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On the Maximum Principle and a Notion of Plurisubharmonicity for Abstract CR Manifolds

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0. Introduction

Let $\mathcal{M}$ be a smooth manifold and let $\mathcal{V}$ be a subbundle of $\mathbb{C}T\mathcal{M}$, the complexified tangent bundle of $\mathcal{M}$. The pair $(\mathcal{M},\mathcal{V})$ is called an *abstract CR manifold* if $\mathcal{V}$ is involutive and if, for each $p \in \mathcal{M}$, $\mathcal{V}_p \cap \bar{\mathcal{V}}_p = \{0\}$. Recall that $\mathcal{V}$ is involutive if the space of smooth sections of $\mathcal{V}$, $C^\infty(\mathcal{M},\mathcal{V})$, is closed under commutators. Let n be the complex dimension of the fibre $\mathcal{V}_p$ of $\mathcal{V}$ at p and write $\dim_{\mathbb{R}} \mathcal{M} = n+m$. The number n is called the *CR dimension* of $\mathcal{M}$, and $d = m-n$ will be called the *CR codimension* of $\mathcal{M}$. If $d = 1$, the CR structure is said to be of hypersurface type. The CR manifold $(\mathcal{M},\mathcal{V})$ is called *integrable* or *locally embeddable* if, for any $p_o \in \mathcal{M}$, there exist m complex-valued C^∞ functions $\mathcal{Z}_1,\dots,\mathcal{Z}_m$ defined near p_o such that (a) $L\mathcal{Z}_j = 0$ for all $L \in C^\infty(\mathcal{M},\mathcal{V})$, $j = 1,\dots,m$, and (b) the differentials $d\mathcal{Z}_1,\dots,d\mathcal{Z}_m$ are $\mathbb{C}$-linearly independent. Any such set of functions $\mathcal{Z}_j$ will be called a *complete set of first integrals.*

If $(\mathcal{M},\mathcal{V})$ is an integrable CR manifold, then the mapping $p \mapsto \mathcal{Z}(p) = (\mathcal{Z}_1(p),\dots,\mathcal{Z}_m(p)) \in \mathbb{C}^m$, where the $\mathcal{Z}_j$ are a complete set of first integrals, is a map of constant rank near p_o and so is an immersion. Thus, if U is a small neighborhood of p_o, then $\mathcal{Z}(U)$ is an embedded real submanifold of $\mathbb{C}^m$ of dimension $m+n$, and its real codimension in $\mathbb{C}^m$ agrees with the CR codimension $d = m-n$. It is easy to see that $\mathcal{Z}(U)$ is a generic CR submanifold of $\mathbb{C}^m$ and that its CR bundle agrees with the push-forward $\mathcal{Z}_*\mathcal{V}$. Conversely, if $\mathcal{M}$ is a CR submanifold of $\mathbb{C}^m$ and $\mathcal{V}$ is its CR bundle, then $(\mathcal{M},\mathcal{V})$ defines an integrable CR structure (see [BER] and [J] for more details).

In an abstract CR manifold $(\mathcal{M},\mathcal{V})$, a smooth section of $\mathcal{V}$ is called a *CR vector field*. A function f on $\mathcal{M}$ is called a *CR function* if $Lf = 0$ for any CR vector field L. The maximum principle for the modulus of CR functions when $(\mathcal{M},\mathcal{V})$ is embeddable has been studied by several authors (see e.g. [Ba; Ber; EHS; Io; Ro; Si]). To our knowledge, very little seems to be known when $(\mathcal{M},\mathcal{V})$ is not necessarily embeddable. The authors of [HNa] have proved a weak maximum principle for almost complex manifolds under some assumptions on the Levi form and minimality of the manifold (see [BER, p. 20]). When $(\mathcal{M},\mathcal{V})$ is locally embeddable, it

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is well known that, near a point $p \in \mathcal{M}$ where the Levi form is strictly definite, one can always find a smooth CR function h whose modulus peaks at p. If we write the coordinates in $\mathbb{C}^2$ as $z = x + iy$ and $w = s + it$ on the 3-dimensional CR submanifold $\mathcal{M} = \{(z, s + i(|z|^4 + s^2))\}$, then the CR function $h = \exp(i(s + i(|z|^4 + s^2)))$ satisfies $|h(0)| > |h(p)|$ for any $p \in \mathcal{M}$, $p \neq 0$. Note that the Levi form of $\mathcal{M}$ is not strictly definite at 0 and so the converse of the preceding result is not true. However, there is a good partial converse: If $(\mathcal{M}, \mathcal{V})$ is locally embeddable and h is a continuous CR function whose modulus peaks at p, then there is a sequence of points p_j in $\mathcal{M}$ converging to p such that the Levi form is strictly definite at each p_j. When h is assumed to be C^2, the result dates back to Rossi [Ro] in the hypersurface case and to Sibony [Si] in the higher-codimensional case. In Section 4 we will relax the regularity to just continuity (see Remark 4.1). In this paper, we show (see Theorem 2) that this partial converse holds for $(\mathcal{M}, \mathcal{V})$ an abstract CR manifold. Observe that in the abstract setting there may be no nonconstant CR functions even on strictly pseudoconvex structures (see [N]), so one will not always get CR functions that peak at strictly definite points. At such points, approximate peak functions exist (see [HaJ, Lemma 1.8]).

For $(\mathcal{M}, \mathcal{V})$ a CR submanifold of $\mathbb{C}^N$, the works [Ro] and [Si] actually establish the following maximum principle for CR functions: For any open set $\Omega \subset\subset \mathcal{M}$, any $z \in \Omega$, and any $h \in C^2(\Omega) \cap C(\bar{\Omega})$ that is CR on Ω,

$$|h(z)| \leq \sup_{\partial\Omega \cup (\Sigma \cap \Omega)} |h|, \tag{0.1}$$

where Σ is the set of strictly definite points of the Levi form in Ω (see Definition 1.3). In this paper we will present a sufficient condition for the validity of the maximum principle for CR functions akin to (0.1) on an abstract CR manifold (Theorem 1). This sufficient condition is always satisfied when the CR manifold $(\mathcal{M}, \mathcal{V})$ is globally embeddable in some $\mathbb{C}^N$. However, we also present a class of examples of locally embeddable (in fact, real-analytic) CR manifolds for which the maximum principle (0.1) fails (Example 4.1). We will also present examples of nonlocally embeddable CR manifolds that satisfy our sufficient condition and hence also the maximum principle as stated in (0.1).

The paper is organized as follows. In Section 1, we recall some basic definitions and state Theorems 1 and 2. In Section 2, we first review Sibony's notion of plurisubharmonicity on an embedded CR manifold and his generalization of the maximum principle for plurisubharmonic functions in $\mathbb{C}^n$. We then extend Sibony's notion of plurisubharmonicity and his results to abstract CR manifolds. Section 3 contains the proofs of Theorems 1 and 2, and Section 4 is devoted to a variety of examples. We conclude with an appendix that presents a result—on the Levi form of an embedded CR manifold—that played a crucial role in Sibony's paper [Si].

1. Preliminaries and Statement of Main Results

Let $(\mathcal{M}, \mathcal{V})$ be an abstract CR manifold. For $p \in \mathcal{M}$, let

$$\pi_p : \mathbb{C}T_p\mathcal{M} \to \mathbb{C}T_p\mathcal{M}/\mathcal{V}_p \oplus \bar{\mathcal{V}}_p$$

denote the quotient map.

DEFINITION 1.1. The *Levi map* at $p \in \mathcal{M}$ is the Hermitian vector-valued form

$$\mathcal{L}_p : \mathcal{V}_p \times \mathcal{V}_p \to \mathbb{C}T_p\mathcal{M}/\mathcal{V}_p \oplus \bar{\mathcal{V}}_p,$$
$$\mathcal{L}_p(X_p, Y_p) = \frac{1}{2i}\pi_p([X, \bar{Y}](p)),$$

where X and Y are CR vector fields that extend X_p and Y_p.

DEFINITION 1.2. The Levi map is called *nondegenerate* at $p \in \mathcal{M}$ if $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y_p \in \mathcal{V}_p$ implies that $X_p = 0$.

DEFINITION 1.3. The Levi map is said to be *strictly definite* at $p \in \mathcal{M}$ if $\mathcal{L}_p(X_p, X_p) \neq 0$ whenever X_p is a nonzero CR vector in $\mathcal{V}_p$.

We will use the notation

$$\Sigma = \{z \in \mathcal{M} : \text{the Levi map is strictly definite at } z\}.$$

Section 4 contains examples that illustrate these concepts (see [Ber; Bo] for details on the Levi form in the embedded case). See Definition 3.1 for the concept of $\mathcal{V}$-convexity used in Theorems 1 and 2.

Our main results are as follows.

THEOREM 1 (Maximum principle). *Suppose $(\mathcal{M}, \mathcal{V})$ is an abstract CR manifold that admits a $\mathcal{V}$-convex function g. Let $\Omega \subseteq \mathcal{M}$ be a relatively compact open set, and let $f \in C^2(\Omega) \cap C(\bar{\Omega})$ be a CR function on Ω. Then, for any $z \in \Omega$,*

$$|f(z)| \leq \max_{\partial\Omega \cup \overline{\Sigma \cap \Omega}} |f|.$$

THEOREM 2. *Let $(\mathcal{M}, \mathcal{V})$ be an abstract CR manifold. Suppose h is a C^2 CR function whose modulus peaks at a point p; that is, suppose $|h(p)| > |h(q)|$ for all $q \neq p$. Then p is in the closure of the set of strictly definite points.*

2. A Notion of Plurisubharmonicity and Some Maximum Principles

In [BeT], Bedford and Taylor proved the following maximum principle for plurisubharmonic functions in $\mathbb{C}^n$.

THEOREM A [BTa]. *Suppose that $\Omega \subseteq \mathbb{C}^n$ is a bounded open set and that φ, ψ are continuous on $\bar{\Omega}$ and plurisubharmonic on Ω. Assume that*

(i) *$\varphi \leq \psi$ on $\partial\Omega$ and*
(ii) *$(dd^c\varphi)^n \geq (dd^c\psi)^n$ in Ω.*

Then $\varphi \leq \psi$ in Ω.

When $\varphi \in C^2(\Omega)$,

$$(dd^c\varphi)^n = dd^c\varphi \wedge \cdots \wedge dd^c\varphi = 4^n n! \det \mathcal{L}(\varphi),$$

where

$$\mathcal{L}(\varphi) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)_{i,j}.$$

In [BeT] and [CLN], this operator was extended to act on plurisubharmonic continuous functions. In [Si], Sibony generalized this theorem to CR manifolds embedded in $\mathbb{C}^N$. His generalization actually implies a strengthened version of Theorem A (see Remark 2.1) because the plurisubharmonicity of ψ is not needed. For the extension of Theorem A to the case of embedded CR manifolds, Sibony introduced a cone of functions that play the role of plurisubharmonic functions and contain the modulus of C^2 CR functions. In this section we briefly review the main results of [Si] and then extend them to abstract CR manifolds.

Let $\mathcal{N} \subseteq \mathbb{C}^{n+d}$ be a generic CR manifold with $\dim_{\mathbb{R}} \mathcal{N} = 2n+d$ and $\dim_{\mathbb{C}} \mathcal{V} = n$, where $\mathcal{V}$ is the CR bundle on $\mathcal{N}$. For ρ a C^2 function defined on an open set in $\mathbb{C}^{n+d}$ ($u, w \in \mathbb{C}^{n+d}$), we define

$$\langle (\mathcal{L}_z \rho) u, w \rangle = \sum_{i,j=1}^{n+d} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z) u_i \bar{w}_j.$$

Let $z_o \in \mathcal{N}$ and let U be a neighborhood of z_o such that

$$\mathcal{N} \cap U = \{z \in U : \rho_j(z) = 0,\ 1 \le j \le d\}$$

for some C^∞ real-valued functions $\rho_1, \dots, \rho_d$ satisfying $d\rho_1 \wedge \cdots \wedge d\rho_d \ne 0$ on U. For any $z \in \mathcal{N} \cap U$, the fiber $\mathcal{V}_z$ is given by

$$\mathcal{V}_z = \left\{ \sum_{j=1}^{n+d} u_j \frac{\partial}{\partial \bar{z}_j}\bigg|_z : \sum_{j=1}^{n+d} u_j \frac{\partial \rho_k}{\partial \bar{z}_j}(z) = 0\ \forall k = 1, \dots, d \right\}.$$

For $z \in \mathcal{N} \cap U$, define

$$K_z = \left\{ \sum_{j=1}^{n+d} u_j \frac{\partial}{\partial \bar{z}_j}\bigg|_z \in \mathcal{V}_z : \langle \mathcal{L}_z(\rho_k)\bar{u}, \bar{w} \rangle = 0\ \forall k \text{ and } \forall \sum_{j=1}^{n+d} w_j \frac{\partial}{\partial \bar{z}_j}\bigg|_z \in \mathcal{V}_z \right\}.$$

We caution the reader that in [Si] the space K_z, denoted there by N_z, was defined as a subspace of $\bar{\mathcal{V}}_z$. This explains the conjugation of u and v in the inner product in our definition of this space.

Recall that K_z is called the Levi space of $\mathcal{N}$ at z. It is well known that K_z is independent of the choice of the defining functions. In fact,

$$K_z = \{u \in \mathcal{V}_z : [U, \bar{V}](z) \in \mathcal{V}_z + \bar{\mathcal{V}}_z \quad \text{for smooth sections } U, V \text{ of } \mathcal{V} \text{ with } U(z) = u\}.$$

The usual way of checking this is by using the embedding $e\colon \mathcal{N} \to \mathbb{C}^{n+d}$ of $\mathcal{N}$ into $\mathbb{C}^{n+d}$. Since $e^* d\rho_j \equiv 0$, $d = \partial + \bar{\partial}$, and $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \ne 0$ ($\mathcal{N}$ is generic) near z_o, it follows that the 1-forms

$$\theta_j = e^*(i\partial\rho_j), \quad 1 \le j \le d,$$

are real and linearly independent. It is easy to see that

$$\langle \theta_j, X \rangle = 0 \quad \forall X \in \mathcal{V} + \bar{\mathcal{V}}. \tag{2.1}$$

If X and Y are CR vector fields then, by Cartan's identity,

$$2\langle d\theta_j, X \wedge \bar{Y} \rangle = X(\langle \theta_j, \bar{Y} \rangle) - \bar{Y}(\langle \theta_j, X \rangle) - \langle \theta_j, [X, \bar{Y}] \rangle = -\langle \theta_j, [X, \bar{Y}] \rangle. \tag{2.2}$$

Since $d\theta_j = e^*(id\partial\rho_j) = e^*(i\bar{\partial}\partial\rho_j)$, it follows that

$$\langle \theta_j, [X, \bar{Y}](p) \rangle = -2\langle e^*(i\bar{\partial}\partial\rho_j)(p), X_p \wedge \bar{Y}_p \rangle. \tag{2.3}$$

That K_z actually is as asserted follows from (2.3) and the observation that, for any $v \in \mathbb{C}T_p\mathcal{N} \setminus \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$, there exists a j such that $\langle \theta_j(p), v \rangle \neq 0$. Let $\sigma = \{z \in \mathcal{N} : K_z = \{0\}\}$. Observe that σ is the set of points where the Levi map is nondegenerate (see Definition 1.2). Equation (2.3) also shows that the Levi map is strictly definite at $z \in \mathcal{N}$ (see Definition 1.3) if and only if, for every nonzero $u \in \mathcal{V}_z$, there exists a j $(1 \leq j \leq d)$ such that $\langle \mathcal{L}_z(\rho_j)u, u \rangle \neq 0$. Recall that $\Sigma = \{z \in \mathcal{N} :$ the Levi map is strictly definite at $z\}$. Clearly, $\Sigma \subseteq \sigma$. We are now ready to recall Sibony's generalization of plurisubharmonicity.

Let φ be a C^2 function defined on an open subset V of $\mathcal{N}$. Now extend φ to a C^2 function $\tilde{\varphi}$ defined on an open subset Ω of $\mathbb{C}^{n+d}$. If $u \in K_z$ and $z \in V$, set

$$\langle \mathcal{L}_z(\varphi)u, u \rangle = \sum_{i,j=1}^{n+d} \frac{\partial^2 \tilde{\varphi}}{\partial z_i \partial \bar{z}_j}(z)\bar{u}_i u_j. \tag{2.4}$$

It was shown in [Si] that the Hermitian form $\mathcal{L}_z(\varphi)$ defined on K_z does not depend on the choice of extension.

Definition 2.1 [Si]. Suppose $\varphi \in C^2(V)$ as before. We say that $\varphi \in \mathcal{P}(V)$ if the Hermitian form $\mathcal{L}_z(\varphi)$ is positive (i.e., ≥ 0) on K_z for each $z \in V$.

Sibony showed that if f is a C^2 CR function on V then $|f|^2 \in \mathcal{P}(V)$. The cone $\mathcal{P}(V)$ also contains the restrictions to $\mathcal{N}$ of plurisubharmonic functions defined on a neighborhood of $\mathcal{N}$. If $\mathcal{N} = \mathbb{C}^m$ then, for each $z \in \mathbb{C}^m$, $K_z = C^m$ and so one recovers the usual notion of plurisubharmonicity.

Suppose now that Ω is a relatively compact and open subset of $\mathcal{N}$ with $\partial\Omega$ its boundary, and let $\delta\Omega = \partial\Omega \cup (\sigma \cap \Omega)$. In [Si] Sibony proved the following generalization of Theorem A.

Theorem B [Si, Thm. 1′]. *Suppose $\varphi, \psi \in C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $\varphi \in \mathcal{P}(\Omega)$, and let*

(i) *$\varphi \leq \psi$ on $\delta\Omega$ and*
(ii) *$\det \mathcal{L}_z(\varphi) \geq \det \mathcal{L}_z(\psi)$ for each $z \in \Omega \setminus \delta\Omega$.*

Then $\varphi \leq \psi$ in Ω.

Remark 2.1. Theorem A follows from Theorem B by observing that, when $\mathcal{N} = \mathbb{C}^m$, $K_z = \mathbb{C}^m$ for all $z \in \mathbb{C}^m$. In fact, one obtains a stronger version of Theorem A when φ and ψ are C^2 since only φ needs to be plurisubharmonic.

REMARK 2.2. The set $\Omega\backslash\delta\Omega$ is

$$\Omega\backslash\delta\Omega = \{z \in \Omega : K_z \neq \{0\}\}.$$

For $z \in \Omega\backslash\delta\Omega$, $\det \mathcal{L}_z(\varphi)$ is not defined because there is no natural basis of K_z. However, it is easy to see that, if $\det \mathcal{L}_z(\varphi) \geq \det \mathcal{L}_z(\psi)$ for a given choice of a basis, then the inequality also holds for any other choice.

Theorem B has the following important corollary.

COROLLARY C [Si, Cor. 1]. *Suppose $f \in C^2(\Omega) \cap C(\bar{\Omega})$ is a CR function. Then, for any $z \in \Omega$,*

$$|f(z)| \leq \max_{\partial\Omega \cup \overline{\Sigma \cap \Omega}} |f|.$$

This in turn implies our next result.

COROLLARY D. *Suppose $\mathcal{N} \subseteq \mathbb{C}^{n+d}$ is also compact. Then, for any C^2 CR function f and any $z \in \mathcal{N}$,*

$$|f(z)| \leq \max_{\bar{\Sigma}} |f|.$$

In order to extend these results to an abstract CR manifold, we will first generalize the cone $\mathcal{P}$.

In what follows, let $(\mathcal{M}, \mathcal{V})$ be an abstract CR manifold, let $\dim_{\mathbb{R}} \mathcal{M} = 2n + d$, and let $\dim_{\mathbb{C}} \mathcal{V}_p = n$ for any $p \in \mathcal{M}$. Pick a C^∞ Hermitian metric on $\mathbb{C}T\mathcal{M}$ such that $\mathcal{V}$ is orthogonal to $\bar{\mathcal{V}}$, and let F be the othogonal complement of $\mathcal{V} \oplus \bar{\mathcal{V}}$ such that

$$\mathbb{C}T\mathcal{M} = \mathcal{V} \oplus \bar{\mathcal{V}} \oplus F.$$

Let $p \colon \mathbb{C}T\mathcal{M} \to \mathcal{V}$ and $\bar{p} \colon \mathbb{C}T\mathcal{M} \to \bar{\mathcal{V}}$ be the orthogonal projections. We will use the following notion of Hessian from Kuranishi [Ku].

DEFINITION 2.2. Let f be a C^2 function on an open subset Ω of $\mathcal{M}$. We define the *$\mathcal{V}$-Hessian* of f by

$$H_{\mathcal{V}}(X, Y) f = X(\bar{Y} f) - \bar{p}[X, \bar{Y}] f,$$

where X and Y are smooth sections of $\mathcal{V}$.

The $\mathcal{V}$-Hessian is not Hermitian. We will show, however, that if f is real-valued then $H_{\mathcal{V}}(X, Y) f$ defines a Hermitian form on each vector space K_z. Observe that the $\mathcal{V}$-Hessian depends only on $\mathcal{V}$ and not on the choice of the metric.

Let $\{L_1, \ldots, L_n\}$ be an orthonormal basis of $\mathcal{V}$ for some Hermitian metric, and let $\{\omega_1, \ldots, \omega_n\}$ its dual basis. That is,

$$\langle \omega_j, L_i \rangle = \delta_{ij} \quad \text{and} \quad \omega_j |_{\bar{\mathcal{V}} \oplus F} = 0 \quad \forall j.$$

Let $\{T_1, \ldots, T_d\}$ be an orthonormal basis of F, and let

$$[L_i, \bar{L}_j] = \sum_{k=1}^{n} a_{ij}^k L_k + \sum_{k=1}^{n} b_{ij}^k \bar{L}_k + \sum_{s=1}^{d} c_{ij}^s T_s \tag{2.5}$$

for some smooth functions a^k_{ij}, b^k_{ij}, and c^s_{ij}. It can then be shown that $H_{\mathcal{V}}(L_i, L_j)$ are the coefficients of $\bar{\partial}_b \partial_b f$ when the $\mathcal{V}$-Hessian is expressed using the orthonormal bases $L_1, \dots, L_n$ and $\omega_1, \dots, \omega_n$ (see [Sh]).

PROPOSITION 2.1. *The $\mathcal{V}$-Hessian of a real-valued f defines a Hermitian form on K_z. In particular, if $X \in C^\infty(\mathcal{M}, \mathcal{V})$ and $X(z) \in K_z$, then $H_{\mathcal{V}}(X, X) f(z)$ is real.*

Proof. Observe that if $X = \sum_{i=1}^n x_i L_i$ is a smooth section of L such that $X(p) \in K_p$, then

$$\sum_{i=1}^n c^m_{ij}(p) x_i(p) = 0 \quad \forall j. \tag{2.6}$$

To see (2.6), use the fact that $[X, \bar{L}_j](p) \in \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$ for all j and then expand $[X, \bar{L}_j]$ using (2.5).

Suppose now $X, Y \in C^\infty(\mathcal{M}, \mathcal{V})$ such that $X(p)$ and $Y(p) \in K_p$. Let $X = \sum_{i=1}^n x_i L_i$ and $Y = \sum_{i=1}^n y_i L_i$. We have

$$H_{\mathcal{V}}(X, Y) f = \sum_{i,j} x_i \bar{y}_j L_i(\bar{L}_j f) - \sum_{i,j} \left(\sum_k b^k_{ij} \bar{L}_k f \right) x_i \bar{y}_j \tag{2.7}$$

and

$$\overline{H_{\mathcal{V}}(Y, X) f} = \sum_{i,j} x_i \bar{y}_j \bar{L}_j(L_i f) - \sum_{i,j} \left(\sum_k \bar{b}^k_{ij} L_k f \right) x_j \bar{y}_i. \tag{2.8}$$

Using (2.5) and (2.6), equation (2.7) can be expressed at the point p as

$$H_{\mathcal{V}}(X, Y) f = \sum_{i,j} x_i \bar{y}_j \bar{L}_j(L_i f) - \sum_{i,j} \left(\sum_k a^k_{ij} L_k f \right) x_i \bar{y}_j. \tag{2.9}$$

Next observe that, since $-[L_j, \bar{L}_i]$ is the complex conjugate of $[L_i, \bar{L}_j]$, equation (2.5) yields

$$a^k_{ij} = -\bar{b}^k_{ij}. \tag{2.10}$$

From (2.8), (2.9), and (2.10), we conclude that

$$H_{\mathcal{V}}(X, Y) f(p) = \overline{H_{\mathcal{V}}(Y, X) f}(p).$$

In particular, $H_{\mathcal{V}}(X, X) f(p)$ is real-valued. □

Proposition 2.1 allows us to introduce the following cone $\mathcal{P}_{\mathcal{V}}$ of functions.

DEFINITION 2.3. Let $V \subseteq \mathcal{M}$ be open and let $f \in C^2(V)$ be real-valued. We say that $f \in \mathcal{P}_{\mathcal{V}}(V)$ if $H_{\mathcal{V}}(X, X) f(p) \geq 0$ for any $X \in C^\infty(V, \mathcal{V})$ such that $X(p) \in K_p$ with $p \in V$.

PROPOSITION 2.2. *Suppose that $V \subseteq \mathcal{M}$ is open and that $f \in C^2(V)$ is a CR function. Then $|f|^2 \in \mathcal{P}_{\mathcal{V}}(V)$.*

Proof. We may assume that $\{L_1, \dots, L_n\}$ is a basis of $\mathcal{V}$ over V as before and that (2.5) holds in V. Since $L_i(f) = 0$ for all i, it follows that

$$L_i\bar{L}_j(|f|^2) = (L_i\bar{f})(\bar{L}_j f) + \bar{f}L_i(\bar{L}_j f) \tag{2.11}$$

and

$$\bar{L}_k(|f|^2) = \bar{f}\bar{L}_k(f). \tag{2.12}$$

Using (2.7), (2.11), and (2.12), for any $X \in C^\infty(V, \mathcal{V})$ we have

$$\begin{aligned} H_{\mathcal{V}}&(X,X)(|f|^2) \\ &= \sum_{i,j}(L_i\bar{f})(\bar{L}_j f)x_i\bar{x}_j + \sum_{i,j}\bar{f}L_i(\bar{L}_j f)x_i\bar{x}_j - \sum_{i,j}\Big(\sum_k b_{ij}^k \bar{f}\bar{L}_k(f)\Big)x_i\bar{x}_j \\ &= \Big|\sum_i (L_i\bar{f})x_i\Big|^2 + \sum_{i,j}\bar{f}L_i(\bar{L}_j f)x_i\bar{x}_j - \sum_{i,j}\Big(\sum_k b_{ij}^k \bar{f}\bar{L}_k(f)\Big)x_i\bar{x}_j, \end{aligned} \tag{2.13}$$

where

$$X = \sum_{i=1}^n x_i L_i.$$

Since $L_i f = 0$ for all i, we obtain

$$L_i(\bar{L}_j f) = [L_i, \bar{L}_j]f = \sum_k b_{ij}^k \bar{L}_k(f) + \sum_s c_{ij}^s T_s(f). \tag{2.14}$$

We now plug (2.14) into (2.13), assume $X(p) \in K_p$, and use (2.6) to conclude that

$$H_{\mathcal{V}}(X,X)(|f|^2)(p) = \Big|\sum_{i=1}^n (L_i\bar{f})(p)x_i(p)\Big|^2 \geq 0. \qquad \square$$

We next show that our cone $\mathcal{P}_{\mathcal{V}}$ agrees with that of Sibony when $(\mathcal{M}, \mathcal{V})$ is an embedded CR submanifold of $\mathbb{C}^{n+d}$.

PROPOSITION 2.3. *Let $\mathcal{M} \subseteq \mathbb{C}^{n+d}$ be a generic embedded CR manifold. A real-valued $f \in C^2(\mathcal{M})$ is in $\mathcal{P}$ if and only if $f \in \mathcal{P}_{\mathcal{V}}$.*

Proof. Let $\mathcal{M}$ be defined by $\rho_1 = \cdots = \rho_d = 0$ near $z_o \in \mathcal{M}$, where

$$\frac{\partial \rho_k}{\partial z_{n+l}}(z_o) = \frac{\delta_{lk}}{2i} \quad \text{for } k, l = 1, \ldots, d.$$

Then $\mathcal{V}$ has a basis near z_o of the form

$$L_i = \frac{\partial}{\partial \bar{z}_i} + \sum_{k=1}^d A_{ik}\frac{\partial}{\partial \bar{z}_{n+k}}, \quad 1 \leq i \leq n,$$

where

$$A_{ik}(z_o) = 0.$$

Pick a Hermitian metric for $\mathbb{C}T\mathcal{M}$ such that $\{L_1, \ldots, L_n\}$ is orthonormal and $\mathcal{V}$ is orthogonal to $\bar{\mathcal{V}}$. Let $X = \sum_i x_i L_i \in C^\infty(\mathcal{M}, \mathcal{V})$ with $X(z_o) \in K_{z_o}$. The proposition will follow if we can show that, for $f \in C^2$ near z_o and $\tilde{f}$ a real-valued C^2 extension of f in a neighborhood in $\mathbb{C}^{n+d}$ of z_o,

$$H_{\mathcal{V}}(X,X)f(z_o) = \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{f}}{\partial \bar{z}_j \partial z_i}(z_o) x_j(z_o) \overline{x_i(z_o)}. \tag{2.15}$$

Computing $H_{\mathcal{V}}(X,X)f = X(\bar{X}f) - \bar{p}([X,\bar{X}])f$ yields

$$X(\bar{X}f) = \sum_{i,j=1}^{n} L_j(\bar{L}_i f) x_j \bar{x}_i + \sum_{i,j=1}^{n} x_j L_j(\bar{x}_i)\bar{L}_i f \tag{2.16}$$

and

$$[X,\bar{X}] = \sum_{i,j} x_i L_i(\bar{x}_j)\bar{L}_j + \sum_{i,j} x_i \bar{x}_j [L_i, \bar{L}_j] - \sum_{i,j} \bar{x}_j \bar{L}_j(x_i) L_i. \tag{2.17}$$

We observe that $[L_i, \bar{L}_j]$ is in the span of $\left\{\frac{\partial}{\partial z_{n+k}}, \frac{\partial}{\partial \bar{z}_{n+k}} : 1 \le k \le d\right\}$ but, at z_o, $\bar{\mathcal{V}}$ is spanned by $\left\{\frac{\partial}{\partial z_i}\Big|_{z_o} : 1 \le i \le n\right\}$. Hence,

$$\bar{p}([X,\bar{X}](z_o)) = \sum_{i,j} x_i(z_o) L_i(\bar{x}_j)(z_o) \bar{L}_j\Big|_{z_o}. \tag{2.18}$$

With $\tilde{f}$ an extension of f and L_j being tangent to $\mathcal{M}$, we then use (2.16), (2.17), and (2.18) to obtain

$$\begin{aligned} H_{\mathcal{V}}(X,X)f(z_0) &= \sum_{i,j=1}^{n} L_j(\bar{L}_i \tilde{f})(z_o) x_j(z_o) \overline{x_i(z_o)} \\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial \bar{z}_j} \left(\frac{\partial \tilde{f}}{\partial z_i} + \sum_{k=1}^{d} \bar{A}_{ik} \frac{\partial \tilde{f}}{\partial z_{n+k}} \right)(z_o) x_j(z_o) \overline{x_i(z_o)} \\ &= \frac{\partial^2 \tilde{f}}{\partial \bar{z}_j \partial z_j}(z_o) x_j(z_o) \overline{x_i(z_o)} \\ &\quad + \sum_{k=1}^{d} \sum_{i,j=1}^{n} \frac{\partial \bar{A}_{ik}}{\partial \bar{z}_j}(z_o) \frac{\partial \tilde{f}}{\partial z_{n+k}}(z_o) x_j(z_o) \overline{x_i(z_o)}, \end{aligned} \tag{2.19}$$

where we have used $A_{ik}(z_o) = 0$.

Consider the last term in (2.19). Differentiating $\bar{L}_j \rho_k = 0$ yields

$$-\frac{\partial^2 \rho_k}{\partial \bar{z}_i \partial z_j}(z_o) = \sum_{l=1}^{d} \frac{\partial \bar{A}_{jl}}{\partial \bar{z}_i}(z_o) \frac{\partial \rho_k}{\partial z_{n+l}}(z_o),$$

in which we use $\frac{\partial \rho_k}{\partial z_{n+l}}(z_o) = \frac{\delta_{lk}}{2i}$ to get

$$\frac{\partial \bar{A}_{ik}}{\partial \bar{z}_j}(z_o) = -2i \frac{\partial^2 \rho_k}{\partial \bar{z}_j \partial z_i}(z_o),$$

so that

$$\sum_{i,j=1}^{n} \frac{\partial \bar{A}_{ik}}{\partial \bar{z}_j}(z_o) x_j(z_o) \overline{x_i(z_o)} = -2i \sum_{i,j=1}^{n} \frac{\partial^2 \rho_k}{\partial \bar{z}_j \partial z_i}(z_o) x_j(z_o) \overline{x_i(z_o)}.$$

Since $X(z_o) \in K_{z_o}$, it follows that this sum equals zero and hence (2.19) simplifies to (2.15) as claimed. $\square$

The proof of Theorem B, Corollary C, and Corollary D in [Si] exploit the function $|z|^2$ or, more generally, the existence of a strictly plurisubharmonic function on a neighborhood in $\mathbb{C}^n$ of the embedded CR manifold. In general, examples in Section 4 will show that these results are not valid even on locally embeddable CR manifolds. We shall demonstrate that Theorem B can be generalized if we assume the existence of a function that behaves like a strictly plurisubharmonic function just in the direction of the vectors $\{v \in K_z : K_z \neq \{0\}\}$.

THEOREM 2.1. *Assume that there is a real-valued function $g \in C^2(\mathcal{M})$ such that, whenever $K_z \neq \{0\}$, $H_{\mathcal{V}}(X,X)g(z) > 0$ for all $X \in C^\infty(\mathcal{M},\mathcal{V})$ with $X(z) \in K_z$. Let Ω be a relatively compact and open subset of $\mathcal{M}$, and let*

$$\delta\Omega = \partial\Omega \cup (\sigma \cap \mathcal{M}).$$

Suppose $\varphi, \psi \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\varphi \in \mathcal{P}_{\mathcal{V}}(\Omega)$. Let

(i) $\varphi \leq \psi$ *on* $\delta\Omega$ *and*
(ii) $\det H_{\mathcal{V}}\varphi(z) \geq \det H_{\mathcal{V}}\psi(z)$ *for each* $z \in \Omega\backslash\delta\Omega$.

Then $\varphi \leq \psi$ in Ω.

Here, for a real-valued function f and $z \in \mathcal{M}$ with $K_z \neq \{0\}$, $H_{\mathcal{V}}f(z)$ denotes the Hermitian form $H_{\mathcal{V}}(X,Y)f(z)$ on K_z.

Proof of Theorem 2.1. Suppose there is a $z \in \Omega\backslash(\partial\Omega \cup (\overline{\sigma \cap \Omega}))$ such that $\varphi(z) > \psi(z)$. Then, for some $\varepsilon > 0$, the function $h = \varphi + \varepsilon g - \psi$ attains its maximum on $\bar{\Omega}$ at a point $z_o \in \Omega\backslash(\partial\Omega \cup (\overline{\sigma \cap \Omega}))$. Let $X \in C^\infty(\mathcal{M},\mathcal{V})$ be such that $X(z_o) \in K_{z_o}$. We will show that $H_{\mathcal{V}}(X,X)h(z_o) \leq 0$. Recall that

$$H_{\mathcal{V}}(X,X)h(z_o) = X_{z_o}(\bar{X}h) - \bar{p}([X,\bar{X}]_{z_o})(h).$$

Since h attains its maximum at z_o, it follows that $\bar{p}([X,\bar{X}]_{z_o})(h) = 0$ and so $H_{\mathcal{V}}(X,X)h(z_o) = X_{z_o}(\bar{X}h)$.

Let $X = A + iB$, where A and B are real vector fields. Then

$$H_{\mathcal{V}}(X,X)h(z_o) = A_{z_o}(Ah) + B_{z_o}(Bh) + i[B,A]_{z_o}(h) = A_{z_o}(Ah) + B_{z_o}(Bh).$$

Let $r(t)$ be an integral curve of A with $r(0) = z_o$. Because the function $t \to h(r(t))$ attains its maximum at 0, we have

$$A_{z_o}(Ah) = \frac{d^2}{dt^2}h(r(t))\Big|_{t=0} \leq 0.$$

Likewise, $B_{z_o}(Bh) \leq 0$ and hence

$$H_{\mathcal{V}}(X,X)h(z_o) \leq 0.$$

(Observe that, in this inequality, the fact that $X(z_o) \in K_{z_o}$ is not used.) Therefore,

$$H_{\mathcal{V}}(X,X)\varphi(z_o) + \varepsilon H_{\mathcal{V}}(X,X)g(z_o) \leq H_{\mathcal{V}}(X,X)\psi(z_o).$$

Since $H_{\mathcal{V}}(X,X)\varphi(z_o) \geq 0$ on K_{z_o} and since $H_{\mathcal{V}}(X,X)g(z_o)$ is strictly positive, it follows that

$$\det H_{\mathcal{V}}\varphi(z_o) < \det H_{\mathcal{V}}\psi(z_o),$$

contradicting assumption (ii). Hence $\varphi \leq \psi$ on Ω. □

Section 4 presents examples of CR manifolds that are not even locally embeddable but that admit functions g as in Theorem 2.1.

3. Proofs of Theorems 1 and 2

For Theorem 1, we use the notion of $\mathcal{V}$-convex functions as defined in [Hö] and [Sh].

DEFINITION 3.1. A real-valued function $g \in C^2(\mathcal{M})$ is called *$\mathcal{V}$-convex* if

$$\Re H_{\mathcal{V}}(L,L)g(p) > 0$$

for any $L \in C^\infty(\mathcal{M},\mathcal{V})$ such that $L(p) \neq 0$.

Observe that a $\mathcal{V}$-convex function is in the cone $\mathcal{P}_{\mathcal{V}}$.

Before we prove our next lemma, we recall the concept of a complete set of approximate first integrals $\{f_1,\dots,f_{n+d}\}$ at $p_o \in \mathcal{M}$ (see [Tr, Thm. IV.1.1]). This means that

(1) the f_j are C^∞;
(2) the differentials $df_1,\dots,df_{n+d}$ are $\mathbb{C}$-linearly independent at p_o; and
(3) for any section $L \in C^\infty(\Omega,\mathcal{V})$, $Lf_j = 0$ to infinite order at p_o for all $j = 1,\dots,n+d$.

Since $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$, the forms df_j and their conjugates $d\bar{f_j}$ span the orthogonal subbundle $\mathcal{V}^\perp \subseteq \mathbb{C}T^*\Omega$ near 0. As a result, after reordering the f_k and replacing some of them with if_k, we may assume that

$$df_1 \wedge \dots \wedge df_n \wedge d\bar{f}_1 \wedge \dots \wedge d\bar{f}_n \wedge d\Re f_{n+1} \wedge \dots \wedge d\Re f_{n+d}(p_o) \neq 0$$

and that n is the largest such integer. Moreover, by taking linear combinations of the f_k, we may also assume that

$$d\Im f_k = 0 \ \text{ at } p_o \quad \text{for } \ k \geq n+1.$$

By replacing the f_j with $f_j - f_j(p_o)$, we also get the f_j vanishing at p_o. After contracting Ω about p_o, it follows that the map $P\colon \Omega \to P(\Omega) = \mathcal{N} \subseteq \mathbb{C}^{n+d}$ given by $P = (f_1,\dots,f_{n+d})$ is an embedding.

The submanifold $\mathcal{N}$ is a generic CR submanifold of $\mathbb{C}^{n+d}$. Denote its CR bundle by $\mathcal{V}$. Since the f_j are approximate first integrals at p_o and since $P(p_o) = 0$, the vector spaces $P_*(\mathcal{V}_{p_o})$ and $\mathcal{V}_0$ are equal and, moreover, the bundles $P_*\mathcal{V}$ and $\mathcal{V}$ agree to infinite order at 0.

LEMMA 3.1. *Let $(\mathcal{M},\mathcal{V})$ be an abstract CR manifold and let $\{f_1,\dots,f_{n+d}\}$ be a complete set of approximate first integrals at p_o. Let Ω be a neighborhood of p_o such that $P = (f_1,\dots,f_{n+d})\colon \Omega \to P(\Omega) = \mathcal{N} \subseteq \mathbb{C}^{n+d}$ is an embedding. Let*

h be a C^2 function near p_o in Ω. Then there exists a C^2 function H defined on a neighborhood of $P(p_o)$ in $\mathbb{C}^{n+d}$ such that

(i) *$h = H \circ P$ near p_o and*
(ii) *$\bar{\partial} H$ vanishes to second order at $P(p_o)$.*

Proof. We may assume that $P(p_0) = 0$. Let $\tilde{h}$ be a C^2 function near 0 in $\mathbb{C}^{n+d}$ such that $\tilde{h} \circ P = h$. We begin by observing that, if $\mathcal{N}$ near 0 is defined by $\rho_1 = \cdots = \rho_d = 0$ and $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d(0) \neq 0$, then it follows that, since $L\tilde{h}(0) = 0$ for all $L \in C^\infty(\mathcal{N}, \mathcal{V})$, there exist constants $\lambda_j \in \mathbb{C}$ such that

$$\bar{\partial}\tilde{h}(0) = \sum_{j=1}^{d} \lambda_j \bar{\partial}\rho_j(0). \tag{3.1}$$

Let $\{u_1, \dots, u_n\}$ be $(0,1)$-forms near 0 such that $\{\bar{\partial}\rho_1, \dots, \bar{\partial}\rho_d, u_1, \dots, u_n\}$ forms a basis of the $(0,1)$-forms in $\mathbb{C}^{n+d}$ near 0. Then (3.1) implies the existence of smooth functions g_j and f_j such that, near 0,

$$\bar{\partial}\tilde{h}(z) = \sum_{j=1}^{d} f_j(z)\bar{\partial}\rho_j(z) + \sum_{j=1}^{n} g_j(z)u_j(z), \tag{3.2}$$
$$f_j(0) = \lambda_j \quad \text{and} \quad g_k(0) = 0.$$

Let $h_1(z) = \tilde{h}(z) - \sum_{j=1}^{d} f_j(z)\rho_j(z)$. Note that $h_1|_{\mathcal{N}} = \tilde{h}$ and

$$\bar{\partial}h_1(z) = \bar{\partial}\tilde{h}(z) - \sum_{j=1}^{d} f_j(z)\bar{\partial}\rho_j(z) - \sum_{j=1}^{d} \rho_j(z)\bar{\partial}f_j(z). \tag{3.3}$$

From (3.2) we see that

$$\bar{\partial}h_1(0) = 0 \tag{3.4}$$

and thus h_1 improves upon $\tilde{h}$. To gain a further improvement, define

$$h_2 = h_1 + \sum_{j,k=1}^{d} g_{j,k}\rho_j\rho_k \tag{3.5}$$

for some $g_{j,k}$ to be determined. Using (3.2) and (3.3), we obtain

$$\bar{\partial}h_2 = \sum_{j=1}^{n} g_j u_j - \sum_{j=1}^{d} \rho_j \bar{\partial} f_j + \sum_{j,k=1}^{d} \rho_k g_{j,k}\bar{\partial}\rho_j + \sum_{j,k=1}^{d} \rho_j g_{j,k}\bar{\partial}\rho_k + O(2), \tag{3.6}$$

where $O(2)$ denotes a term that vanishes to second order at 0. Next we apply $\bar{\partial}$ to (3.2), which yields

$$0 = \bar{\partial}^2\tilde{h} = \sum_{j=1}^{d} \bar{\partial}f_j \wedge \bar{\partial}\rho_j + \sum_{j=1}^{n} \bar{\partial}g_j \wedge u_j + \sum_{j=1}^{n} g_j\bar{\partial}u_j. \tag{3.7}$$

Since $g_j(0) = 0$ for all j and since the $\bar{\partial}\rho_k$ annihilate $\mathcal{V}$, (3.7) implies that $\bar{\partial}g_j(L_0) = 0$ for all j and for all $L_0 \in \mathcal{V}_0$. That is, the forms $\bar{\partial}g_j(0)$ are in the span of $\{\bar\partial\rho_1(0),\dots,\bar\partial\rho_d(0)\}$ and so there exist smooth functions a_{jk} and b_{jm} such that, near 0 and for each $j=1,\dots,n$,

$$\bar\partial g_j(z)=\sum_{k=1}^{d}a_{jk}(z)\bar\partial\rho_k(z)+\sum_{m=1}^{n}b_{jm}(z)u_m(z), \tag{3.8}$$

where $b_{mj}(0)=0$ for all m,j.

We now plug (3.8) into (3.7) to arrive at

$$0=\sum_{j=1}^{d}\bar\partial f_j\wedge\bar\partial\rho_j+\sum_{j=1}^{n}\sum_{k=1}^{d}a_{jk}\bar\partial\rho_k\wedge u_j+\sum_{j=1}^{n}\sum_{m=1}^{n}b_{jm}u_m\wedge u_j+\sum_{j=1}^{n}g_j\bar\partial u_j. \tag{3.9}$$

In particular, at 0 we have

$$\sum_{k=1}^{d}\left(\bar\partial f_k(0)-\sum_{j=1}^{n}a_{jk}(0)u_j(0)\right)\wedge\bar\partial\rho_k(0)=0. \tag{3.10}$$

For each $k=1,\dots,d$, let

$$\bar\partial f_k-\sum_{j=1}^{n}a_{jk}u_j=\sum_{j=1}^{d}c_{kj}\bar\partial\rho_j+\sum_{s=1}^{n}d_{ks}u_s. \tag{3.11}$$

Plugging (3.11) into (3.10) and using the linear independence of the forms $\bar\partial\rho_1,\dots,\bar\partial\rho_d$ and $u_1,\dots,u_n$ shows that

$$c_{kj}(0)-c_{jk}(0)=0\quad\text{and}\quad d_{ks}(0)=0. \tag{3.12}$$

Next, plug the formula for $\bar\partial f_k$ from (3.11) into (3.6) to get:

$$\begin{aligned}\bar\partial h_2=&\sum_{j=1}^{n}g_ju_j-\sum_{j=1}^{n}\sum_{k=1}^{d}\rho_ka_{jk}u_j-\sum_{j,k=1}^{d}\rho_kc_{kj}\bar\partial\rho_j-\sum_{k=1}^{d}\sum_{s=1}^{n}\rho_kd_{ks}u_s\\&+\sum_{j,k=1}^{d}g_{jk}\rho_k\bar\partial\rho_j+\sum_{j,k=1}^{d}g_{jk}\rho_j\bar\partial\rho_k+O(2).\end{aligned} \tag{3.13}$$

Recall that $d_{ks}(0)=0$ and so the term $\sum_{k,s}\rho_kd_{ks}u_s=O(2)$, which leads to

$$\bar\partial h_2=\sum_{j=1}^{n}\left(g_j-\sum_{k=1}^{d}\rho_ka_{jk}\right)u_j+\sum_{k}\left(\sum_{j}(g_{kj}+g_{jk}-c_{jk})\rho_j\right)\bar\partial\rho_k+O(2). \tag{3.14}$$

We now set $g_{kj}=c_{kj}/2$. The equation $c_{kj}(0)=c_{jk}(0)$ in (3.12) then implies that

$$\bar\partial h_2=\sum_{j=1}^{n}\eta_ju_j+O(2), \tag{3.15}$$

where each

$$\eta_j=g_j-\sum_{k=1}^{d}\rho_ka_{jk}. \tag{3.16}$$

Clearly, $\eta_j(0) = 0$ for all j. Next we show that $\bar\partial\eta_j(0) = 0$ for all j. We have

$$\bar\partial\eta_j = \bar\partial g_j - \sum_{k=1}^{d} a_{jk}\bar\partial\rho_k - \sum_{k=1}^{d} \rho_k\bar\partial a_{jk} = \sum_{m=1}^{n} b_{jm}u_m - \sum_{k=1}^{d} \rho_k\bar\partial a_{jk}, \tag{3.17}$$

where (3.8) was used in the last equality. It follows that $\bar\partial\eta_j(0) = 0$ for all j.

We may assume that each $u_j = \bar\partial F_j$, where the F_j are C^∞ near 0 and where $F_j(0) = 0$ for all j. Then we can write (recall (3.15))

$$\bar\partial h_2 = \sum_{j=1}^{n} \eta_j\bar\partial F_j + O(2).$$

Define $h_3 = h_2 - \sum_{j=1}^{n}\eta_j F_j$. Then $h_3 \circ P = h$ near p_o in Ω and

$$\bar\partial h_3 = -\sum_{j=1}^{n} F_j\bar\partial\eta_j + O(2).$$

Since each $F_j(0) = 0$ and $\bar\partial\eta_j(0) = 0$, we have shown that $\bar\partial h_3$ vanishes to second order at 0. Then $H = h_3$ is as desired.

Proof of Theorem 1. Suppose that, for some $\varepsilon > 0$, the function $h = |f|^2 + \varepsilon g$ attains its maximum on $\bar\Omega$ at some $z_o \in \Omega$. We will show that $z_o \in \Sigma$. The theorem will follow from this.

Choose a complete set of approximate first integrals $\{f_1, \dots, f_{n+d}\}$ at z_o as in the proof of Lemma 3.1, so that $f_j(z_o) = 0$,

$$df_1 \wedge \cdots \wedge df_n \wedge d\bar f_1 \wedge \cdots \wedge d\bar f_n \wedge d\Re f_{n+1} \wedge \cdots \wedge d\Re f_{n+d}(z_o) \neq 0,$$

and

$$d(\Im f_{n+k}(z_o)) = 0 \quad \text{for } 1 \le k \le d.$$

After contracting Ω about z_o, recall that the map $P = (f_1, \dots, f_{n+d})\colon \Omega \to P(\Omega) = \mathcal{N} \subseteq \mathbb{C}^{n+d}$ is an embedding and, if $\mathcal{V}$ is the CR bundle on $\mathcal{N}$, then the bundles $P_*\mathcal{V}$ and $\mathcal{V}$ agree to infinite order at 0. We may assume $\mathcal{N}$ is defined near 0 by

$$\rho_1 = \cdots = \rho_d = 0,$$

where

$$\frac{\partial\rho_k}{\partial z_{n+l}}(0) = \frac{\delta_{lk}}{2i} \quad \text{for } k, l = 1, \dots, d. \tag{3.18}$$

Then $\mathcal{V}$ has a basis near 0 of the form

$$L_i = \frac{\partial}{\partial\bar z_i} + \sum_{k=1}^{d} A_{ik}\frac{\partial}{\partial\bar z_{n+k}}, \quad 1 \le i \le n,$$

where

$$A_{ik}(0) = 0.$$

Let $\{L_1', \dots, L_n'\}$ be a basis of $\mathcal{V}$ near z_o such that

$$P_*(L'_j|_{z_o}) = L_j|_0, \quad 1 \le j \le n.$$

Pick a Hermitian metric for $\mathbb{C}T\mathcal{M}$ such that $\{L'_1, \ldots, L'_n\}$ is orthonormal and $\mathcal{V}$ is orthogonal to $\bar{\mathcal{V}}$ as before. Let a^k_{ij}, b^k_{ij}, and c_{ij} be as before. Let $\tilde{h}$, $\tilde{g}$, and $\tilde{f}$ be defined in a neighborhood of $\mathcal{N}$ in $\mathbb{C}^{n+d}$ so that

$$\tilde{h} \circ P = h, \quad \tilde{g} \circ P = g, \quad \text{and} \quad \tilde{f} \circ P = f.$$

Since $P_*\mathcal{V}$ and $\mathcal{V}$ agree to infinite order at 0, it follows that

$$H_{\mathcal{V}}(L'_i, L'_j)h(z_o) = L_i(\bar{L}_j(\tilde{h}))(0) - \sum_k b^k_{ij}(z_o)\bar{L}_k(\tilde{h})(0). \tag{3.19}$$

Observe that the commutators

$$[L_i, \bar{L}_j] = \left[\frac{\partial}{\partial \bar{z}_i} + \sum_{k=1}^{d} A_{ik}\frac{\partial}{\partial \bar{z}_{n+k}}, \frac{\partial}{\partial z_j} + \sum_{k=1}^{d} \bar{A}_{jk}\frac{\partial}{\partial z_{n+k}}\right]$$

have no component in $\bar{\mathcal{V}}$, so

$$b^k_{ij}(z_o) = 0 \quad \forall i, j, k. \tag{3.20}$$

Using (3.20) and (3.18), we may simplify (3.19) to

$$H_{\mathcal{V}}(L'_i, L'_j)h(z_o) = \frac{\partial^2 \tilde{h}}{\partial \bar{z}_i \partial z_j}(0) + \sum_{k=1}^{d} \frac{\partial \bar{A}_{jk}}{\partial \bar{z}_i}(0)\frac{\partial \tilde{h}}{\partial z_{n+k}}(0).$$

Hence, if $L = \sum_{i=1}^n \bar{u}_i L'_i$ for some $(u_1, \ldots, u_n) \in \mathbb{C}^n$ then

$$\begin{aligned} H_{\mathcal{V}}(L, L)h(0) &= \sum_{i,j} H_{\mathcal{V}}(L'_i, L'_j)h\bar{u}_i u_j \\ &= \sum_{i,j} \frac{\partial^2 \tilde{h}}{\partial \bar{z}_i \partial z_j}(0)\bar{u}_i u_j + \sum_{k=1}^{d}\sum_{i,j} \frac{\partial \bar{A}_{jk}}{\partial \bar{z}_i}(0)\bar{u}_i u_j \frac{\partial \tilde{h}}{\partial z_{n+k}}(0). \end{aligned} \tag{3.21}$$

Since

$$\frac{\partial \bar{A}_{jk}}{\partial \bar{z}_i}(0) = -2i\frac{\partial^2 \rho}{\partial \bar{z}_i \partial z_j}(0),$$

as we saw in the proof of Proposition 2.3, we can express (3.21) as

$$H_{\mathcal{V}}(L, L)h(0) = \sum_{i,j} \frac{\partial^2 \tilde{h}}{\partial \bar{z}_i \partial z_j}(0)\bar{u}_i u_j - 2i\sum_{k=1}^{d} \frac{\partial \tilde{h}}{\partial z_{n+k}}(0)\langle \mathcal{L}_0(\rho_k)u, u\rangle. \tag{3.22}$$

Likewise, we have

$$\begin{aligned} &H_{\mathcal{V}}(L, L)(|f|^2)(0) \\ &\qquad = \sum_{i,j} \frac{\partial^2 |\tilde{f}|^2}{\partial \bar{z}_i \partial z_j}(0)\bar{u}_i u_j - 2i\sum_{k=1}^{d} \frac{\partial |\tilde{f}|^2}{\partial z_{n+k}}(0)\langle \mathcal{L}_0(\rho_k)u, u\rangle. \end{aligned} \tag{3.23}$$

Since f is a CR function, by Lemma 3.1 we may assume that $\bar{\partial}\tilde{f} = 0$ at 0 to second order, which implies that

$$\sum_{i,j} \frac{\partial^2 |\tilde{f}|^2}{\partial \bar{z}_i \partial z_j}(0)\bar{u}_i u_j \geq 0. \tag{3.24}$$

Since $h = |f|^2 + \varepsilon g$, we have

$H_{\mathcal{V}}(L, L)h(z_o)$

$$= -2i \sum_{k=1}^{d} \frac{\partial |\tilde{f}|^2}{\partial z_{n+k}}(0)\langle \mathcal{L}_0(\rho_k)u, u\rangle + \sum_{i,j} \frac{\partial^2 |\tilde{f}|^2}{\partial \bar{z}_i \partial z_j}(0)\bar{u}_i u_j + \varepsilon H_g(L, L).$$

Suppose the vector $L(0) = \sum_{i=1}^{n} \bar{u}_i L_i|_0 \neq 0$. Then $\Re H_g(L, L)(z_o) > 0$ by hypothesis, and the proof of Theorem 2.1 shows that, since h has a maximum at z_o,

$$H_{\mathcal{V}}(L, L)h(z_o) \leq 0. \tag{3.25}$$

These observations and (3.24) imply that

$$\sum_{k=1}^{d} \frac{\partial |\tilde{f}|^2}{\partial z_{n+k}}(0)\langle \mathcal{L}_0(\rho_k)u, u\rangle \neq 0;$$

therefore, by the equality at 0 to infinite order of $P_*\mathcal{V}$ and $\mathcal{V}$, we conclude that $z_o \in \Sigma$ as desired.

Proof of Theorem 2. Let V be any neighborhood of p. By Lemma 4 in [Sh], there exist a neighborhood $V_1 \subset V$ of p and $g \in C^\infty(V_1)$ such that g is $\mathcal{V}$-convex in V_1. We may assume V_1 to be small enough that

$$\max_{\partial V_1} |h| < |h(p)|. \tag{3.26}$$

By Theorem 1 and (3.26), since V_1 admits a $\mathcal{V}$-convex function,

$$|h(p)| \leq \max_{\Sigma \cap \bar{V}_1} |h|$$

with $\Sigma \cap \bar{V}_1 \neq \emptyset$. Since V can be chosen arbitrarity small, there exists a $p_j \in \mathcal{M}$ such that $(\mathcal{M}, \mathcal{V})$ is strictly definite at p_j and $p_j \to p$.

4. Examples

Our first example will show that, even on real analytic CR manifolds (which are necessarily locally embeddable), the analogues of Theorem B, Corollary C, and Corollary D may not hold. Thus it shows that such manifolds do not admit a function g as in Theorem 1 or Theorem 2.1.

EXAMPLE 4.1. We will use the examples constructed by Kaup and Zaitsev in their recent paper [KZ]. They exhibited a class of compact, globally nonembeddable, real-analytic, strongly pseudoconvex CR manifolds of arbitrary CR codimension that are nontrivial in the sense that they are not locally products of lower-dimensional CR manifolds. The manifolds in this class are finite covers of embedded CR submanifolds of $\mathbb{C}^n$ and therefore all have "many" nonconstant CR functions. Let $\mathcal{M}_1$ be one of these manifolds. Theorems 1 and 2.1 and Corollaries C and D all hold on $\mathcal{M}_1$ for the trivial reason that $\Sigma(\mathcal{M}_1) = \mathcal{M}_1$.

Let $\mathcal{M}_2$ be a real-analytic, Levi-flat CR manifold that is compact, and let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with the CR structure coming from $\mathcal{M}_1$ and $\mathcal{M}_2$. It is clear that $\mathcal{M}$ is locally embeddable but not globally embeddable. Since $\mathcal{M}_2$ is Levi flat, there are no strictly definite points on $\mathcal{M}$; in fact, $\sigma = \emptyset$ on $\mathcal{M}$.

Theorems 1 and 2.1 and Corollaries C and D cannot hold on $\mathcal{M}$ because, for example, Corollary D would imply that every CR function on $\mathcal{M}$ is constant—contrary to the existence of nonconstant CR functions on $\mathcal{M}_1$ and hence on $\mathcal{M}$.

EXAMPLE 4.2. Let $\mathcal{M} = \mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,s) = (z_1, \dots, z_n, s)$. There is a smooth function $a(z,s)$ (see [JTr1; JTr2]) defined on a neighborhood of the origin and vanishing to infinite order at $z_1 = 0$ such that the bundle $\mathcal{V}$ generated by the vector fields

$$L_1 = \frac{\partial}{\partial \bar{z}_1} + iz_1(1 + a(z,s))\frac{\partial}{\partial s}$$

and

$$L_j = \frac{\partial}{\partial \bar{z}_j} - iz_j(1 + a(z,s))\frac{\partial}{\partial s}, \quad 2 \le j \le n,$$

is a CR structure that is not locally embeddable in any neighborhood of the origin.

Let $g(z,s) = \sum_{j=1}^n |z_j|^2$. Equip $\mathcal{V}$ with a Hermitian metric so that the L_j form an orthonormal basis and $\bar{\mathcal{V}}$ is orthogonal to $\mathcal{V}$. For any smooth CR vector field $L = \sum_{j=1}^n c_j(z,s)L_j$, we have

$$H_{\mathcal{V}}(L,L)g = L(\bar{L}g) - \bar{p}([L,\bar{L}])(g) = L\biggl(\sum_j \bar{c}_j(\bar{L}_j g)\biggr) - \sum_j c_j L_j(\bar{c}_j)(\bar{L}_j g);$$

since $[L_j, \bar{L}_j]$ has no component in $\bar{\mathcal{V}}$, it follows that

$$H_{\mathcal{V}}(L,L)g = \sum_{j=1}^n |c_j|^2 L_j(\bar{L}_j g) = \sum_{j=1}^n |c_j|^2.$$

Hence, if $L(p) \ne 0$ then $H_g(L,L)(p) > 0$ and so g is $\mathcal{V}$-convex on $\mathcal{M}$. By Theorem 1 and the fact that the Levi form is never strictly definite on $\mathcal{M}$, CR functions on $\mathcal{M}$ satisfy the following maximum principle: If Ω is a relatively compact and open subset of $\mathcal{M}$ and if $f \in C^2(\Omega) \cap C(\bar{\Omega})$ is CR on Ω, then

$$|f(z)| \le \max_{\partial\Omega} |f| \quad \forall z \in \Omega.$$

EXAMPLE 4.3. Let $(\mathcal{M}, \mathcal{V})$ be an abstract CR manifold. Recall that, for $z \in \mathcal{M}$,

$$K_z = \{u \in \mathcal{V}_z : [U, \bar{L}](z) \in \mathcal{V}_z \oplus \bar{\mathcal{V}}_z \quad \text{for all } C^\infty \text{ sections } U, L \text{ of } \mathcal{V} \text{ such that } U(z) = u\}.$$

Assume that, for some $k \ge 1$, we have $\dim_{\mathbb{C}} K_z = k$ for all $z \in \mathcal{M}$. Define $K = \bigcup_{z\in\mathcal{M}} K_z$. We can easily check that the bundle K is involutive by using Jacobi's identity. Since

$$K \oplus \bar{K} = \{W \in \mathcal{V} \oplus \bar{\mathcal{V}} : [W, \mathcal{V} \oplus \bar{\mathcal{V}}] \subseteq \mathcal{V} \oplus \bar{\mathcal{V}}\},$$

Jacobi's identity also shows that $K \oplus \bar{K}$ is involutive. It follows that $\Re K = \{X + \bar{X} : X \in K\}$ is a real subbundle of $T\mathcal{M}$ of fiber dimension $2k$. By the Frobenius theorem, $\mathcal{M}$ is foliated by the leaves of $\Re K$. If $p \in \mathcal{M}$ and if $\mathcal{S}$ is the leaf through p, then

$$\mathbb{C}T_q\mathcal{S} = K_q \oplus \bar{K}_q \quad \text{and} \quad K_q = \mathcal{V}_q \cap \mathbb{C}T_q\mathcal{S}.$$

By the Newlander–Nirenberg theorem, the pair $(\mathcal{S}, K|_{\mathcal{S}})$ has the structure of a complex manifold of dimension k. We can thus conclude as follows: A real-valued φ on $\mathcal{M}$ is in the cone $\mathcal{P}_{\mathcal{V}}$ if and only if the restriction of φ to each leaf $\mathcal{S}$ is plurisubharmonic.

REMARK 4.1. Suppose that $(\mathcal{M}, \mathcal{V})$ is a locally embeddable CR manifold and that h is a continuous CR function whose modulus peaks at a point p in $\mathcal{M}$. By the Baouendi–Treves approximation theorem [BT], in a sufficiently small neighborhood V of p we can find a sequence of $f_j \in C^\infty(V)$ that are CR such that $f_j \to h$ uniformly on $\bar{V}$. Hence, for m large enough, $|f_m|$ attains its maximum in $\bar{V}$ on a compact subset $K \subseteq V$. That is, for any $z \in \bar{V} \setminus K$,

$$|f_m(z)| < \max_K |f_m|. \tag{4.1}$$

We may assume V is small enough that there is a $g \in C^2(V)$, which is $\mathcal{V}$-convex by [Sh, Lemma 4]. Then, by Theorem 1, since f_m is CR it follows that, for any $z \in V$,

$$|f_m(z)| \le \max_{(\Sigma \cap \bar{V}) \cup \partial V} |f_m|. \tag{4.2}$$

From (4.1) and (4.2), we conclude that $K \subseteq \Sigma \cap \bar{V}$ and hence $p \in \bar{\Sigma}$.

Appendix

Let $\mathcal{M} \subset \mathbb{C}^{n+d}$ be a CR submanifold of the form

$$\mathcal{M} = \{(z, s + i\varphi(z, \bar{z}, s))\},$$

where we denote the coordinates of $\mathbb{C}^{n+d}$ by $(z, w) = (x + iy, s + it)$, φ is a smooth real-valued function defined near the origin with $\varphi(0) = 0$, and $d\varphi(0) = 0$. Let $\rho_j = t_j - \varphi_j(z, \bar{z}, s)$ for $j = 1, \ldots, d$. In [Si], Sibony used the following inequality, which involves the Levi form of $\mathcal{M}$. The inequality played a key role in the proofs of his results.

SIBONY'S LEMMA. *Let θ be a C^2 function defined in a neighborhood V of 0 in $\mathbb{C}^{n+d}$. Suppose $\theta \le 1$ on $\mathcal{M}$ and $\theta(0) = 1$. Then, for all $v \in \mathcal{V}_o(\mathcal{M})$,*

$$\langle L_o(\theta)v, v\rangle \le \sum_{i=1}^{d} \frac{\partial \theta}{\partial t_i}(0)\langle L_o(\rho_i)v, v\rangle.$$

Here we will present this inequality as an equation in a slightly more general setup.

LEMMA A.1. *Let $(\mathcal{N},\mathcal{V}) \subseteq \mathbb{C}^{n+d}$ be a generic CR manifold defined near $0 \in \mathcal{N}$ by $\rho_1 = \cdots = \rho_d = 0$. Let $\Omega \subseteq \mathbb{R}^{2n+d}$ be a neighborhood of 0 and let $P : \Omega \to \mathcal{N}$ be a parameterization, $P(0) = 0$. Suppose $\theta^o : \Omega \to \mathbb{R}$ is a C^2 function for which 0 is a critical point, and suppose $\theta : U \subseteq \mathbb{C}^{n+d} \to \mathbb{R}$ is C^2, where U is a neighborhood of $\mathcal{N}$ in $\mathbb{C}^{n+d}$ and $\theta \circ P = \theta^o$. Then there exist real numbers $a_1, \ldots, a_d$ such that, for any tangent vector $u \in T_0\Omega$ with $P_* u \in \Re\mathcal{V}|_0$,*

$$\langle D^2\theta(0)P_*u, P_*u\rangle = \langle D^2\theta^o(0)u, u\rangle + \sum_{i=1}^{d} a_i \langle \mathcal{L}_0\rho_i P_*u, P_*u\rangle. \qquad \text{(A.1)}$$

REMARK A.1. The term $\langle \mathcal{L}_0\rho P_*u, P_*u\rangle$ is to be understood in the following sense. The real tangent vector $P_*u \in T_0\mathbb{C}^{n+d}$ has a complex vector realization $w \in \mathbb{C}^{n+d}$—once we identify $T_0\mathbb{C}^{n+d} = \mathbb{R}^{2(n+d)}$ with $\mathbb{C}^n$ by means of

$$\begin{aligned}\sum_{j=1}^{n+d} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n+d} b_j \frac{\partial}{\partial y_j} &\mapsto (a_1, b_1, \ldots, a_{n+d}, b_{n+d}) \\ &\mapsto (a_1 + ib_1, \ldots, a_{n+d} + ib_{n+d}).\end{aligned}$$

With this convention,

$$\langle \mathcal{L}_0\rho P_*u, P_*u\rangle = \sum_{i,j=1}^{n+d} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(0) w_i \bar{w}_j.$$

In the statement of the lemma,

$$\Re\mathcal{V} = \{L + \bar{L} : L \in C^\infty(\mathcal{N},\mathcal{V})\}.$$

Proof of Lemma A.1. Let $A : \mathbb{C}^{n+d} \to \mathbb{C}^{n+d}$ be a $\mathbb{C}$-linear isomorphism such that, near 0,

$$\mathcal{M} = A(\mathcal{N}) = \{(x + iy, s + i\varphi(x, y, s))\},$$

where $z = x + iy \in \mathbb{C}^n$, $s \in \mathbb{R}^d$, and $\varphi = (\varphi_1, \ldots, \varphi_d)$ is real-valued and C^∞ with

$$\varphi(0) = 0 \quad \text{and} \quad d\varphi(0) = 0.$$

Let Ω' be a neighborhood of 0 in $\mathbb{R}^{2n+d}$ and define $P' : \Omega' \to A(\mathcal{N})$ by

$$P'(x_1, y_1, \ldots, x_n, y_n, s_1, \ldots, s_d) = (x + iy, s + i\varphi(x, y, s)).$$

Then there is a diffeomorphism $R : \Omega' \to \Omega$ with $R(0) = 0$ such that $P' = A \circ P \circ R$. Let $B = A^{-1}$, $\eta^o = \theta^o \circ R$, and $\eta = \theta \circ B$. Then $\eta^o = \eta \circ P'$.

Differentiating the equation

$$\begin{aligned}&\eta^o(x_1, y_1, \ldots, x_n, y_n, s_1, \ldots, s_d) \\ &\qquad = \eta(x_1, y_1, \ldots, x_n, y_n, s_1, \ldots, s_d, \varphi_1(x, y, s), \ldots, \varphi_d(x, y, s)),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\partial^2\eta^o}{\partial x_j \partial x_k} &= \frac{\partial^2\eta}{\partial x_j \partial x_k} + \sum_{l=1}^{d} \frac{\partial^2\eta}{\partial x_j \partial t_l}\frac{\partial\phi_l}{\partial x_k} + \sum_{l=1}^{d} \frac{\partial\eta}{\partial t_l \partial x_k}\frac{\partial\phi_l}{\partial x_j} \\ &\quad + \sum_{l,m}^{d} \frac{\partial^2\eta}{\partial t_l \partial t_m}\frac{\partial\phi_l}{\partial x_k}\frac{\partial\phi_m}{\partial x_j} + \sum_{l=1}^{d} \frac{\partial\eta}{\partial t_l}\frac{\partial^2\phi_l}{\partial x_j \partial x_k},\end{aligned}$$

and similarly for $\frac{\partial^2\eta^o}{\partial x_j \partial y_k}$ and $\frac{\partial^2\eta^o}{\partial y_j \partial y_k}$.

At 0 we have $d\phi_j(0) = 0$ for all $j = 1, \dots, d$. Hence, at 0,

$$\frac{\partial^2 \eta^o}{\partial x_j \partial x_k} = \frac{\partial^2 \eta}{\partial x_j \partial x_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 \phi_l}{\partial x_j \partial x_k}, \tag{A.2}$$

and similarly

$$\frac{\partial^2 \eta^o}{\partial x_j \partial y_k} = \frac{\partial^2 \eta}{\partial x_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 \phi_l}{\partial x_j \partial y_k},$$

$$\frac{\partial^2 \eta^o}{\partial y_j \partial y_k} = \frac{\partial^2 \eta}{\partial y_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 \phi_l}{\partial y_j \partial y_k}.$$

Let $r_m(x, y, s, t) = t_m - \varphi_m(x, y, s)$ for $1 \leq m \leq d$. We apply (A.2) to the function $\eta = r_m$. Since $r_m^o = r_m|_{A(\mathcal{N})} = 0$, at 0 we have

$$\frac{\partial^2 r_m^o}{\partial x_j \partial x_k} = \frac{\partial^2 r_m}{\partial x_j \partial x_k} + \sum_{l=1}^{d} \frac{\partial r_m}{\partial t_l} \frac{\partial^2 \phi_l}{\partial x_j \partial x_k}$$

and also

$$\frac{\partial^2 r_m^o}{\partial x_j \partial y_k} = \frac{\partial^2 r_m}{\partial x_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial r_m}{\partial t_l} \frac{\partial^2 \phi_l}{\partial x_j \partial y_k},$$

$$\frac{\partial^2 r_m^o}{\partial y_j \partial y_k} = \frac{\partial^2 r_m}{\partial y_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial r_m}{\partial t_l} \frac{\partial^2 \phi_l}{\partial y_j \partial y_k}.$$

Since $r_m = t_m - \varphi_m(x, y, s)$, we have $\frac{\partial r_m}{\partial t_l} = \delta_{ml}$. Hence

$$-\frac{\partial^2 r_m}{\partial x_j \partial x_k}(0) = \frac{\partial^2 \phi_m}{\partial x_j \partial x_k}(0),$$

and similarly

$$-\frac{\partial^2 r_m}{\partial x_j \partial y_k}(0) = \frac{\partial^2 \phi_m}{\partial x_j \partial y_k}(0) \quad \text{and} \quad -\frac{\partial^2 r_m}{\partial y_j \partial y_k}(0) = \frac{\partial^2 \phi_m}{\partial y_j \partial y_k}(0).$$

These equations, together with (A.2), lead to the following equations at 0:

$$\frac{\partial^2 \eta}{\partial x_j \partial x_k} = \frac{\partial^2 \eta^o}{\partial x_j \partial x_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 r_l}{\partial x_j \partial x_k}, \tag{A.3}$$

$$\frac{\partial^2 \eta}{\partial x_j \partial y_k} = \frac{\partial^2 \eta^o}{\partial x_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 r_l}{\partial x_j \partial y_k},$$

$$\frac{\partial^2 \eta}{\partial y_j \partial y_k} = \frac{\partial^2 \eta^o}{\partial y_j \partial y_k} + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} \frac{\partial^2 r_l}{\partial y_j \partial y_k}.$$

Let v be a tangent vector at 0 in Ω' that is in the span of $\left\{ \frac{\partial}{\partial x_j}\Big|_0, \frac{\partial}{\partial y_j}\Big|_0, 1 \leq j \leq n \right\}$. For such v, by using the interpretation stated in Remark A.1 and (A.3) we obtain, at 0,

$$\langle D^2\eta(P'_*v), P'_*v\rangle = \langle D^2\eta^o v, v\rangle + \sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l}\langle \mathcal{L}_0(r_l)P'_*v, P'_*v\rangle. \tag{A.4}$$

Because the functions $r_l \circ A$ $(1 \le l \le d)$ define $\mathcal{N}$ near 0, we can find C^∞ functions c_{lk} such that

$$r_l \circ A = \sum_{k=1}^{d} c_{lk}\rho_k \quad \text{for } 1 \le l \le d.$$

By invariance of the Levi form under biholomorphic maps, this implies that

$$\begin{aligned}\langle \mathcal{L}_0(r_l)P'_*v, P'_*v\rangle &= \langle \mathcal{L}_0(r_l \circ A)(BP'_*v), BP'_*v\rangle \\ &= \sum_k c_{lk}\langle \mathcal{L}_0(\rho_k)(BP'_*v), BP'_*v\rangle. \end{aligned}\tag{A.5}$$

From (A.4) and (A.5) it follows that

$$\langle D^2\eta(P'_*v), P'_*v\rangle = \big\langle D^2\eta^o v, v\big\rangle + \sum_{k=1}^{d}\left(\sum_{l=1}^{d} \frac{\partial \eta}{\partial t_l} c_{lk}\right)\langle \mathcal{L}_0(\rho_k)BP'_*v, BP'_*v\rangle. \tag{A.6}$$

Observe next that $BP'_*v = P_*R_*v$ and, since θ^o has a critical point at 0,

$$\langle D^2\eta^o v, v\rangle = \langle D^2\theta^o(R_*v), R_*v\rangle.$$

Moreover, since $\eta = \theta \circ B$, we have

$$\langle D^2\eta(P'_*v), P'_*v\rangle = \langle D^2\theta(P_*(R_*v)), (P_*(R_*v))\rangle.$$

These observations and (A.6) establish (A.1).

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