable if and only if τ is a solution of the Hirota equations. Let us define these equations precisely.

Consider two partitions λ and μ of an integer d. Denote by $\chi_\mu(\lambda)$ the character of any permutation with cycle type λ in the irreducible representation assigned to μ. Denote by $|\mathrm{Aut}(\lambda)|$ the number of permutations of the elements of λ that preserve their values. For instance, $|\mathrm{Aut}(7,6,6,4,1,1,1,1,1)| = 2! \cdot 5!$.

Let $d_i = \partial/\partial p_i$. Let D_μ be the differential operator

$$D_\mu = \sum_{\lambda, |\lambda|=d} \chi_\mu(\lambda) \frac{d_{\lambda_1} \cdots d_{\lambda_k}}{|\mathrm{Aut}(\lambda)|},$$

where k is the number of elements of λ. For instance:

$$D_{()} = 1, \quad D_{(1)} = d_1, \quad D_{(2)} = \tfrac{1}{2}d_1^2 + d_2, \quad D_{(1,1)} = \tfrac{1}{2}d_1^2 - d_2,$$
$$D_{(3)} = \tfrac{1}{6}d_1^3 + d_1 d_2 + d_3, \quad D_{(2,1)} = \tfrac{1}{3}d_1^3 - d_3, \quad D_{(1,1,1)} = \tfrac{1}{6}d_1^3 - d_1 d_2 + d_3.$$

Let τ be a formal power series in $p_1, p_2, \dots$. If the constant term of τ does not vanish, we can also consider its logarithm $F = \ln \tau$.

DEFINITION 3.1. The *Hirota hierarchy* is the following family of bilinear differential equations (see [1, Prop. 1]):

$$\mathrm{Hir}_{i,j}(\tau) = D_{()}\tau \cdot D_{(j,i)}\tau - D_{(i-1)}\tau \cdot D_{(j,1)}\tau + D_{(j)}\tau \cdot D_{(i-1,1)}\tau \qquad (10)$$

for $2 \le i \le j$.

Substituting $\tau = e^F$ and dividing by τ^2, we obtain a family of equations on F. It is called the *Kadomtsev–Petviashvili* (or KP) hierarchy:

$$\mathrm{KP}_{i,j}(F) = \frac{\mathrm{Hir}_{i,j}(e^F)}{e^{2F}}. \qquad (11)$$

Finally, by leaving only the linear terms in the KP hierarchy we obtain its *dispersionless limit*:

$$\mathrm{DKP}_{i,j}(F) = \text{linear part of } \mathrm{KP}_{i,j}(F). \qquad (12)$$

EXAMPLE 3.2. Denoting the derivative with respect to p_i by the index i, we have

$$\begin{aligned}
\mathrm{Hir}_{2,2} &= \tau\tau_{2,2} - \tau_2^2 - \tau\tau_{1,3} + \tau_1\tau_3 + \tfrac{1}{4}\tau_{1,1}^2 - \tfrac{1}{3}\tau_1\tau_{1,1,1} + \tfrac{1}{12}\tau\tau_{1,1,1,1},\\
\mathrm{KP}_{2,2} &= F_{2,2} - F_{1,3} + \tfrac{1}{2}F_{1,1}^2 + \tfrac{1}{12}F_{1,1,1,1},\\
\mathrm{DKP}_{2,2} &= F_{2,2} - F_{1,3} + \tfrac{1}{12}F_{1,1,1,1},\\
\mathrm{Hir}_{2,3} &= \tau\tau_{2,3} - \tau_2\tau_3 - \tau\tau_{1,4} + \tau_1\tau_4 + \tfrac{1}{2}\tau_{1,1}\tau_{1,2} - \tfrac{1}{2}\tau_1\tau_{1,1,2} - \tfrac{1}{6}\tau_{1,1,1}\tau_2\\
&\quad + \tfrac{1}{6}\tau\tau_{1,1,1,2} - \tfrac{1}{2}\tau_1\tau_{2,2} + \tfrac{1}{2}\tau\tau_{1,2,2} - \tau_{1,2}\tau_2 - \tfrac{1}{2}\tau\tau_{1,1,3} + \tfrac{1}{2}\tau_{1,1}\tau_3\\
&\quad + \tfrac{1}{24}\tau\tau_{1,1,1,1,1} - \tfrac{1}{8}\tau_1\tau_{1,1,1,1} + \tfrac{1}{12}\tau_{1,1}\tau_{1,1,1},\\
\mathrm{KP}_{2,3} &= F_{2,3} - F_{1,4} + F_{1,1}F_{1,2} + \tfrac{1}{6}F_{1,1,1,2} + \tfrac{1}{2}F_1F_{2,2} - \tfrac{1}{2}F_1F_{1,3} + \tfrac{1}{4}F_1F_{1,1}^2\\
&\quad + \tfrac{1}{24}F_1F_{1,1,1,1} + \tfrac{1}{2}F_{1,2,2} - \tfrac{1}{2}F_{1,1,3} + \tfrac{1}{2}F_{1,1}F_{1,1,1} + \tfrac{1}{24}F_{1,1,1,1,1},\\
\mathrm{DKP}_{2,3} &= F_{2,3} - F_{1,4} + \tfrac{1}{6}F_{1,1,1,2} + \tfrac{1}{2}F_{1,2,2} - \tfrac{1}{2}F_{1,1,3} + \tfrac{1}{24}F_{1,1,1,1,1}.
\end{aligned}$$

REMARK 3.3. Every Hirota equation can be simplified by adding to it some partial derivatives of lower equations. This, in turn, leads to simplified KP and DKP equations. For instance, we have

$$\begin{aligned}
\mathrm{Hir}_{2,3} - \frac{1}{2}\frac{\partial \mathrm{Hir}_{2,2}}{\partial p_1} &= \tau\tau_{2,3} - \tau_2\tau_3 - \tau\tau_{1,4} + \tau_1\tau_4 + \frac{1}{2}\tau_{1,1}\tau_{1,2}\\
&\quad - \frac{1}{2}\tau_1\tau_{1,1,2} - \frac{1}{6}\tau_{1,1,1}\tau_2 + \frac{1}{6}\tau\tau_{1,1,1,2},\\
\mathrm{KP}_{2,3} - \frac{1}{2}F_1\,\mathrm{KP}_{2,2} - \frac{1}{2}\frac{\partial \mathrm{KP}_{2,2}}{\partial p_1} &= F_{2,3} - F_{1,4} + F_{1,1}F_{1,2} + \frac{1}{6}F_{1,1,1,2},\\
\mathrm{DKP}_{2,3} - \frac{1}{2}\frac{\partial \mathrm{DKP}_{2,2}}{\partial p_1} &= F_{2,3} - F_{1,4} + \frac{1}{6}F_{1,1,1,2}.
\end{aligned}$$

Thus we obtain a simplified hierarchy that is, of course, equivalent to the initial one. Sometimes it is the equations of this simplified hierarchy that are called Hirota equations. However, for our purposes it is easier to use the equations as we defined them.

Proof of Theorem 1. We will actually prove that, *for any function $c = c(\beta)$, the series $c + L_p^2 H$ satisfies the Hirota hierarchy.* Let us show that the element of W assigned to $c + L_p^2 H$ is decomposable. Consider the following Laurent series in z:

$$\varphi_1 = cz + \sum_{n\geq 0} \beta^{n(n+1)/2} z^{-n}, \qquad \varphi_i = z^i - e^{(i-1)\beta} z^{i-1} \quad \text{for } i \geq 2.$$

The coefficients of these series are shown in the following matrix.

…	−4	−3	−2	−1	0	1	2	3	4	5	…
…	$e^{10\beta}$	$e^{6\beta}$	$e^{3\beta}$	e^{β}	1	c	0	0	0	0	…
…	0	0	0	0	0	$-e^{-\beta}$	1	0	0	0	…
…	0	0	0	0	0	0	$-e^{-2\beta}$	1	0	0	…
…	0	0	0	0	0	0	0	$-e^{-3\beta}$	1	0	…
…	0	0	0	0	0	0	0	0	$-e^{-4\beta}$	1	…

We claim that expanding the wedge product $\varphi_1 \wedge \varphi_2 \wedge \cdots$ and then replacing every Young diagram by the corresponding Schur polynomial will yield the series $c + L_p^2 H$. The proof goes as in [7].

We introduce the *cut-and-join operator*

$$A = \frac{1}{2} \sum_{i,j=1}^{\infty} \left[(i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right].$$

Then $L_p H$ satisfies the equation $\partial(L_p H)/\partial\beta = A(L_p H)$ (see [2]). Since the operator L_p commutes both with A and with $\partial/\partial\beta$, the series $L_p^2 H$ satisfies the same equation.

The Schur polynomials s_λ are eigenvectors of A. The eigenvalue corresponding to a Young diagram λ equals $f_\lambda = \frac{1}{2} \sum \lambda_i(\lambda_i - 2i + 1)$, where the λ_i are the column lengths. This allows one to reconstitute the whole series $L_p^2 H$ starting with its β-free terms $L_p^2 H|_{\beta=0}$: if

$$L_p^2 H|_{\beta=0} = \sum c_\lambda s_\lambda$$

then

$$L_p^2 H = \sum c_\lambda s_\lambda e^{f_\lambda \beta}.$$

It is apparent from the form of the preceding matrix that the coefficients of s_λ in the expansion are nonzero in only two cases: (i) for the empty diagram, where the coefficient equals c; and (ii) for the *hook Young diagrams* $\lambda = \mathrm{hook}(a, b)$ with column lengths

$$a+1, \underbrace{1, 1, \ldots, 1}_{b}.$$

For a Young diagram like that, the coefficient of $s_{\mathrm{hook}(a,b)}$ equals

$$(-1)^b e^{[a(a+1)/2 - b(b+1)/2]\beta}.$$

For the β-free terms we have

$$L_p^2 H|_{\beta=0} = \sum_{i \geq 1} p_i = \sum_{a,b \geq 0} (-1)^b s_{\mathrm{hook}(a,b)}$$

(the second equality is an exercise in representation theory). To this we add the remark that $a(a+1)/2 - b(b+1)/2$ is precisely the eigenvalue f_λ for $\lambda = \mathrm{hook}(a, b)$. It follows that the series corresponding to $\varphi_1 \wedge \varphi_2 \wedge \cdots$ equals $L_p^2 H$ as claimed. Thus we have proved that the series $c + L_p^2 H$ satisfies the Hirota hierarchy.

The claim about the DKP equations is a simple corollary. Indeed, it follows from Definition 3.1 that, for any series G,

$$\mathrm{Hir}_{i,j}(1 + G) - \mathrm{Hir}_{i,j}(G) = \mathrm{DKP}_{i,j}(G).$$

Hence, because both $L_p^2 H$ and $1 + L_p^2 H$ satisfy the Hirota equations, it follows immediately that $L_p^2 H$ satisfies the dispersionless KP equations. ∎

4. Hirota Equations and the Change of Variables

Now we shall study the effect of the change of variables (7) on the Hirota hierarchy before proving Theorem 2.

In Section 3 we assigned to each Young diagram μ an operator D_μ and then used these operators as building blocks to define the Hirota equations. It turns out that the change of variables (7) acts on D_μ by "biting off" the corners of μ. From this we will deduce that each Hirota equation becomes, after the change of variables, a linear combination of lower Hirota equations.

As in Theorem 2, we rescale the variable t_i by setting $t_i = i!\, T_{i+1}$. Then we have

$$\frac{\partial}{\partial p_i} = \binom{i-1}{0}\beta^{1/2}\frac{\partial}{\partial T_1} + \binom{i-1}{1}\beta\frac{\partial}{\partial T_2} + \cdots + \binom{i-1}{i-1}\beta^{i/2}\frac{\partial}{\partial T_i} \tag{13}$$

instead of equation (9).

Consider a linear differential operator D with constant coefficients in variables p_i. Denote $\partial/\partial p_i$ by d_i and consider d_i as a new set of variables. Introduce the differential operator

$$S = \sum_{i\geq 1} i d_i \frac{\partial}{\partial d_{i+1}}$$

in these variables. Then we have the following lemma.

LEMMA 4.1. *Assume that D is a quasihomogeneous polynomial in the variables d_i with total weight n. Applying the change of variables* (13) *to D viewed as a differential operator is then equivalent to applying the differential operator $\beta^{n/2}e^{S/\sqrt{\beta}}$ to D viewed as a polynomial in variables d_i.*

The proof is a simple check.

What we actually want is to apply the change of variables (13) to the operators D_μ defined in Section 3. Indeed, these operators are the building blocks of the Hirota equations (Definition 3.1). The answer is given in Proposition 4.3.

DEFINITION 4.2. A square of a Young diagram μ is called a *corner* if, when we erase it, we obtain another Young diagram. In other words, a corner is a square with coordinates (i, μ_i) such that either $\mu_{i+1} < \mu_i$ or μ_i is the last column of μ. If (i, μ_i) is a corner of μ then we will denote by $\mu - \square_i$ the diagram obtained by erasing this corner.

PROPOSITION 4.3. *We have*

$$SD_\mu = \sum_{(i,\mu_i)=\text{corner of }\mu} (\mu_i - i)\cdot D_{\mu-\square_i}.$$

Proof. Let μ be a Young diagram with d squares and let λ be a partition of $d-1$. Assume that λ has k parts. Then, for $1 \leq i \leq k$, denote by $\lambda + 1_i$ the partition of d obtained from λ by replacing λ_i by $\lambda_i + 1$.

Writing down explicitly the action of S on D_μ, one finds that the assertion of the proposition is equivalent to the following identity:

$$\sum_{(i,\mu_i)=\text{corner of }\mu} (\mu_i - i) \cdot \chi_{\mu-\square_i}(\lambda) = \sum_{i=1}^{k} \lambda_i \cdot \chi_\mu(\lambda + 1_i). \tag{14}$$

In order to prove this identity we need a short digression into the representation theory of the symmetric group as presented in [10].

Consider the subgroup $S_{d-1} \subset S_d$ consisting of the permutations that fix the last element d. The irreducible representation of S_d assigned to μ is then also a representation of S_{d-1}, although not necessarily irreducible. It turns out that this representation is isomorphic to

$$\bigoplus_{\text{corners of }\mu} \mu - \square_i,$$

where, by abuse of notation, $\mu - \square_i$ stands for the irreducible representation of S_{d-1} assigned to this Young diagram.

Further, consider the following element of the group algebra $\mathbb{C}S_d$:

$$X = (1,d) + (2,d) + \cdots + (d-1,d).$$

This element (the sum of all transpositions involving d) is called the first Jucys–Murphy–Young element, and it obviously commutes with the subgroup S_{d-1}. Hence its eigenspaces in the representation μ coincide with the irreducible subrepresentations of S_{d-1}; that is, they are also in one-to-one correspondence with the corners of μ. The eigenvalue corresponding to the corner (i,μ_i) equals $\mu_i - i$ (see [10]).

Using this information, let us choose a permutation $\sigma \in S_{d-1}$ with cycle type λ and compute in two different ways the character

$$\chi_\mu(\sigma \cdot X),$$

where $\sigma \cdot X \in \mathbb{C}S_d$.

First way. Both σ and X leave invariant the irreducible subrepresentations of S_{d-1}. For X, such a subrepresentation is an eigenspace with eigenvalue $\mu_i - i$; the character of σ in the same subrepresentation equals $\chi_{\mu-\square_i}(\lambda)$. We obtain the left-hand side of equation (14).

Second way. Let us see what happens when we multiply σ by X. Each transposition in X increases the length of precisely one cycle of σ by 1. This is equivalent to increasing one of the λ_i by 1. Moreover, if the ith cycle of σ has length λ_i, then it will be touched by a transposition from X exactly λ_i times. Thus we obtain the right-hand side of (14).

This completes the proof. ■

PROPOSITION 4.4. *The change of variables* (13) *transforms the Hirota equation* $\mathrm{Hir}_{i,j}$ *into an equation of the form*

$$\sum_{\substack{2\le i'\le i,\, 2\le j'\le j\\ i'\le j'}} c_{i',j'} \beta^{(i'+j')/2}\, \mathrm{Hir}_{i',j'}$$

for some rational constants $c_{i',j'}$. *The constant* $c_{i,j}$ *of the leading term equals* 1.

Proof. By Definition 3.1, the equation $\mathrm{Hir}_{i,j}$ has the form

$$\mathrm{Hir}_{i,j}(\tau) = D_{()}\tau \cdot D_{(j,i)}\tau - D_{(i-1)}\tau \cdot D_{(j,1)}\tau + D_{(j)}\tau \cdot D_{(i-1,1)}\tau.$$

According to Lemma 4.1, applying the change of variables to the equation is the same as applying to each D_μ in this expression the operator

$$\beta^{(i+j)/2} e^{S/\sqrt{\beta}}.$$

To simplify the computations, consider the flow e^{tS} applied to $\mathrm{Hir}_{i,j}$. We will compute the derivative of this flow with respect to t. If $\mathcal{D}$ is the vector space of all polynomials in variables d_i, then $\mathrm{Hir}_{i,j}$ lies in $\mathcal{D} \otimes \mathcal{D}$. The flow e^{tS} acts as $e^{tS} \otimes e^{tS}$, while its derivative with respect to t is $1 \otimes S + S \otimes 1$.

We will prove that $1 \otimes S + S \otimes 1$ applied to $\mathrm{Hir}_{i,j}$ is a linear combination of lower Hirota equations ($i' < i$, $j' < j$). Since this is true for all i, j, when we integrate the flow we see that $\mathrm{Hir}_{i,j}$ will have changed by a linear combination of lower Hirota equations.

It remains to apply $1 \otimes S + S \otimes 1$ to $\mathrm{Hir}_{i,j}$. For this we use Proposition 4.3. If $i < j$ we obtain

$$\begin{aligned} &D_{()} \cdot [(i-2)D_{(j,i-1)} + (j-1)D_{(j-1,i)}] + 0 \cdot D_{(j,i)} - D_{(i-1)} \cdot (j-1)D_{(j-1,1)} \\ &\quad - (i-2)D_{(i-2)} \cdot D_{(j,1)} + D_{(j)} \cdot (i-2)D_{(i-2,1)} + (j-1)D_{(j-1)} \cdot D_{(i-1,1)} \\ &\qquad = (i-2)\,\mathrm{Hir}_{i-1,j} + (j-1)\,\mathrm{Hir}_{i,j-1}. \end{aligned}$$

If $i = j$ we obtain

$$\begin{aligned} &D_{()} \cdot (i-2)D_{(i,i-1)} + 0 \cdot D_{(i,i)} - D_{(i-1)} \cdot (i-1)D_{(i-1,1)} \\ &\quad - (i-2)D_{(i-2)} \cdot D_{(i,1)} + D_{(i)}(i-2) \cdot D_{(i-2,1)} + (i-1)D_{(i-1)} \cdot D_{(i-1,1)} \\ &\qquad = (i-2)\,\mathrm{Hir}_{i-1,i}. \end{aligned}$$

This completes the proof. ■

REMARK 4.5. The family of equations given in Proposition 4.4 is equivalent to the Hirota hierarchy. Indeed, the equations of the Hirota hierarchy can be obtained from these equations by linear combinations and vice versa.

Proof of Theorem 2. The series $c + L_p^2 H$ satisfies the Hirota equations by Theorem 1. Therefore, by Proposition 4.4 and Remark 4.5, the series obtained from it under the change of variables (7) also satisfies the Hirota hierarchy. According to Corollary 2.4, this new series has the form

$$c + \frac{1}{\sqrt{\beta}} U + O_\beta(1).$$

Taking $c = c'/\sqrt{\beta}$ and considering the lowest-order terms in β, we obtain that $c' + U$ satisfies the Hirota hierarchy for any constant c'. It follows that U satisfies the dispersionless limit of the KP hierarchy. ■

Appendix: On Hodge Integrals

In this section we use the change of variables suggested in [7] to study Hodge integrals over the moduli spaces of curves. We consider the integrals involving a unique λ-class and arbitrary powers of ψ-classes.

A.1. Hurwitz Numbers and Hodge Integrals

Here we study intersection theory on moduli spaces rather than on Picard varieties. We follow the same path as in Sections 1 and 2 but with different intersection numbers and Hurwitz numbers. Our aim is to extend the results of [7].

Instead of (1), we define the brackets

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{(k)} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_k \tag{15}$$

for $k + \sum d_i = 3g - 3 + n$ (otherwise the bracket vanishes).

Instead of (2), we use the generating series

$$F^{(k)}(t_0, t_1, \ldots) = \sum_n \frac{1}{n!} \sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{(k)} t_{d_1} \cdots t_{d_n}. \tag{16}$$

We can also regroup these series into a unique series

$$\boldsymbol{F}(z; t_0, t_1, \ldots) = \sum_{k \geq 0} (-1)^k z^k F^{(k)}.$$

Instead of the Hurwitz numbers of Definition 1.1 we now use different Hurwitz numbers. Fix n positive integers $b_1, \ldots, b_n$, and let $d = \sum b_i$ be their sum.

Definition A.6. The number of degree-d ramified coverings of the sphere by a genus-g surface possessing n numbered preimages of ∞ with multiplicities $b_1, \ldots, b_n$ and $d + n + 2g - 2$ fixed simple branch points is called a *Hurwitz number* and is denoted by $h_{g; b_1, \ldots, b_n}$.

We introduce the following generating series for Hurwitz numbers:

$$H(\beta, p_1, p_2, \ldots) = \sum_{g,n} \frac{1}{n!} \frac{\beta^{d+n+2g-2}}{(d+n+2g-2)!} \sum_{b_1, \ldots, b_n} h_{g; b_1, \ldots, b_n} p_{b_1} \cdots p_{b_n}.$$

This series is divided in two parts. The unstable part, corresponding to $g = 0, n = 1, 2$, equals

$$H_{\text{unst}} = \sum_{b \geq 1} \beta^{b-1} \frac{b^{b-2}}{b!} p_b + \frac{1}{2} \sum_{b_1, b_2 \geq 1} \beta^{b_1+b_2} \frac{b_1^{b_1} b_2^{b_2}}{(b_1 + b_2) b_1! \, b_2!} p_{b_1} p_{b_2};$$

the stable part equals $H_{\text{st}} = H - H_{\text{unst}}$.

Finally, instead of Conjecture 1.2 we use the so-called ELSV formula proved in [2].

THEOREM 3 (ELSV formula). *We have*

$$h_{g;b_1,\dots,b_n} = (d+n+2g-2)!\prod_{i=1}^{n}\frac{b_i^{b_i}}{b_i!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{1-\lambda_1+\lambda_2-\cdots\pm\lambda_g}{(1-b_1\psi_1)\cdots(1-b_n\psi_n)}.$$

As before, it turns out that the series $\boldsymbol{F}$ and H are related via a change of variables based on equation (8). However, the change of variables differs from (7) owing to (i) the factors $b_i^{b_i}/b_i!$ in the ELSV formula and (ii) a different relation between the number of simple ramification points and the dimension of the Picard/moduli space. Namely, following [7], we let

$$p_b = \sum_{d=b-1}^{\infty}\frac{(-1)^{d-b+1}}{(d-b+1)!\,b^{b-1}}\beta^{-b-(2d+1)/3}t_d. \tag{17}$$

Thus

$$\begin{aligned}
p_1 &= \beta^{-4/3}t_0 - \beta^{-6/3}t_1 + \tfrac{1}{2}\beta^{-8/3}t_2 - \cdots,\\
p_2 &= \tfrac{1}{2}\beta^{-9/3}t_1 - \tfrac{1}{2}\beta^{-11/3}t_2 + \cdots,\\
p_3 &= \tfrac{1}{9}\beta^{-14/3}t_2 - \cdots.
\end{aligned}$$

This change of variables transforms H into a series in variables $t_0, t_1, \dots,$ and $\beta^{2/3}$.

We will also need a more detailed version of equation (8):

$$\sum_{b=1}^{d+1}\frac{(-1)^{d-b+1}}{(d-b+1)!\,(b-1)!}\cdot\frac{1}{1-b\psi} = \psi^d + \sum_{k=1}^{\infty}a_{d,d+k}\psi^{d+k},$$

where $a_{d,d+k}$ are some rational constants that actually happen to be integers. For instance,

$$\begin{aligned}
\frac{1}{1-\psi} &= 1+\psi+\psi^2+\cdots,\\
-\frac{1}{1-\psi}+\frac{1}{1-2\psi} &= \psi+3\psi^2+7\psi^3+\cdots,\\
\frac{1/2}{1-\psi}-\frac{1}{1-2\psi}+\frac{1/2}{1-3\psi} &= \psi^2+6\psi^3+25\psi^4+\cdots.
\end{aligned}$$

Using these constants, we introduce the following differential operators:

$$\begin{aligned}
L_1 &= \sum_{n=0}^{\infty}a_{n,n+1}t_{n+1}\frac{\partial}{\partial t_n},\\
L_2 &= \sum_{n=0}^{\infty}a_{n,n+2}t_{n+2}\frac{\partial}{\partial t_n} + \frac{1}{2!}\sum_{n_1,n_2=0}^{\infty}a_{n_1,n_1+1}a_{n_2,n_2+1}t_{n_1+1}t_{n_2+1}\frac{\partial^2}{\partial t_{n_1}\partial t_{n_2}},\\
L_3 &= \sum_{n=0}^{\infty}a_{n,n+3}t_{n+3}\frac{\partial}{\partial t_n} + \frac{1}{2!}\sum_{n_1,n_2=0}^{\infty}a_{n_1,n_1+2}a_{n_2,n_2+1}t_{n_1+2}t_{n_2+1}\frac{\partial^2}{\partial t_{n_1}\partial t_{n_2}}\\
&\quad+\frac{1}{2!}\sum_{n_1,n_2=0}^{\infty}a_{n_1,n_1+1}a_{n_2,n_2+2}t_{n_1+1}t_{n_2+2}\frac{\partial^2}{\partial t_{n_1}\partial t_{n_2}}\\
&\quad+\frac{1}{3!}\sum_{n_1,n_2,n_3=0}^{\infty}a_{n_1,n_1+1}a_{n_2,n_2+1}a_{n_3,n_3+1}t_{n_1+1}t_{n_2+1}t_{n_3+1}\frac{\partial^3}{\partial t_{n_1}\partial t_{n_2}\partial t_{n_3}},
\end{aligned}$$

and so on. We can also regroup these operators in a unique operator

$$\boldsymbol{L} = 1 + zL_1 + z^2L_2 + \cdots .$$

These operators and the change of variables (17) were designed to make the following proposition work.

PROPOSITION A.7. *Performing the change of variables* (17) *on the series* H_{st} *and replacing* $\beta^{2/3}$ *by* z, *we obtain the series* $\boldsymbol{LF}$.

Proof. Using the ELSV formula, one can check that the change of variables (17) transforms H into the series

$$\sum_{n,g}\frac{1}{n!}\sum_{d_1,\ldots,d_n}\int_{\overline{\mathcal{M}}_{g,n}}(1-\beta^{2/3}\lambda_1+\beta^{4/3}\lambda_2-\cdots)\prod_{i=1}^{n}(\psi_1^{d_i}+\beta^{2/3}a_{d_i,d_i+1}\psi_1^{d_i+1}+\cdots).$$

The proposition follows. ■

A.2. Hierarchies and Operators

PROPOSITION A.8. *We have that* $\boldsymbol{L} = e^{\boldsymbol{l}}$, *where*

$$\boldsymbol{l} = zl_1 + z^2l_2 + \cdots$$

is a first-order linear differential operator:

$$l_k = \alpha_{n,n+k}t_{n+k}\frac{\partial}{\partial t_n}.$$

Proof. Consider the operators l_k and $\boldsymbol{l}$ as just described with indeterminate coefficients $\alpha_{n,n+k}$. Consider the expansion of $e^{\boldsymbol{l}}$ and denote by $a_{n,n+k}$ the coefficient of $t_{n+k}\frac{\partial}{\partial t_n}$ in this expansion. We have

$$\begin{aligned}
a_{n,n+1} &= \alpha_{n,n+1},\\
a_{n,n+2} &= \alpha_{n,n+2} + \tfrac{1}{2}\alpha_{n,n+1}\alpha_{n+1,n+2},\\
a_{n,n+3} &= \alpha_{n,n+3} + \tfrac{1}{2}\alpha_{n,n+1}\alpha_{n+1,n+3} + \tfrac{1}{2}\alpha_{n,n+2}\alpha_{n+2,n+3}\\
&\quad + \tfrac{1}{6}\alpha_{n,n+1}\alpha_{n+1,n+2}\alpha_{n+2,n+3},
\end{aligned}$$

and so on. Note that these equalities allow us to determine the coefficients α unambiguously once the coefficients a are known.

Now consider the coefficient of a monomial

$$\prod_{i=1}^{p} t_{n_i+k_i}\frac{\partial}{\partial t_{n_i}}$$

in the same expansion of $e^{\boldsymbol{l}}$. It is equal to

$$\frac{|\mathrm{Aut}\{(n_1,k_1),\ldots,(n_p,k_p)\}|}{p!}\prod \alpha_{n_i,n_i+k_i} + \text{higher-order terms},$$

and we claim that this sum can be factorized as

$$\frac{|\mathrm{Aut}\{(n_1,k_1),\dots,(n_p,k_p)\}|}{p!}\prod a_{n_i,n_i+k_i}.$$

Indeed, suppose that we have already chosen a power of $\boldsymbol{l}$, say $\boldsymbol{l}^q$ with a term in each of the q factors that contribute to the coefficient of

$$\prod_{i=1}^{p-1} t_{n_i+k_i}\frac{\partial}{\partial t_{n_i}}.$$

Now we must choose some additional power $\boldsymbol{l}^r$ of $\boldsymbol{l}$ and a term in each of the r factors that will contribute to the coefficient of

$$t_{n_p+k_p}\frac{\partial}{\partial t_{n_p}} = t_{n+k}\frac{\partial}{\partial t_n}.$$

Moreover, we must choose the positions of the r new factors among the q that are already chosen. This can be done in

$$\binom{q+r}{q}$$

ways. (The operator $t_{n+k}\frac{\partial}{\partial t_n}$ acts by replacing t_n by t_{n+k}; hence the r terms in question divide the segment $[n, n+k]$ into r parts and should be ordered in a uniquely determined way.)

In the end we must divide the coefficient so obtained by $(q+r)!$, since we are looking at $e^{\boldsymbol{l}}$. Thus we obtain a coefficient of

$$\frac{1}{q!}\cdot\frac{1}{r!}$$

for any choice of r terms.

Now, if $q = 0$ then what we have finally obtained is precisely the expression for $a_{n,n+k}$. For a general q we will therefore obtain the same expression for $a_{n,n+k}$ divided by $q!$. Thus we have proved that $a_{n,n+k} = a_{n_p,n_p+k_p}$ can be factored out in the coefficient of

$$\prod_{i=1}^{p} t_{n_i+k_i}\frac{\partial}{\partial t_{n_i}}.$$

The same is true for a_{n_i,n_i+k_i} for all i. Hence the coefficient is the product of a_{n_i,n_i+k_i} as claimed.

In other words, we have shown that the coefficients of $\exp(\boldsymbol{l})$ coincide with those of $\boldsymbol{L}$. ■

CONJECTURE A.9. *The operators l_k have the form*

$$l_k = c_k\sum_{n\geq 0}\binom{n+k+1}{k+1} t_n\frac{\partial}{\partial t_{n+k}}$$

for some sequence of rational constants c_k.

The sequence c_k seems to be quite irregular and starts as follows:

$$1, -\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}, \frac{11}{12}, -\frac{3}{4}, -\frac{11}{6}, \frac{29}{4}, \frac{493}{12}, -\frac{2711}{6}, -\frac{12406}{15}, \frac{2636317}{60}, \dots.$$

We shall now establish a hierarchy of partial differential equations satisfied by $\boldsymbol{F}$. We use Propositions A.7 and A.8 together with the following fact.

THEOREM 4 (see [9] or [7]). *The series H satisfies the KP hierarchy in variables $p_1, p_2, \ldots$.*

This theorem is proved in the same manner as Theorem 2 once we observe that $\exp(H)$ satisfies the cut-and-join equation.

Applying the change of variables (17) to the KP equations yields the partial differential equations that are satisfied by $\boldsymbol{LF}$. These equations can, of course, be considered as equations on $\boldsymbol{F}$, since the coefficients of $\boldsymbol{L}$ are known. However, the equations thus obtained are infinite; that is, they have an infinite number of terms. Our goal is to prove that we can combine them in a way that leads to finite differential equations.

The derivatives $\partial/\partial p_b$ are expressed via $\partial/\partial t_d$ by computing the inverse of the matrix of the change of variables (17). We have

$$\frac{\partial}{\partial p_b} = b^{b-1} \sum_{d=0}^{b-1} \frac{\beta^{b+(2d+1)/3}}{(b-d-1)!} \frac{\partial}{\partial t_d}. \tag{18}$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial p_1} &= \beta^{4/3} \frac{\partial}{\partial t_0}, \\
\frac{\partial}{\partial p_2} &= 2\beta^{7/3} \frac{\partial}{\partial t_0} + 2\beta^{9/3} \frac{\partial}{\partial t_1}, \\
\frac{\partial}{\partial p_3} &= \frac{9}{2}\beta^{10/3} \frac{\partial}{\partial t_0} + 9\beta^{12/3} \frac{\partial}{\partial t_1} + 9\beta^{14/3} \frac{\partial}{\partial t_2}.
\end{aligned}$$

Now using Theorem 4 and Proposition A.7 allows us to transform the KP hierarchy into a system of equations on $\boldsymbol{F}$. We will illustrate the procedure on the example of $\mathrm{KP}_{2,2}$.

We know that $\mathrm{KP}_{i,j}(H) = 0$. For $i = j = 2$ this means

$$\frac{\partial^2 H}{\partial p_2^2} - \frac{\partial^2 H}{\partial p_1 \partial p_3} + \frac{1}{2}\left(\frac{\partial^2 H}{\partial p_1^2}\right)^2 + \frac{1}{12}\frac{\partial^4 H}{\partial p_1^4} = 0.$$

Using $H = H_{\mathrm{st}} + H_{\mathrm{unst}}$ and the explicit expression of H_{unst}, we transform $\mathrm{KP}_{i,j}$ into a (finite) equation $\widehat{\mathrm{KP}}_{i,j}$ on H_{st}. For instance, for $i = j = 2$ we obtain

$$\frac{\partial^2 H_{\mathrm{st}}}{\partial p_2^2} - \frac{\partial^2 H_{\mathrm{st}}}{\partial p_1 \partial p_3} + \frac{1}{2}\left(\frac{\partial^2 H_{\mathrm{st}}}{\partial p_1^2}\right)^2 + \frac{1}{12}\frac{\partial^4 H_{\mathrm{st}}}{\partial p_1^4} + \frac{1}{2}\beta^2 \frac{\partial^2 H_{\mathrm{st}}}{\partial p_1^2} = 0.$$

Applying the change of variables (18) and replacing $\beta^{2/3}$ by z, we transform this into an equation $\overline{\mathrm{KP}}_{i,j}$ on $\boldsymbol{LF}$. For $i = j = 2$ we have

$$-\frac{\partial^2(\boldsymbol{LF})}{\partial t_0 \partial t_1} + \frac{1}{2}\left(\frac{\partial^2(\boldsymbol{LF})}{\partial t_0^2}\right)^2 + \frac{1}{12}\frac{\partial^4(\boldsymbol{LF})}{\partial t_0^4} + z\left(4\frac{\partial^2(\boldsymbol{LF})}{\partial t_1^2} - 9\frac{\partial^2(\boldsymbol{LF})}{\partial t_0 \partial t_2}\right) = 0.$$

In principle, we could have stopped here. However, in this form the equation is useful only for studying the z-free part $F^{(0)}$ of $\boldsymbol{F}$, which was already done in [7].

Indeed, the operators L_i for $i \geq 1$ are composed of infinitely many terms. This means that if we develop this equation and take its coefficient of z^1, we will obtain an *infinite* equation on $F^{(0)}$ and $F^{(1)}$. Such an equation is quite useless if we want to compute $F^{(1)}$. Therefore we continue with the following theorem (recall that $\boldsymbol{L} = e^{\boldsymbol{l}}$).

Theorem 5. *Consider the expression*

$$e^{-\boldsymbol{l}}\, \overline{\mathrm{KP}}_{i,j}(e^{\boldsymbol{l}}\boldsymbol{F})$$

as a series in z. Then its coefficient of z^k is a finite differential equation on $F^{(0)}, \dots, F^{(k)}$.

Example A.10. The coefficient of z^1 in $e^{-\boldsymbol{l}}\, \overline{\mathrm{KP}}_{2,2}(e^{\boldsymbol{l}}\boldsymbol{F})$ gives the following equation:

$$-\frac{\partial^2 F^{(1)}}{\partial t_0 \partial t_1} + \frac{\partial^2 F^{(0)}}{\partial t_0^2}\frac{\partial^2 F^{(1)}}{\partial t_0^2} + \frac{1}{12}\frac{\partial^4 F^{(1)}}{\partial t_0^4}$$
$$+ 12\frac{\partial^2 F^{(0)}}{\partial t_0 \partial t_2} - 3\frac{\partial^2 F^{(0)}}{\partial t_1^2} - 2\frac{\partial^2 F^{(0)}}{\partial t_0^2}\frac{\partial^2 F^{(0)}}{\partial t_0 \partial t_1} - \frac{1}{3}\frac{\partial^4 F^{(0)}}{\partial t_0^3 \partial t_1} = 0.$$

Assuming that we know $F^{(0)}$, this equation—together with the string and dilation equations—allows us to compute all the coefficients of $F^{(1)}$ (i.e., all Hodge integrals involving λ_1).

Proof of Theorem 5. Let Q be a linear differential operator (in variables t_d) whose coefficients are polynomials in z. Then

$$e^{-\boldsymbol{l}} Q e^{\boldsymbol{l}} = Q + [Q, \boldsymbol{l}] + \frac{1}{2}[[Q, \boldsymbol{l}], \boldsymbol{l}] + \cdots$$

is a series in z whose coefficients are finite differential operators. We will denote this series by $\hat{Q}$. Now suppose we have several linear operators $Q_1, \dots, Q_r$ as before. Since $\boldsymbol{l}$ is a first-order operator, we obtain

$$e^{-\boldsymbol{l}} Q_1(e^{\boldsymbol{l}}\boldsymbol{F}) \cdots Q_r(e^{\boldsymbol{l}}\boldsymbol{F}) = \hat{Q}_1(\boldsymbol{F}) \cdots \hat{Q}_r(\boldsymbol{F}).$$

This is, once again, a series in z whose coefficients are finite differential equations on the $F^{(k)}$. The theorem now follows from the fact that every equation $\overline{\mathrm{KP}}_{i,j}$ is a finite linear combination of expressions of the form

$$Q_1(e^{\boldsymbol{l}}\boldsymbol{F}) \cdots Q_r(e^{\boldsymbol{l}}\boldsymbol{F}). \qquad \blacksquare$$

Thus every equation $\mathrm{KP}_{i,j}$ and every power of z gives us a finite differential equation on the functions $F^{(k)}$. We now describe some facts concerning these equations that we have observed but not proved.

For any $F^{(k)}$ and any $(d', d'') \neq (0, 0)$, by taking linear combinations of the equations in question we can obtain an equation of the form

$$\frac{\partial F^{(k)}}{\partial t_{d'}\partial t_{d''}} = \text{terms with more than two derivations.}$$

For homogeneity reasons the sum of indices in these derivatives will be smaller than $d'+d''$. Hence we use similar equations with smaller $d'+d''$ in order to simplify the right-hand part by substitutions. After a finite number of substitutions we will obtain an expression of $\partial F^{(k)}/\partial t_{d'}\partial t_{d''}$ exclusively via partial derivatives with respect to t_0. Moreover, these expressions can themselves be organized into equations on $\boldsymbol{F}$:

$$\begin{aligned}
\boldsymbol{F}_{0,1} &= \left(\frac{1}{2}\boldsymbol{F}_{0,0}^2 + \frac{1}{12}\boldsymbol{F}_{0,0,0,0}\right) - z\left(\frac{1}{24}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{720}\boldsymbol{F}_{0,0,0,0,0,0}\right)\\
&\quad + z^2\left(\frac{1}{720}\boldsymbol{F}_{0,0,0}\boldsymbol{F}_{0,0,0,0,0} + \frac{1}{360}\boldsymbol{F}_{0,0,0,0}^2 + \frac{1}{30240}\boldsymbol{F}_{0,0,0,0,0,0,0,0}\right) + \cdots,\\
\boldsymbol{F}_{0,2} &= \left(\frac{1}{6}\boldsymbol{F}_{0,0}^3 + \frac{1}{12}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0} + \frac{1}{24}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{240}\boldsymbol{F}_{0,0,0,0,0,0}\right)\\
&\quad - z\left(\frac{1}{24}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{720}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0,0,0} + \frac{7}{720}\boldsymbol{F}_{0,0,0}\boldsymbol{F}_{0,0,0,0,0}\right.\\
&\qquad \left. + \frac{1}{180}\boldsymbol{F}_{0,0,0,0}^2 + \frac{1}{7560}\boldsymbol{F}_{0,0,0,0,0,0,0,0}\right) + \cdots,\\
\boldsymbol{F}_{1,1} &= \left(\frac{1}{3}\boldsymbol{F}_{0,0}^3 + \frac{1}{6}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0} + \frac{1}{24}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{144}\boldsymbol{F}_{0,0,0,0,0,0}\right)\\
&\quad - z\left(\frac{1}{12}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{360}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0,0,0} + \frac{13}{720}\boldsymbol{F}_{0,0,0}\boldsymbol{F}_{0,0,0,0,0}\right.\\
&\qquad \left. + \frac{1}{120}\boldsymbol{F}_{0,0,0,0}^2 + \frac{1}{4320}\boldsymbol{F}_{0,0,0,0,0,0,0,0}\right) + \cdots,\\
\boldsymbol{F}_{0,3} &= \left(\frac{1}{24}\boldsymbol{F}_{0,0}^4 + \frac{1}{24}\boldsymbol{F}_{0,0}^2\boldsymbol{F}_{0,0,0,0} + \frac{1}{24}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{240}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0,0,0}\right.\\
&\qquad \left. + \frac{1}{120}\boldsymbol{F}_{0,0,0}\boldsymbol{F}_{0,0,0,0,0} + \frac{1}{160}\boldsymbol{F}_{0,0,0,0}^2 + \frac{1}{6720}\boldsymbol{F}_{0,0,0,0,0,0,0,0}\right) + \cdots,\\
\boldsymbol{F}_{1,2} &= \left(\frac{1}{8}\boldsymbol{F}_{0,0}^4 + \frac{1}{8}\boldsymbol{F}_{0,0}^2\boldsymbol{F}_{0,0,0,0} + \frac{1}{12}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0}^2 + \frac{1}{90}\boldsymbol{F}_{0,0}\boldsymbol{F}_{0,0,0,0,0,0}\right.\\
&\qquad \left. + \frac{1}{60}\boldsymbol{F}_{0,0,0}\boldsymbol{F}_{0,0,0,0,0} + \frac{23}{1440}\boldsymbol{F}_{0,0,0,0}^2 + \frac{1}{2880}\boldsymbol{F}_{0,0,0,0,0,0,0,0}\right) + \cdots.
\end{aligned}$$

It is not entirely unexpected that the free terms of these equations turn out to form the well-known Korteweg–de Vries (KdV) hierarchy on $F^{(0)}$.

The first equation listed (expressing $\boldsymbol{F}_{0,1}$), together with the string and dilation equations, is sufficient to determine the values of all Hodge integrals involving a single λ-class. This approach seems to be simpler than the method of [8] based on the study of double Hurwitz numbers.

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