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A Dirichlet Problem for the Complex Monge–Ampère Operator in $\mathcal{F}(f)$

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Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a hyperconvex domain: a connected open set that admits a negative plurisubharmonic exhaustion function. Throughout this paper it is always assumed that Ω is bounded. The class of plurisubharmonic functions defined on Ω will be denoted $\mathcal{PSH}(\Omega)$. In the theory of distributions, the smooth functions with compact support—the so-called test functions—play an important role. Because there exist no plurisubharmonic functions with compact support in Ω that are not identically zero, it is useful to introduce $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$. This class has a role similar to that of the class of test functions, $C_0^\infty(\Omega)$, since $C_0^\infty(\Omega) \subset \mathcal{E}_0 \cap C(\bar{\Omega}) - \mathcal{E}_0 \cap C(\bar{\Omega})$ [9, Lemma 3.1]. A bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial\Omega$ and $\int_\Omega (dd^c\varphi)^n < +\infty$, where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator. The maximum principle for plurisubharmonic functions implies that if $\varphi \in \mathcal{E}_0$ then $\varphi < 0$ or $\varphi = 0$. Bedford and Taylor proved in [4] that $(dd^c\cdot)^n$ is well-defined on $\mathcal{PSH}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. This implies that the definition of $\mathcal{E}_0$ is well-posed and that $(dd^c\cdot)^n$ is well-defined on $\mathcal{E}_0$.

Assume that u is a plurisubharmonic function defined on Ω and that $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, is a decreasing sequence that converges pointwise to u on Ω as j tends to $+\infty$. If there can be no misinterpretation, a sequence $[\cdot]_{j=1}^\infty$ will be denoted by $[\cdot]$. For fixed $p \geq 1$, consider the following assertions:

(1) $\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < +\infty$;
(2) $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$.

If the sequence $[\varphi_j]$ can be chosen such that (1) holds, then u is said to be in $\mathcal{E}_p$ $(= \mathcal{E}_p(\Omega))$; if (2) holds, then u is in $\mathcal{F}$ $(= \mathcal{F}(\Omega))$. Finally, if both (1) and (2) are satisfied then $u \in \mathcal{F}_p$ $(= \mathcal{F}_p(\Omega))$. In [9], Cegrell proved that the complex Monge–Ampère operator is well-defined on the subset $\mathcal{E}$ of nonpositive plurisubharmonic functions containing both $\mathcal{F}$ and $\mathcal{E}_p$ (see Section 1 or [9] for the definition of $\mathcal{E}$).

It is proved in Section 1 that, for $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$,

$$\limsup_{z\to\xi} u(z) = 0$$

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for every $\xi \in \partial\Omega$. This is a generalization of [8, Lemma 3.12]. Example 1.6 shows that there exists a function $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$ such that $\liminf_{z\to\xi} u(z) = -\infty$ for every $\xi \in \partial\Omega$.

The following construction will play a central role. Let Ω be a domain in $\mathbb{C}^n$, let f be a continuous real-valued function defined on $\partial\Omega$, and let μ be a nonnegative measure defined on Ω. The envelope $U(\mu, f)$ is then defined by

$$U(\mu, f)(z) = \sup\{v(z) : v \in B(\mu, f)\},$$

where

$$\begin{aligned} &B(\mu, f) \\ &\quad = \{w \in \mathcal{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) : (dd^c w)^n \geq \mu, \\ &\qquad\qquad\qquad \limsup_{z\to\xi} w(z) \leq f(\xi) \text{ for every } \xi \in \partial\Omega\}. \end{aligned}$$

In Section 2, $\mathcal{E}_p(f)$, $\mathcal{F}(f)$, and $\mathcal{E}(f)$ will be defined using the envelope $U(0, f)$ in a similar manner to how $\mathcal{E}_0(f)$ and $\mathcal{F}_p(f)$ were defined in [8]. From the boundary behavior of functions from the classes $\mathcal{E}_p$ and $\mathcal{F}$ as proved in Section 1, it follows that, if $u \in \mathcal{F}(f) \cup_{p\geq 1} \mathcal{E}_p(f)$, then $\limsup_{z\to\xi} u(z) = f(\xi)$ for every $\xi \in \partial\Omega$ (Proposition 2.2). The main goal of Section 2 is to prove that it is possible to define the complex Monge–Ampère operator on these new classes in an appropriate way.

Let $\Omega \subseteq \mathbb{C}^n$ $(n \geq 2)$ be a bounded hyperconvex domain, and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function such that $\lim_{z\to\xi} U(0, f)(z) = f(\xi)$ for every $\xi \in \partial\Omega$. Assuming that μ is a nonnegative measure on Ω with finite total mass and that μ vanishes on pluripolar sets, it will be proved in Theorem 3.4 that there exists a uniquely determined function $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \mu$ as measures defined on Ω. In [9], Cegrell solved this Dirichlet problem for $f = 0$. This paper ends with a comparison principle, which is proved by using methods from the proof of Theorem 3.4.

For an introduction to classical and pluripotential theory, the monographs *Pluripotential Theory* by Klimek [14] and *Classical Potential Theory* by Armitage and Gardiner [3] are recommended. For further information about these *Cegrell classes* see, for example, [11; 12; 13] and the references therein. This paper is an enhanced and revised version of a part of the author's Ph.D. thesis (see [1]).

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1. On the Boundary Behavior of Functions in $\mathcal{E}_p$ and $\mathcal{F}$

Let $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ be the class of plurisubharmonic functions φ defined on Ω such that, for each $z_0 \in \Omega$, there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to φ on ω, as $j \to +\infty$, and for which

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty.$$

Theorem 1.1 will be used extensively throughout this article.

THEOREM 1.1. *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$, $u \in \mathcal{K}$, and $v \in \mathcal{PSH}(\Omega)$ for $v \leq 0$. Then*

$$\max\{u, v\} \in \mathcal{K}.$$

Proof. See [8] and [9]. □

Lemma 3.12 in [8] states that if Ω is a bounded, strictly pseudoconvex domain and $u \in \mathcal{E}_1$, then

$$\limsup_{\substack{z\to\xi \\ z\in\Omega}} u(z) = 0 \tag{1.1}$$

for every $\xi \in \partial\Omega$. In this section it will be proved that this holds for any function $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$, where Ω is a bounded hyperconvex domain. Recall that a bounded hyperconvex domain Ω, viewed as a domain in $\mathbb{R}^{2n}$, is always regular with respect to the Dirichlet problem for the Laplace operator; therefore, (1.1) holds for any subharmonic function defined on Ω whose smallest harmonic majorant is the zero function. Theorem 1.4 shows that any function in $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$ has smallest harmonic majorant the zero function.

LEMMA 1.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and $h\colon \Omega \to (-\infty, 0]$ a harmonic function. Define $\Psi(z) = \sup\{w(z) : w \in \mathcal{PSH}(\Omega), w \leq h$ on $\Omega\}$. If $\Psi \in \mathcal{E}$, then $(dd^c\Psi)^n = 0$.*

Proof. Let B be an open ball such that $\bar{B} \subseteq \Omega$ and let $\varepsilon > 0$ be such that $B \subset \Omega_\varepsilon \subset \Omega$, where $\Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$. Let χ_ε be the standard regularization kernel and let $\Psi_\varepsilon = (u * \chi_\varepsilon)$, where $*$ denotes the convolution. Then $\Psi_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon)$ and $[\Psi_\varepsilon]$ is a decreasing sequence such that $\lim_{\varepsilon\to 0^+} \Psi_\varepsilon(z) = \Psi(z)$ for every $z \in \Omega$. Solving the Dirichlet problem with boundary values Ψ_ε yields a function $g_\varepsilon \in \mathcal{PSH}(B) \cap C(\bar{B})$ such that $g_\varepsilon = \Psi_\varepsilon$ on ∂B and

$$(dd^c g_\varepsilon)^n = 0 \tag{1.2}$$

on B (see e.g. [4]). Define a function H_ε on Ω_ε by

$$H_\varepsilon(z) = \begin{cases} g_\varepsilon(z) & \text{if } z \in B, \\ \Psi_\varepsilon(z) & \text{if } z \in (\Omega_\varepsilon \backslash B). \end{cases} \tag{1.3}$$

Then $H_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon)$ and $[H_\varepsilon]$ decrease as ε decreases to 0. Let $\varepsilon \to 0^+$. The limit function H of $[H_\varepsilon]$ exists and is plurisubharmonic on Ω or identically $-\infty$. It also follows that $\Psi \leq H_\varepsilon$ on Ω_ε, which yields that

$$\Psi(z) \leq H(z) \tag{1.4}$$

for every $z \in \Omega$. The definition of Ψ implies that $\Psi \leq h$ on Ω and hence $H = \Psi \leq h$ on $\Omega \backslash B$. Therefore, $H \leq h$ on Ω because H is, in particular, subharmonic. Thus,

$$H(z) \leq \Psi(z) \tag{1.5}$$

for every $z \in \Omega$. Inequalities (1.4) and (1.5) imply that $\Psi = H$ on Ω. This, together with (1.2), (1.3), and the assumption that $\Psi \in \mathcal{E}$, yields $(dd^c\Psi)^n = (dd^cH)^n = 0$ on B. Since B was arbitrary, the lemma is proved. □

Example 1.3 was kindly suggested to the author by Alexander Rashkovskii [16]; it shows that the set $\{w(z) : w \in \mathcal{PSH}(\Omega),\ w \leq h \text{ on } \Omega\}$ might be empty.

EXAMPLE 1.3. Let $B \subseteq \mathbb{C}^2$ be the unit ball, and let $p = (1,0) \in \mathbb{C}^2$. For $z \in B$, define

$$h(z) = \frac{|z|^2 - 1}{|z-p|^4}.$$

Then $-h$ is the Poisson kernel for B. Therefore, h is harmonic and $h \leq 0$. It can be proved that there does not exist a function $\varphi \in \mathcal{PSH}(B)$ such that $\varphi \leq h$, which implies that $\{w(z) : w \in \mathcal{PSH}(B),\ w \leq h \text{ on } B\} = \emptyset$.

THEOREM 1.4. *If $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$, then the smallest harmonic majorant of u is identically zero on Ω.*

Proof. Assume that $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$. The zero function is harmonic and thus is a harmonic majorant of u; hence there exists a smallest harmonic majorant of u (see e.g. [3, Thm. 3.6.3]). Assume that there exists a smaller harmonic majorant of u; in other words, assume there exists a harmonic function h defined on Ω such that

$$u \leq h \leq 0 \tag{1.6}$$

and $h(z) \neq 0$ for at least one z in Ω. Let the function Ψ be defined as in Lemma 1.2. Then the definition of Ψ and (1.6) imply that $u \leq \Psi \leq 0$, so $\Psi \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$ by Theorem 1.1. Moreover, $(dd^c\Psi)^n = 0$ by Lemma 1.2. If $\Psi \in \mathcal{F}$ then $\Psi = 0$ by the uniqueness part of [9, Lemma 5.14], and if $\Psi \in \bigcup_{p\geq 1} \mathcal{E}_p$ then $\Psi = 0$ by the uniqueness part of [8, Thm. 6.2]. By construction it holds that $\Psi \leq h \leq 0$, which implies that $h = 0$. This contradicts the assumption that there exists a $z \in \Omega$ such that $h(z) \neq 0$. Thus the smallest harmonic majorant of u is equal to zero on Ω. □

COROLLARY 1.5. *Suppose that $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$. Then*

$$\limsup_{\substack{z\to\xi \\ z\in\Omega}} u(z) = 0$$

for every $\xi \in \partial\Omega$.

REMARK. Theorem 1.4 and Corollary 1.5 are not generally valid for functions from $\mathcal{E}$. Consider, for example, the function that is identically -1.

Corollary 1.5 implies that $\limsup_{z\to\xi} \tilde{u}(z) = 0$ for every $\xi \in \partial\Omega$, and Example 1.6 shows that there exists a function $\tilde{u} \in \mathcal{F}_p$ such that $\liminf_{z\to\xi} \tilde{u}(z) = -\infty$ for every $\xi \in \partial\Omega$.

EXAMPLE 1.6. Let $[z_j]$, $z_j \in \Omega$, be a sequence such that every point on $\partial\Omega$ is a limit point to $[z_j]$. The set $\{z_j\}$ is pluripolar because it is countable. Theorem 5.8 in [9] implies that there exists a function $u \in \mathcal{F}_1$ such that $\{z_j\} \subseteq \{u = -\infty\}$. For each $p \geq 1$, let the function $\tilde{u}$ be defined by $\tilde{u} = \max\{u, -(-u)^{1/p}\}$. Then Theorem 1.1 implies that $\tilde{u} \in \mathcal{F}_1$, since $-(-u)^{1/p} \in \mathcal{PSH}(\Omega)$ and $u \leq \tilde{u} \leq 0$. It therefore follows that

$$\int_\Omega (-\tilde{u})^p (dd^c\tilde{u})^n < +\infty,$$

since $u \in \mathcal{F}_1$. Theorem 5.6 in [8] yields that $\tilde{u} \in \mathcal{F}_p$. The constructions of $[z_j]$ and $\tilde{u}$ imply that $\liminf_{z\to\xi} u(z) = -\infty$ and $\liminf_{z\to\xi} -(-u(z))^{1/p} = -\infty$ for every $\xi \in \partial\Omega$. Thus

$$\liminf_{\substack{z\to\xi \\ z\in\Omega}} \tilde{u}(z) = -\infty$$

for every $\xi \in \partial\Omega$. Corollary 1.5 then concludes this example.

2. Definition of the Complex Monge–Ampère Operator on $\mathcal{E}(f)$

The classes $\mathcal{E}_0(f)$ and $\mathcal{F}_p(f)$ were first defined in [8]. Here those definitions will be recalled, and $\mathcal{E}_p(f)$, $\mathcal{F}(f)$, and $\mathcal{E}(f)$ will be defined in a similar manner. If $\mathcal{K}(f)$ is one of these classes, where $f = 0$, it follows immediately that $\mathcal{K}(0) = \mathcal{K}$, where $\mathcal{K}$ is the corresponding class defined in the Introduction and Section 1. Hence the classes defined in this section are generalizations of those in Section 1. Proposition 2.2 is a direct consequence of the definition of these classes and Corollary 1.5. Therefore, functions from $\mathcal{E}_p(f)$ and $\mathcal{F}(f)$ essentially have their boundary values given by the function f. The main goal of this section is to prove that it is possible to define the complex Monge–Ampère operator in an appropriate way on $\mathcal{E}(f)$. The class $\mathcal{E}(f)$ contains $\mathcal{E}_0(f)$, $\mathcal{F}_p(f)$, $\mathcal{E}_p(f)$, and $\mathcal{F}(f)$; the complex Monge–Ampère operator is well-defined on these classes as well.

DEFINITION 2.1. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$, let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain, and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function such that

$$\lim_{z\to\xi} U(0, f)(z) = f(\xi) \quad \text{for every } \xi \in \partial\Omega.$$

A plurisubharmonic function u defined on Ω belongs to $\mathcal{K}(f)$ $(= \mathcal{K}(\Omega, f))$ if there exists a function $\varphi \in \mathcal{K}$ such that

$$U(0, f) \geq u \geq \varphi + U(0, f).$$

REMARKS. (1) Under the assumptions in Definition 2.1, the Perron–Bremermann envelope $U(0, f)$ belongs to $\mathcal{E}_0(f) \cap C(\bar{\Omega})$. Moreover, $\mathcal{E}_0(f) \subseteq L^\infty(\Omega)$ and $(dd^c U(0, f))^n = 0$.

(2) If $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$ then $\mathcal{K}(0) = \mathcal{K}$. The class $\mathcal{K}(f)$ is a convex set, but in general it is not a convex cone.

(3) Let p and f be fixed; then $\mathcal{E}_0(f) \subseteq \mathcal{F}_p(f) \subseteq \mathcal{F}(f) \subseteq \mathcal{E}(f)$ and $\mathcal{E}_0(f) \subseteq \mathcal{F}_p(f) \subseteq \mathcal{E}_p(f) \subseteq \mathcal{E}(f)$.

(4) There exists a function $u \in \mathcal{E}_0(f)$ such that

$$\int_\Omega (dd^c u)^n = +\infty$$

(see [2; 13]).

In the rest of this section, let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and let $f\colon \partial\Omega \to \mathbb{R}$ be a continuous function such that $\lim_{z\to\xi} U(0, f)(z) = f(\xi)$ for every $\xi \in \partial\Omega$.

PROPOSITION 2.2. *Let $u \in \mathcal{F}(f) \cup_{p\geq 1} \mathcal{E}_p(f)$. Then*

$$\limsup_{\substack{z\to\xi \\ z\in\Omega}} u(z) = f(\xi) \tag{2.1}$$

for every $\xi \in \partial\Omega$. If $u \in \mathcal{E}_0(f)$, then

$$\lim_{z\to\xi} u(z) = f(\xi) \tag{2.2}$$

for every $\xi \in \partial\Omega$.

Proof. Assume that $u \in \mathcal{F}(f) \cup_{p\geq 1} \mathcal{E}_p(f)$, that is, $u \in \mathcal{PSH}(\Omega)$, and that there exists a function $\varphi \in \mathcal{E}_p$ (or in $\mathcal{F}$) such that $U(0, f) \geq u \geq \varphi + U(0, f)$. Then

$$\varphi \leq u - U(0, f) \leq 0. \tag{2.3}$$

It follows from Corollary 1.5 that

$$\limsup_{\substack{z\to\xi \\ z\in\Omega}} \varphi(z) = 0 \tag{2.4}$$

for every $\xi \in \partial\Omega$. Hence (2.3) and (2.4) yield that (2.1) holds. Let $u \in \mathcal{E}_0(f)$. Using the definition of $\mathcal{E}_0$ instead of Corollary 1.5 in the preceding method yields the desired result—that is, (2.2) holds. □

PROPOSITION 2.3. (1) *If $f \leq 0$ and $u \in \mathcal{E}(f)$, then $u \in \mathcal{E}$.*

(2) *If $v \in \mathcal{E}(f)$, then there exists a constant $c_1 \leq 0$ such that $(v + c_1) \in \mathcal{E}$.*

(3) *If $w \in \mathcal{E}$, then there exists a constant $c_2 \leq 0$ such that $(w + c_2) \in \mathcal{E}(f)$.*

Proof. (1) This follows from the definition of $\mathcal{E}(f)$ and Theorem 1.1.

(2) This is a consequence of (1).

(3) Let $w \in \mathcal{E}$ and consider the function $w - |\max_{\xi\in\partial\Omega} f(\xi)|$. This function belongs to $\mathcal{E}(f)$, which completes the proof of this proposition. □

It is possible to define the complex Monge–Ampère operator on $\mathcal{E}(f)$ by using property 2 in Proposition 2.3. Yet by applying the method used here, the information in Theorem 2.4 and Theorem 2.5 is gained.

Theorem 7.2 in [8] proves that $(dd^c\cdot)^n$ is well-defined on $\mathcal{F}_p(f)$. The same method will be used here to prove that this operator is well-defined on $\mathcal{E}(f)$. This implies, in particular, that the complex Monge–Ampère operator is well-defined on $\mathcal{F}_p(f)$, $\mathcal{E}_p(f)$, and $\mathcal{F}(f)$.

THEOREM 2.4. *Let $u \in \mathcal{E}(f)$. Then there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(f)$, that converges pointwise to u as j tends to $+\infty$.*

Proof. Let $u \in \mathcal{E}(f)$, that is, $u \in \mathcal{PSH}(\Omega)$, and let there exist a function $\varphi \in \mathcal{E}$ such that

$$U(0, f) \geq u \geq \varphi + U(0, f). \tag{2.5}$$

It follows from [9, Thm. 2.1] that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, such that φ_j converges pointwise to φ as $j \to +\infty$. Let the sequence $[u_j]$, $j \in \mathbb{N}$, be defined by $u_j = \max\{u, \varphi_j + U(0, f)\}$. It is a decreasing sequence of plurisubharmonic functions, since $[\varphi_j]$ is decreasing, and it converges pointwise to u as $j \to +\infty$. The definition of u_j implies that

$$u_j \geq \varphi_j + U(0, f), \tag{2.6}$$

and by (2.5) it follows that $U(0, f) \geq u_j$, since $(\varphi_j + U(0, f)) \in B(0, f)$. Therefore, inequality (2.6) yields that $U(0, f) \geq u_j \geq \varphi_j + U(0, f)$ for every $j \in \mathbb{N}$. Hence $[u_j]$, $u_j \in \mathcal{E}_0(f)$, is a decreasing sequence that converges pointwise to u as $j \to +\infty$. □

THEOREM 2.5. *Let $[u_j]$, $u_j \in \mathcal{E}_0(f)$, be a decreasing sequence that converges pointwise to $u \in \mathcal{E}(f)$ as j tends to $+\infty$. Then $(dd^c u_j)^n$ is weak*-convergent and the limit measure does not depend on the particular sequence $[u_j]$.*

Proof. Assume that $[u_j]$, $u_j \in \mathcal{E}_0(f)$, is a decreasing sequence that converges pointwise to $u \in \mathcal{E}(f)$ as $j \to +\infty$. Let $K \subseteq \Omega$ ($K \neq \emptyset$) be a compact set. By Definition 2.1, $u \in \mathcal{PSH}(\Omega)$ and there exists a function $\varphi \in \mathcal{E}$ such that

$$U(0, f) \geq u \geq \varphi + U(0, f). \tag{2.7}$$

There is no loss of generality in assuming that $\varphi < 0$, because if $\varphi = 0$ then (2.7) implies that $u = U(0, f) \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, and $u_j = U(0, f)$ for every $j \in \mathbb{N}$ by Definition 2.1. The function $U(0, f)$ is continuous on $\bar{\Omega}$ and $\varphi < 0$; hence there exists a constant $c \geq 0$ such that $U(0, f) - \alpha > c\varphi$ on K, where α is the constant defined by

$$\alpha = \begin{cases} 0 & \text{if } \max_{\xi \in \partial\Omega} f(\xi) \leq 0, \\ \max_{\xi \in \partial\Omega} f(\xi) & \text{otherwise.} \end{cases}$$

This and (2.7) imply that

$$u - \alpha \geq (1 + c)\varphi \tag{2.8}$$

in a neighborhood of K. Theorem 2.1 in [9] yields that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathcal{E}_0$, that converges pointwise to φ as $j \to +\infty$. Let

$$v_j = \max\{u_j - \alpha, (1 + c)\varphi_j\}.$$

The assumption $u_j \in \mathcal{E}_0(f)$ implies that $u_j \in \mathcal{PSH}(\Omega)$, so $(u_j - \alpha)$ is plurisubharmonic and $(u_j - \alpha) \leq 0$. The class $\mathcal{E}_0$ is a convex cone; hence $(1 + c)\varphi_j \in \mathcal{E}_0$ and therefore $v_j \in \mathcal{E}_0$ by Theorem 1.1. Moreover, the sequence $[v_j]$ is a decreasing sequence that converges pointwise to $\max\{u - \alpha, (1 + c)\varphi\}$. Note that, by (2.8), $\max\{u - \alpha, (1+c)\varphi\} = u - \alpha$ in a neighborhood of K, and Theorem 1.1 yields that $\max\{u - \alpha, (1 + c)\varphi\} \in \mathcal{E}$. Theorem 4.2 in [9] implies that $[(dd^c v_j)^n]$ is weak*-convergent and the limit measure does not depend on the particular sequence $[v_j]$. Hence $[(dd^c(u_j - \alpha))^n]$ is weak*-convergent, since K was arbitrarily chosen. But $(dd^c(u_j - \alpha))^n = (dd^c u_j)^n$. Thus $(dd^c u_j)^n$ is weak*-convergent and the limit measure does not depend on the particular sequence $[u_j]$. □

DEFINITION 2.6. Let $u \in \mathcal{E}(f)$. Define $(dd^c u)^n u$ to be the limit measure in Theorem 2.5.

Let $u \in \mathcal{E}(f)$. Then, by Theorem 2.4, there exists a decreasing sequence $[u_j]$, $u_j \in \mathcal{E}_0(f)$, that converges pointwise to u as j tends to $+\infty$. If $[v_j]$, $v_j \in \mathcal{E}_0(f)$, is any decreasing sequence that converges pointwise to u as j tends to $+\infty$, then Theorem 2.5 ensures that $(dd^c v_j)^n$ is weak*-convergent and the limit measure does not depend on the particular sequence $[v_j]$. This implies that Definition 2.6 is well-posed.

Suppose $f : \partial\Omega \to \mathbb{R}$ is a continuous function such that $f \leq 0$ and $u \in \mathcal{E}(f)$. Proposition 2.3 implies that $u \in \mathcal{E}$. Consider $u = u_{\mathcal{E}(f)}$ to be a function only in $\mathcal{E}(f)$ and $u = u_{\mathcal{E}}$ to be a function only in $\mathcal{E}$. Then $(dd^c u_{\mathcal{E}(f)})^n$ is a nonnegative measure by Definition 2.6, and $(dd^c u_{\mathcal{E}})^n$ is also a nonnegative measure according to [9, Def. 4.3]. Fortunately, the proof of Theorem 2.5 implies that these two measures are the same.

Let $u_1, u_2, \ldots, u_n \in \mathcal{E}(f)$. Then it is possible, using the idea of the proof of Theorem 2.5, to define

$$(dd^c u_1) \wedge (dd^c u_2) \wedge \cdots \wedge (dd^c u_n)$$

in the same way as $(dd^c u)^n$ was defined in Definition 2.6.

Proposition 2.7 is obtained by using Proposition 2.3 together with [9, Cor. 5.2]; this proposition will later be used in the proof of Theorem 3.4.

PROPOSITION 2.7. *Let $u \in \mathcal{F}(f)$ and let $[u_j]$, $u_j \in \mathcal{E}_0(f)$, be a decreasing sequence that converges pointwise to u as j tends to $+\infty$. If $\varphi \in \mathcal{PSH}(\Omega)$, $\varphi \leq 0$, and if*

$$\int_\Omega (-\varphi)(dd^c u)^n < +\infty,$$

then $\lim_{j\to+\infty}(-\varphi)(dd^c u_j)^n = (-\varphi)(dd^c u)^n$ *in the weak*-topology.*

3. A Dirichlet Problem for the Complex Monge–Ampère Operator

Assume that $\Omega \subseteq \mathbb{C}^n$ is a bounded hyperconvex domain, and assume that $f : \partial\Omega \to \mathbb{R}$ is a continuous function such that $\lim_{z\to\xi} U(0, f)(z) = f(\xi)$ for every $\xi \in \partial\Omega$.

In this section, a Dirichlet problem for the complex Monge–Ampère operator is proved. More precisely: assume that μ is a nonnegative measure that vanishes on pluripolar sets and has finite total mass. Then there exists a uniquely determined function $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \mu$ (Theorem 3.4). This paper ends with a comparison principle, which is proved by using methods from the proof of Theorem 3.4. In [9], Cegrell solved this Dirichlet problem for $f = 0$. By using the existence part of Theorem 3.4 and the Bedford–Taylor comparison principle for bounded plurisubharmonic functions, Cegrell [10] proves a comparison principle in the class $\mathcal{F}^a(f)$; as a corollary, the uniqueness part of Theorem 3.4 follows.

LEMMA 3.1. *Let $u \in \mathcal{E}_0(f)$ and $\phi \in \mathcal{E}_0(f) \cap C(\Omega)$. If*

$$A = \{z \in \Omega : u(z) > \phi(z)\},$$

then $\chi_A (dd^c u)^n = \chi_A (dd^c \max\{u, \phi\})^n$. Here χ_A is the characteristic function for the set A.

Proof. If $u = U(0, f)$, the lemma follows immediately. Hence assume that u is not the function $U(0, f)$. It is sufficient to prove the equality of two measures on an arbitrary compact set $K \subseteq \Omega$ ($K \neq \emptyset$). Let α be the constant defined by

$$\alpha = \begin{cases} 0 & \text{if } \max_{\xi \in \partial\Omega} f(\xi) \leq 0, \\ \max_{\xi \in \partial\Omega} f(\xi) & \text{otherwise.} \end{cases}$$

The proof of Theorem 2.5 yields that there exists a function $u_\omega \in \mathcal{E}_0$ such that $u_\omega = u - \alpha$ in a neighborhood $\omega \subseteq \Omega$ of the given set K. If $\tilde{A} = \{z \in \Omega : u_\omega > \phi - \alpha\}$, then [8, Lemma 5.4] yields that $\chi_{\tilde{A}} (dd^c u_\omega)^n = \chi_{\tilde{A}} (dd^c(\max\{u_\omega, \phi - \alpha\}))^n$ on Ω and thus, in particular, on ω. Therefore,

$$\begin{aligned} \chi_A (dd^c u)^n &= \chi_A (dd^c(u - \alpha))^n = \chi_A (dd^c(\max\{u - \alpha, \phi - \alpha\}))^n \\ &= \chi_A (dd^c(\max\{u, \phi\} - \alpha))^n = \chi_A (dd^c(\max\{u, \phi\}))^n \end{aligned}$$

on K, since $A \cap \omega = \tilde{A} \cap \omega$. □

THEOREM 3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded open set and let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$. If*

$$\liminf_{\substack{z \to \xi \\ z \in \Omega}} (u(z) - v(z)) \geq 0$$

for every $\xi \in \partial\Omega$ and $(dd^c u)^n \leq (dd^c v)^n$, then $u \geq v$.

Proof. See for example [5]. □

THEOREM 3.3. *Assume that μ is a nonnegative measure defined on a bounded hyperconvex domain Ω. Then there exist functions $\psi \in \mathcal{E}_0$ and $\varphi \in L^1_{\mathrm{loc}}((dd^c \psi)^n)$, $\varphi \geq 0$, such that $\mu = \varphi (dd^c \psi)^n + \nu$. The nonnegative measure ν is such that there exists a pluripolar set $A \subseteq \Omega$ with $\nu(\Omega \setminus A) = 0$.*

Proof. See [9, Thm. 5.11]. □

THEOREM 3.4. *Let $\Omega \subseteq \mathbb{C}^n$ $(n \geq 2)$ be a bounded hyperconvex domain. Assume that μ is a nonnegative measure defined on Ω with $\mu(\Omega) < +\infty$ and $\mu(A) = 0$ for every pluripolar set $A \subseteq \Omega$. Then, for every continuous function $f : \partial\Omega \to \mathbb{R}$ such that $\lim_{z\to\xi} U(0, f)(z) = f(\xi)$ for every $\xi \in \partial\Omega$, there exists a uniquely determined function $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \mu$.*

Proof. The existence part of the theorem will be proved first. Since μ vanishes on pluripolar sets and has a finite total mass, it follows from Theorem 3.3 that there exist functions $\psi \in \mathcal{E}_0$ and $\varphi \in L^1((dd^c\psi)^n)$, $\varphi \geq 0$, such that $\mu = \varphi(dd^c\psi)^n$. For each $k \in \mathbb{N}$, let μ_k be the measure defined by $\mu_k = \min\{\varphi, k\}(dd^c\psi)^n$. Then $\mu_k \leq (dd^c(k^{1/n}\psi))^n$ and so, by Kołodziej's theorem (see [15]; see also [8, Prop. 6.1]), there exists a uniquely determined function $w_k \in \mathcal{E}_0$ such that

$$(dd^c w_k)^n = \mu_k. \tag{3.1}$$

The sequence $[w_k]$ is decreasing. This construction implies that $(w_k + U(0, f)) \in L^\infty(\Omega) \cap \mathcal{PSH}(\Omega)$, that $\lim_{z\to\xi}(w_k + U(0, f))(z) = f(\xi)$ for every $\xi \in \partial\Omega$, and that $U((dd^c(w_k + U(0, f)))^n, f) = w_k + U(0, f)$. Equality (3.1) implies that $(dd^c(w_k + U(0, f)))^n \geq \mu_k$. Theorem 8.1 in [8] yields that $(dd^c U(\mu_k, f))^n = \mu_k$ and

$$U(0, f) \geq U(\mu_k, f) \geq w_k + U(0, f). \tag{3.2}$$

Therefore, $U(\mu_k, f) \in \mathcal{E}_0(f)$. It also follows that $[U(\mu_k, f)]$ is a decreasing sequence. Since $\mu(\Omega) < +\infty$ by assumption, it follows that

$$\sup_k \int_\Omega (dd^c w_k)^n = \sup_k \int_\Omega (dd^c U(\mu_k, f))^n \leq \sup_k \mu_k(\Omega) \leq \mu(\Omega) < +\infty$$

and so $\lim_{k\to+\infty} w_k \in \mathcal{F}$. Let $u = \lim_{k\to+\infty} U(\mu_k, f)$; then $u \in \mathcal{PSH}(\Omega)$ and $U(0, f) \geq u \geq (\lim_{k\to+\infty} w_k) + U(0, f)$ by inequality (3.2). As a result, $u \in \mathcal{F}(f)$. From Theorem 2.5 it follows that $(dd^c u)^n = \mu$.

Now for the uniqueness part of the theorem. Assume that $v \in \mathcal{F}(f)$ is such that $(dd^c v)^n = \mu$ and assume (by the first part of this proof) that there exists a function $u \in \mathcal{F}(f)$ such that $(dd^c u)^n = \mu$. The assumption $\mu(\Omega) < +\infty$ then implies that $\int_\Omega (dd^c u)^n < +\infty$ and $\int_\Omega (dd^c v)^n < +\infty$. The aim is to prove that $u = v$.

The comparison principle has not been shown to be valid in $\mathcal{F}^a(f)$. This fact suggests the use of approximating sequences of the solutions u and v and then using the comparison principle (Theorem 3.2) on these approximants. For the function u the sequence $[u_k]$, $u_k \in \mathcal{E}_0(f)$, from the existence part is used. Let $[K_j]$ with $K_j \subseteq \Omega$ and $\operatorname{int}(K_j) \neq \emptyset$ be a sequence of compact sets such that, for every $j \in \mathbb{N}$, $K_j \subseteq \operatorname{int}(K_{j+1})$ and $\bigcup_{j=1}^\infty K_j = \Omega$. Moreover, let h_{K_j} denote the relative extremal function and let s_j be a positive integer. The sequence $[\max\{v, s_j h_{K_j} + U(0, f)\}]$ is then constructed such that $\max\{v, s_j h_{K_j} + U(0, f)\} \in \mathcal{E}_0(f)$ and such that it decreases to v on Ω as $j \to +\infty$. By using the auxiliary function a_j (to be defined shortly), it is possible to obtain

$$x_{jk} + \max\{v, s_j h_{K_j} + U(0, f)\} \leq u_k \leq y_{jk},$$

where $x_{jk} \in \mathcal{E}_0(0)$ and $y_{jk} \in \mathcal{E}_0(f)$ are constructed in a suitable way. When constructing the function a_j, an idea from the proof of [9, Lemma 5.14] is used. To complete this proof it is then sufficient to prove that x_{jk} converges to 0 and y_{jk} converges to v on Ω as k and j tend to $+\infty$.

By Theorem 3.3, there exist functions $\psi \in \mathcal{E}_0$ and $\varphi \in L^1((dd^c\psi)^n)$, $\varphi \geq 0$, such that

$$\mu = \varphi(dd^c\psi)^n; \tag{3.3}$$

this follows because μ vanishes on pluripolar sets and $\mu(\Omega) < \infty$, by assumption. For each $k \in \mathbb{N}$, let μ_k be the measure defined by

$$\mu_k = \min\{\varphi, k\}(dd^c\psi)^n. \tag{3.4}$$

From the first part of this proof it follows that there exists a decreasing sequence $[u_k]$, $u_k \in \mathcal{E}_0(f)$, such that

$$(dd^c u_k)^n = \mu_k \tag{3.5}$$

and $u = \lim_{k\to+\infty} u_k$. The sequence $[K_j]$ of compacts should also have the property that the relative extremal function h_{K_j} is in $\mathcal{E}_0 \cap C(\bar{\Omega})$. Recall that

$$h_{K_j}(z) = \sup\{\vartheta(z) : \vartheta \in \mathcal{PSH}(\Omega),\ \vartheta < 0 \text{ and } \vartheta \leq -1 \text{ on } K_j\}.$$

Let $[s_j]$ be a strictly increasing sequence of positive integers, and define the function a_j by

$$a_j = -h_{K_j} + \max\left\{\frac{v - U(0,f)}{s_j}, h_{K_j}\right\}.$$

Note that the function a_j is, in general, not plurisubharmonic. The definition of a_j yields that $\lim_{j\to+\infty}(1 - a_j) = 0$ on $\Omega\backslash\{v = -\infty\}$. It is thus possible to choose an increasing sequence $[l_j]_{j=1}^{\infty}$ of positive integers such that, for each $j \in \mathbb{N}$, the inequality

$$\int_\Omega (1 - a_{l_j})(dd^c v)^n \leq \frac{1}{j} \tag{3.6}$$

holds by the monotone convergence theorem and the assumption that $(dd^c v)^n$ vanishes on pluripolar sets. To simplify the notation, $[K_j]$ and $[s_j]$ will be used instead of $[K_{l_j}]$ and $[s_{l_j}]$ (the original sequences will no longer be used). If $A_j = \{v > s_j h_{K_j} + U(0,f)\}$ then

$$0 \leq a_j \leq \chi_{A_j} \leq 1, \tag{3.7}$$

where χ_{A_j} is the characteristic function for the set A_j. Since $s_j h_{K_j} \in \mathcal{E}_0$, it follows that $(s_j h_{K_j} + U(0,f)) \in \mathcal{E}_0(f)$. The sequence $[\max\{v, s_j h_{K_j} + U(0,f)\}]$ decreases to v as $j \to +\infty$. Let $j \in \mathbb{N}$ be fixed and let $s \in \mathbb{N}$ be such that $s \geq s_j$. Then Lemma 3.1 implies that

$$\chi_{A_j}(dd^c \max\{v, sh_{K_j} + U(0,f)\})^n = \chi_{A_j}(dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n, \tag{3.8}$$

since $\max\{\max\{v, sh_{K_j} + U(0,f)\}, s_j h_{K_j} + U(0,f)\} = \max\{v, s_j h_{K_j} + U(0,f)\}$. From (3.7) and (3.8) it follows that

$$\begin{aligned} 0 &\leq a_j(dd^c \max\{v, sh_{K_j} + U(0,f)\})^n \\ &\leq \chi_{A_j}(dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \\ &\leq (dd^c \max\{v, sh_{K_j} + U(0,f)\})^n. \end{aligned} \tag{3.9}$$

The following weak*-limits hold:

$$\lim_{s\to+\infty}(dd^c\max\{v, sh_{K_j}+U(0,f)\})^n=(dd^cv)^n,$$

$$\lim_{s\to+\infty}(-h_{K_j})(dd^c\max\{v, sh_{K_j}+U(0,f)\})^n=(-h_{K_j})(dd^cv)^n,$$

$$\begin{aligned}\lim_{s\to+\infty}\max\left\{\frac{v}{s_j}, h_{K_j}+\frac{U(0,f)}{s_j}\right\}&(dd^c\max\{v, sh_{K_j}+U(0,f)\})^n\\ &=\max\left\{\frac{v}{s_j}, h_{K_j}+\frac{U(0,f)}{s_j}\right\}(dd^cv)^n,\end{aligned}$$

$$\lim_{s\to+\infty}\left(-\frac{U(0,f)}{s_j}\right)(dd^c\max\{v, sh_{K_j}+U(0,f)\})^n=\left(-\frac{U(0,f)}{s_j}\right)(dd^cv)^n.$$

The first limit follows by Theorem 2.5 and the other three by Proposition 2.7. It is possible to write the function a_j as

$$a_j=-h_{K_j}+\max\left\{\frac{v}{s_j}, h_{K_j}+\frac{U(0,f)}{s_j}\right\}-\frac{U(0,f)}{s_j};$$

then, given (3.9) together with the preceding limits, it follows that

$$a_j(dd^cv)^n\le\chi_{A_j}(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n\le(dd^cv)^n \tag{3.10}$$

when $s\to+\infty$. Inequality (3.7) and (3.10) imply that

$$\begin{aligned}&(1-a_j)(dd^cv)^n+(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n\\ &\quad\ge(1-a_j)(dd^cv)^n+\chi_{A_j}(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n\\ &\quad\ge(dd^cv)^n\ge a_j(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n.\end{aligned} \tag{3.11}$$

The assumption that $(dd^cv)^n=\mu$ together with (3.3)–(3.5) yields that

$$\min\{\varphi, k\}(dd^cv)^n=\varphi(dd^cu_k)^n. \tag{3.12}$$

Define

$$\varrho_k(z)=\begin{cases}1 & \text{if } \varphi(z)=0,\\ \dfrac{\min\{\varphi(z), k\}}{\varphi(z)} & \text{otherwise;}\end{cases}$$

then $0\le\varrho_k\le 1$. By (3.11) and (3.12) it follows that

$$\begin{aligned}&\varrho_k(1-a_j)(dd^cv)^n+\varrho_k(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n\\ &\quad\ge\varrho_k(dd^cv)^n=(dd^cu_k)^n\\ &\quad\ge\varrho_ka_j(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n.\end{aligned} \tag{3.13}$$

Kołodziej's theorem again implies that, for each $j, k\in\mathbb{N}$, there exist functions $x_{jk}\in\mathcal{E}_0(0)$ such that $(dd^cx_{jk})^n=\varrho_k(1-a_j)(dd^cv)^n$, since $\varrho_k(dd^cv)^n=(dd^cu_k)^n$. From the first part of this proof it follows that there exist functions $y_{jk}\in\mathcal{E}_0(f)$ such that $(dd^cy_{jk})^n=\varrho_ka_j(dd^c\max\{v, s_jh_{K_j}+U(0,f)\})^n$. Let $j\in\mathbb{N}$ be fixed. Then the sequences $[(dd^cx_{jk})^n]_{k=1}^\infty$ and $[(dd^cy_{jk})^n]_{k=1}^\infty$ are increasing and so $[x_{jk}]_{k=1}^\infty$ and $[y_{jk}]_{k=1}^\infty$ are decreasing by Theorem 3.2. For each $j\in\mathbb{N}$, define

$$x_j = \lim_{k\to+\infty} x_{jk} \quad \text{and} \quad y_j = \lim_{k\to+\infty} y_{jk}.$$

Now the aim is to prove that, as $j \to +\infty$, the sequence $[x_j]$ converges to 0 on Ω and the sequence $[y_j]$ converges to v on Ω. From construction (3.6) it follows that $\sup_k \int_\Omega (dd^c x_{jk})^n \le 1/j$, which implies that $x_j \in \mathcal{F}$. There exists a function $\phi \in \mathcal{PSH}(\Omega) \cap C(\bar\Omega)$ such that

$$(dd^c\phi)^n = dV, \qquad \lim_{\substack{z\to\xi\\ z\in\Omega}} \phi(z) = 0 \quad \text{for every } \xi \in \partial\Omega$$

(see [7]). It is a consequence of [6, Cor. 2.2] and the definition of $\mathcal{F}$ that

$$\int_\Omega (-x_j)^n\, dV = \int_\Omega (-x_j)^n (dd^c\phi)^n \le C_\phi \int_\Omega (dd^c x_j)^n \le C_\phi \frac{1}{j},$$

where $C_\phi \ge 0$ is a constant depending only on ϕ and the dimension n. Therefore,

$$\lim_{j\to+\infty} x_j = 0 \tag{3.14}$$

weakly on Ω. Inequality (3.13) then yields that

$$\begin{aligned}(dd^c(x_{jk} &+ \max\{v, s_j h_{K_j} + U(0,f)\}))^n \\ &\ge (dd^c x_{jk})^n + \varrho_k (dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \\ &\ge (dd^c u_k)^n \ge (dd^c y_{jk})^n\end{aligned}$$

for every $j,k \in \mathbb{N}$. Then, by Theorem 3.2,

$$x_{jk} + \max\{v, s_j h_{K_j} + U(0,f)\} \le u_k \le y_{jk}. \tag{3.15}$$

Since $(dd^c y_{jk})^n \le (dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n$, it follows that $U(0,f) \ge y_{jk} \ge \max\{v, s_j h_{K_j} + U(0,f)\}$ by Theorem 3.2. Thus,

$$U(0,f) \ge y_j = \lim_{k\to+\infty} y_{jk} \ge \max\{v, s_j h_{K_j} + U(0,f)\}.$$

Hence $y_j \in L^\infty(\Omega)$ and, by Proposition 2.2, it follows that $\lim_{z\to\xi} y_j(z) = f(\xi)$ for every $\xi \in \partial\Omega$. For each $j \in \mathbb{N}$, [8, Prop. 6.1] implies that there exists a function $w_j \in \mathcal{F}_1 \cap L^\infty(\Omega)$ such that

$$(dd^c w_j)^n = (1-a_j)(dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \tag{3.16}$$

and therefore

$$\begin{aligned}(dd^c(y_j + w_j))^n &\ge (dd^c y_j)^n + (dd^c w_j)^n \\ &= (dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \ge (dd^c y_j)^n.\end{aligned}$$

As a result,

$$y_j + w_j \le \max\{v, s_j h_{K_j} + U(0,f)\} \le y_j \tag{3.17}$$

by Theorem 3.2, since $y_j, w_j \in L^\infty(\Omega)$ and $y_j + w_j \le \max\{v, s_j h_{K_j} + U(0,f)\} = y_j$ on $\partial\Omega$. Theorem 2.5 yields that $(dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \le (dd^c v)^n$; after multiplying the left inequality in (3.10) by a_j it follows that $a_j^2 (dd^c v)^n \le a_j (dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n$, so

$$\int_\Omega (1-a_j)(dd^c \max\{v, s_j h_{K_j} + U(0,f)\})^n \leq \int_\Omega (1-a_j^2)(dd^c v)^n.$$

Now it follows by (3.6) and (3.16) that

$$\int_\Omega (dd^c w_j)^n \leq \int_\Omega (1-a_j^2)(dd^c v)^n \leq 2\int_\Omega (1-a_j)(dd^c v)^n \leq \frac{2}{j}.$$

Hence, by [6, Cor. 2.2],

$$\int_\Omega (-w_j)^n\, dV = \int_\Omega (-w_j)^n (dd^c\phi)^n \leq C'_\phi \int_\Omega (dd^c w_j)^n \leq C'_\phi \frac{2}{j},$$

where $C'_\phi \geq 0$ is a constant depending only on ϕ and the dimension n. This implies that

$$\lim_{j\to+\infty} w_j = 0 \tag{3.18}$$

weakly on Ω. It follows from (3.14), (3.15), (3.17), and (3.18) that $u = v$ on Ω after letting k and j tend to $+\infty$. □

DEFINITION 3.5. Define $\mathcal{F}^a(f)$ to be the class of plurisubharmonic functions $u \in \mathcal{F}(f)$ such that $(dd^c u)^n$ vanishes on all pluripolar sets.

COROLLARY 3.6. *Let $\Omega \subseteq \mathbb{C}^n$ $(n \geq 2)$ be a bounded hyperconvex domain, and let $f, g\colon \partial\Omega \to \mathbb{R}$ be continuous functions such that $\lim_{z\to\xi} U(0,f)(z) = f(\xi)$ and $\lim_{z\to\xi} U(0,g)(z) = g(\xi)$ for every $\xi \in \partial\Omega$. If $u \in \mathcal{F}(f)$ and $v \in \mathcal{F}^a(g)$ where $f \leq g$, $\int_\Omega (dd^c u)^n < +\infty$, and $(dd^c u)^n \geq (dd^c v)^n$, then $u \leq v$.*

Proof. There exist functions $\psi_1, \psi_2 \in \mathcal{E}_0$, with $\varphi_1 \in L^1((dd^c\psi_1)^n)$, $\varphi_1 \geq 0$, and $\varphi_2 \in L^1((dd^c\psi_2)^n)$, $\varphi_2 \geq 0$, such that

$$\begin{aligned} (dd^c u)^n &= \varphi_1 (dd^c\psi_1)^n + \nu, \\ (dd^c v)^n &= \varphi_2 (dd^c\psi_2)^n; \end{aligned} \tag{3.19}$$

here ν is a nonnegative measure, which (by Theorem 3.3) is carried by a pluripolar set. Moreover, $(dd^c(\psi_1+\psi_2))^n \geq (dd^c\psi_1)^n$ and $(dd^c(\psi_1+\psi_2))^n \geq (dd^c\psi_2)^n$. The measures $(dd^c\psi_1)^n$ and $(dd^c\psi_2)^n$ are thus absolutely continuous with respect to $(dd^c(\psi_1+\psi_2))^n$. Hence there exist functions $\tau_1 \in L^1((dd^c(\psi_1+\psi_2))^n)$, $\tau_1 \geq 0$, and $\tau_2 \in L^1((dd^c(\psi_1+\psi_2))^n)$, $\tau_2 \geq 0$, such that

$$\begin{aligned} \tau_1 (dd^c(\psi_1+\psi_2))^n &= (dd^c\psi_1)^n, \\ \tau_2 (dd^c(\psi_1+\psi_2))^n &= (dd^c\psi_2)^n. \end{aligned} \tag{3.20}$$

By the equality of measures in (3.19) and (3.20) it follows that

$$\begin{aligned} (dd^c u)^n &= \varphi_1\tau_1 (dd^c(\psi_1+\psi_2))^n + \nu, \\ (dd^c v)^n &= \varphi_2\tau_2 (dd^c(\psi_1+\psi_2))^n. \end{aligned} \tag{3.21}$$

Therefore, $\varphi_1\tau_1 \geq \varphi_2\tau_2$ on Ω because $(dd^c u)^n \geq (dd^c v)^n$, by assumption. Consider the measure $\varphi_1\tau_1(dd^c(\psi_1+\psi_2))^n$; it has finite total mass and vanishes on every pluripolar set. Hence Theorem 3.4 implies that there exists a uniquely determined function $w \in \mathcal{F}^a(g)$ such that $(dd^c w)^n = \varphi_1\tau_1(dd^c(\psi_1 + \psi_2))^n$, and from (3.21) it follows that

$$(dd^c v)^n \leq (dd^c w)^n = \varphi_1\tau_1(dd^c(\psi_1 + \psi_2))^n \leq (dd^c u)^n,$$

since $\varphi_1\tau_1 \geq \varphi_2\tau_2$ on Ω. For each $j \in \mathbb{N}$, let the measures μ_j^v and μ_j^w be defined by

$$\mu_j^v = \min\{\varphi_2\tau_2, j\}(dd^c(\psi_1 + \psi_2))^n,$$
$$\mu_j^w = \min\{\varphi_1\tau_1, j\}(dd^c(\psi_1 + \psi_2))^n.$$

By the proof of the existence part of Theorem 3.4, there exist uniquely determined functions $v_j, w_j \in \mathcal{E}_0(g)$ such that $(dd^c v_j)^n = \mu_j^v$ and $(dd^c w_j)^n = \mu_j^w$. As a result, $(dd^c v_j)^n \leq (dd^c w_j)^n$. Theorem 3.2 then yields that

$$v_j \geq w_j \tag{3.22}$$

and that $[v_j]$ and $[w_j]$ are decreasing sequences. Let

$$\tilde{v} = \lim_{j\to+\infty} v_j \quad \text{and} \quad \tilde{w} = \lim_{j\to+\infty} w_j.$$

Using the same idea used in the existence part of the proof of Theorem 3.4, it is possible to prove that $\tilde{v}, \tilde{w} \in \mathcal{F}^a(g)$, $(dd^c\tilde{v})^n = \varphi_2\tau_2(dd^c(\psi_1 + \psi_2))^n$, and $(dd^c\tilde{w})^n = \varphi_1\tau_1(dd^c(\psi_1 + \psi_2))^n$. But v and w were uniquely determined and so $v = \tilde{v}$ and $w = \tilde{w}$. It follows from (3.22) that

$$v \geq w. \tag{3.23}$$

Let $[s_j]$ and $[K_j]$ be as in the proof of the uniqueness part of Theorem 3.4. In a similar manner, define the function b_j on Ω by

$$b_j = -h_{K_j} + \max\left\{\frac{u - U(0, f)}{s_j}, h_{K_j}\right\}.$$

Note that $u \in \mathcal{F}(f)$ and therefore $(dd^c u)^n$ may put mass on pluripolar sets. Inequality (3.10) yields that

$$b_j(dd^c u)^n \leq (dd^c \max\{u, s_j h_{K_j} + U(0, f)\})^n. \tag{3.24}$$

This implies, in particular, that the nonnegative measure $b_j(dd^c u)^n$ vanishes on pluripolar sets and so

$$b_j(dd^c u)^n = b_j\varphi_1\tau_1(dd^c(\psi_1 + \psi_2))^n = b_j(dd^c w)^n.$$

There exists a uniquely determined function $w_j' \in \mathcal{E}_0(g)$ such that $(dd^c w_j')^n = \varrho_j b_j(dd^c u)^n$, where

$$\varrho_j(z) = \begin{cases} 1 & \text{if } \varphi_1(z)\tau_1(z) = 0, \\ \dfrac{\min\{\varphi_1(z)\tau_1(z), j\}}{\varphi_1(z)\tau_1(z)} & \text{otherwise.} \end{cases}$$

Theorem 3.2 and (3.24) imply that

$$w_j' \geq \max\{u, s_j h_{K_j} + U(0, f)\} \tag{3.25}$$

on Ω. Recall that $f \leq g$ by assumption. Let $\tilde{w}' = \lim_{j\to+\infty} w_j'$. Then $\tilde{w}' \in \mathcal{F}(g)$ and $(dd^c\tilde{w}')^n = (dd^c w)^n$. Therefore, $\tilde{w}' \in \mathcal{F}^a(g)$ and $\tilde{w}' = w$ on Ω, since w was uniquely determined. It thus follows that $w \geq u$ on Ω, by (3.25). The proof of the corollary is completed since $v \geq w$ on Ω by (3.23). □

References

[1] P. Åhag, *The complex Monge–Ampère operator on bounded hyperconvex domains,* Ph.D. Thesis, Umeå Univ., 2002.

[2] P. Åhag and R. Czyż, *The connection between the Cegrell classes and compliant functions,* Math. Scand. 99 (2006), 87–98.

[3] D. H. Armitage and S. J. Gardiner, *Classical potential theory,* Springer Monogr. Math., Springer-Verlag, London, 2001.

[4] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation,* Invent. Math. 37 (1976), 1–44.

[5] ———, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.

[6] Z. Błocki, *Estimates for the complex Monge–Ampère operator,* Bull. Polish Acad. Sci. Math. 41 (1993), 151–157.

[7] ———, *On the L^p stability for the complex Monge–Ampère operator,* Michigan Math. J. 42 (1995), 269–275.

[8] U. Cegrell, *Pluricomplex energy,* Acta Math. 180 (1998), 187–217.

[9] ———, *The general definition of the complex Monge–Ampère operator,* Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[10] ———, personal communication.

[11] U. Cegrell and S. Kołodziej, *The equation of complex Monge–Ampère type and stability of solutions,* Math. Ann. 334 (2006), 713–729.

[12] U. Cegrell, S. Kołodziej, and A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularites,* Math. Z. 250 (2005), 7–22.

[13] U. Cegrell and J. Wiklund, *A Monge–Ampère norm for delta-plurisubharmonic functions,* Math. Scand. 97 (2005), 201–216.

[14] M. Klimek, *Pluripotential theory,* London Math. Soc. Monogr. (N.S.), 6, Oxford Univ. Press, New York, 1991.

[15] S. Kołodziej, *The range of the complex Monge–Ampère operator, II,* Indiana Univ. Math. J. 44 (1995), 765–782.

[16] A. Rashkovskii, personal communication.

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