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Limiting Weak-type Behavior for the Riesz Transform and Maximal Operator When $\lambda \to \infty$

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1. Introduction

The goal of this paper is to analyze the limiting weak-type behavior of important operators in harmonic analysis when they act on singular measures in $\mathbb{R}^n$. Consider the jth Riesz transform R_j defined on appropriate functions by

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}\, \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy. \tag{1.1}$$

Here R_j is bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$ and from $L^1(\mathbb{R}^n)$ into the weak-L^1 space $L^{1,\infty}(\mathbb{R}^n)$. That is, there exist constants C_p for each $1 < p < \infty$ and C_1 such that, for all functions $f \in L^p(\mathbb{R}^n)$,

$$\|R_j f\|_p \le C_p \|f\|_p; \tag{1.2}$$

moreover, for all $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$,

$$\lambda m\{x \in \mathbb{R}^n : |R_j f(x)| > \lambda\} \le C_1 \|f\|_1. \tag{1.3}$$

These are referred to as the strong-type (p, p) and weak-type $(1, 1)$ inequalities, respectively. See Stein [12] for the basic theory.

The strong-type (p, p) constant C_p is

$$C_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \le 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \le p < \infty. \end{cases}$$

This is proved by Pichorides [11] for $n = 1$ and completed by Iwaniec and Martin [5] for higher dimensions. When $n = 1$, the weak-type constant C_1 is

$$C_1 = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.$$

This is proved by Davis [2] and Baernstein [1]. However, for higher dimensions the question remains open. One conjecture regarding the weak-type constant is that it is independent of dimension n. A recent result [6] proved by the present

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author is that the constant C_1 is at worst logarithmic with respect to n. The proof entails a modification of the Calderón–Zygmund theory; however, an observation leads to the following precise limit result, proved in [7] for singular integral operators of convolution type.

THEOREM 1.1. *Let $f \in L^1(\mathbb{R}^n)$. Then*

$$\lim_{\lambda \to 0} \lambda m\{x \in \mathbb{R}^n : |R_j f(x)| > \lambda\} = \frac{2}{\pi n} \left| \int_{\mathbb{R}^n} f(x)\, dx \right|. \tag{1.4}$$

It is easy to check that the limit is 0 when $\lambda \to \infty$. However, as shown in this paper, the situation changes dramatically when the Riesz transforms are considered as operators on singular measures. In particular, the next theorem is proved in Section 2. The notations and facts needed are as follows:

1. Let k be an integer less than n, and let ν denote a singular measure supported on a k-dimensional Lipschitz surface Γ (or a countable union of surfaces). In general, $d\nu = f d\mathcal{H}^k_\Gamma$, where $\mathcal{H}^k_\Gamma$ is the k-dimensional Hausdorff measure restricted to Γ.
2. A countable collection of sets $\{A_i\}_i$ in $\mathbb{R}^n$ is said to be $\mathcal{H}^k$*-disjoint* if $\mathcal{H}^k(A_i \cap A_j) = 0$ whenever $i \neq j$.
3. An arbitrary Lipschitz surface is locally a Lipschitz graph and hence may be decomposed into a countable union of graphs that are $\mathcal{H}^k$-disjoint.
4. The constant in front of the Riesz transform integral is denoted by C_n.
5. The Riesz transform is defined on the measure ν by
$$R_j \nu(x) = C_n \int_\Gamma \frac{x_j - y_j}{|x-y|^n}\, d\nu(y).$$
Observe that, when $k < n$, this is a usual integral and is not principle valued.
6. The vector Riesz transform R is defined as $R\nu(x) = (R_1\nu(x), \dots, R_n\nu(x))$.

THEOREM 1.2. *Let $\Gamma = \bigcup_i \Gamma_i$ be the countable union of k-dimensional Lipschitz surfaces that are $\mathcal{H}^k$-disjoint. Let $d\nu_i = f_i d\mathcal{H}^k_{\Gamma_i}$, where $f_i \in L^1(\Gamma_i)$, be a measure supported on Γ_i satisfying $\|\nu\|_1 = \sum_i \|\nu_i\|_1 < \infty$. Then*

$$\lim_{\lambda \to \infty} \lambda m\{x \in \mathbb{R}^n : |R\nu(x)| > \lambda\} = C_{n,k} \|\nu\|_1, \tag{1.5}$$

where

$$C_{n,k} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n-k+2}{2}\right)}. \tag{1.6}$$

Thus the behavior of the weak-type inequality at both ends ($\lambda = 0$ and $\lambda = \infty$) are now precisely known (for certain classes of measures). The motivation for Theorem 1.2 comes from a theorem of Varopolous [14].

THEOREM 1.3. *There exists a $c > 0$ such that, for any finite Radon measure ν,*

$$\liminf_{\lambda \to \infty} \lambda m\{x \in \mathbb{R}^n : |R\nu(x)| > \lambda\} \geq c \|\nu_s\|_1, \tag{1.7}$$

where ν_s is the singular part of ν.

Varopolous's theorem is true for a larger class of measures. Thus the question for exact constant c remains open in the general case although (1.6) is one candidate at least when the dimension of ν is k. The techniques used to prove Theorem 1.2 depend on the tangency properties available for Lipschitz surfaces and hence cannot be directly extended to arbitrary measures.

On the other hand, they are not specific to the Riesz transform and can be used to prove analogous results for other operators that occur in harmonic analysis. The key property required is that an operator T map to a function $T\nu$ that has a specific approach rate near the surface Γ. In general, let $\varphi\colon [0,\infty) \to [0,\infty)$ be an increasing surjective function and let T satisfy two conditions as follows.

1. The weak-type inequality:

$$\sup_{\lambda>0} \varphi(\lambda) m\{x \in \mathbb{R}^n : |T\nu(x)| > \lambda\} \leq C_1 \int_\Gamma \varphi(|f(x_0)|)\, d\mathcal{H}_\Gamma(x_0), \tag{1.8}$$

with C_1 depending only on n and k and not on Γ.

2. The limit equality: for a.e. $x_0 \in \Gamma$,

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} \varphi(|T\nu(x_0 + t\vec{n})|) = C \frac{\varphi(|f(x_0)|)}{(\sin\Theta)^{n-k}}, \tag{1.9}$$

where Θ is the smallest angle that $\vec{n}$ (see Section 2(2)) makes with the tangent space at x_0 and where γ_{n-k} is the volume of the unit ball in $\mathbb{R}^{n-k}$.

Then

$$\lim_{\lambda\to\infty} \varphi(\lambda) m\{x \in \mathbb{R}^n : |T\nu(x)| > \lambda\} = C \int_\Gamma \varphi(|f(x_0)|)\, d\mathcal{H}_\Gamma(x_0). \tag{1.10}$$

The Riesz transform and the Hardy–Littlewood maximal operator (described after Remark 1.1) satisfy conditions (1.8) and (1.9) with $\varphi(\lambda) = \lambda$. The Riesz and Bessel potentials (see Grafakos [4, pp. 414–420]) I_s and J_s satisfy (1.9) when $0 < s < n-k$ and $\varphi(\lambda) = \lambda^p$ for $p = (1 - s/(n-k))^{-1}$. At present, however, it is known only that they satisfy (1.8) when Γ is a Lipschitz graph and with C_1 depending on the Lipschitz constant. Hence the main result (1.10) is not yet known for these operators in full generality.

Remark 1.1. This question of whether the weak-type inequality is independent of the Lipschitz constant is of independent interest and should be explored for such operators in general.

The Hardy–Littlewood centered maximal operator M acts on a measure ν by

$$M\nu(x) = \sup_{r>0} \frac{|\nu|(B(x,r))}{m(B(x,r))}, \tag{1.11}$$

where $B(x,r)$ is the ball centered at x with radius r. This sublinear operator is strong-type (p,p) and weak-type $(1,1)$ and satisfies results analogous to (1.4) and (1.7). The best constants in these cases are not known in general. The exception is when $n = 1$, for which case the weak-type constant is found by Melas [10]. Stein and Stromberg [13] have shown that the strong-type (p,p) constant is independent of dimension n and that the weak-type $(1,1)$ constant is at worst linear in n. It is an open question whether the latter is independent of n. The following theorem, analogous to Theorem 1.2, is proved for the maximal operator in Section 3.

THEOREM 1.4. *Let $\Gamma = \bigcup_i \Gamma_i$ be the countable union of k-dimensional Lipschitz surfaces that are $\mathcal{H}^k$-disjoint. Let $d\nu_i = f_i \, d\mathcal{H}^k_{\Gamma_i}$ (where $0 \le f_i \in L^1(\Gamma_i)$) be a measure supported on Γ_i and satisfying $\|\nu\|_1 = \left\| \sum_i \nu_i \right\|_1 < \infty$. Then*

$$\lim_{\lambda\to\infty} \lambda m\{x \in \mathbb{R}^n : M\nu(x) > \lambda\} = \tilde{C}_{n,k}\|\nu\|_1, \tag{1.12}$$

where

$$\tilde{C}_{n,k} = \frac{\left(\gamma_{n-k}\sqrt{n-k}^{\,n-k}\right)\left(\gamma_k\sqrt{k}^{\,k}\right)}{\gamma_n\sqrt{n}^{\,n}}. \tag{1.13}$$

Besides finding the approach rate to Γ (done in Lemmas 2.1 and 3.1), there are two important approximations that simplify the proofs of these theorems. First, since every Lipschitz surface is locally a Lipschitz graph, it suffices to prove the theorems for countable unions of Lipschitz graphs. Second, it suffices to prove the theorem for a single Lipschitz graph and then use the weak-type inequality available for these operators to extend to the general case. These arguments are given for the Riesz transform.

2. Proof of Theorem 1.2

The proof proceeds in five steps as follows.

(1) Assume $f \in C_c(\Gamma)$ and Γ is a Lipschitz graph; in other words, assume there exists a $\Phi\colon D \subset \mathbb{R}^k \to \Gamma \subset \mathbb{R}^n$ such that $\Phi(x) = (x, \phi(x))$, where $\phi\colon D \to \mathbb{R}^{n-k}$ is Lipschitz. Without loss of generality, assume that $D = \mathbb{R}^k$. Note that D may be considered as $\mathbb{R}^k \times \{0\}^{n-k}$.

(2) Assume $x \in \mathbb{R}^k$ is a point where $D\Phi(x)$ exists and hence the tangent plane $\Pi_{\Phi(x)}$ exists at the point $\Phi(x)$. Let $N^\perp = \{0\}^k \times \mathbb{R}^{n-k}$ and $S^{n-k-1} = S^{n-1} \cap N^\perp$. The symbol $\vec{n}$ denotes any element of S^{n-k-1}.

(3) Let

$$t^1_\lambda = \inf\{t > 0 : |R\nu(\Phi(x) + t\vec{n})| < \lambda\}$$

and

$$t^2_\lambda = \sup\{t > 0 : |R\nu(\Phi(x) + t\vec{n})| > \lambda\}.$$

For $i \in \{1,2\}$, let

$$E^i_\lambda(x) = \{\Phi(x) + t\vec{n} : \vec{n} \in S^{n-k-1} \text{ and } 0 < t < t^i_\lambda(x,\vec{n})\}$$

and $E_\lambda = \{x \in \mathbb{R}^n : |R\nu(x)| > \lambda\}$. It follows from Fubini's theorem that

$$\int_{\mathbb{R}^k} m_{n-k}(E^1_\lambda(x))\, dx \le m_n(E_\lambda) \le \int_{\mathbb{R}^k} m_{n-k}(E^2_\lambda(x))\, dx$$

and

$$\begin{aligned} m_{n-k}(E^i_\lambda(x)) &= \int_{S^{n-k-1}} \int_0^{t^i_\lambda(x,\vec{n})} r^{n-k-1}\, dr\, d\sigma(\vec{n}) \\ &= \int_{S^{n-k-1}} \frac{t^i_\lambda(x,\vec{n})^{n-k}}{n-k}\, d\sigma(\vec{n}). \end{aligned}$$

It is an easy exercise to show that $R\nu$ is continuous away from Γ and hence $|R\nu(\Phi(x) + t^i_\lambda(x, \vec{n})\vec{n})| = \lambda$ whenever $t^i_\lambda(x, \vec{n}) > 0$. Therefore,

$$\lambda m_{n-k}(E^i_\lambda(x)) = \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} H^i_\lambda(x, \vec{n})\, d\sigma(\vec{n}),$$

where $H^i_\lambda(x, \vec{n}) = \gamma_{n-k} t^i_\lambda(x, \vec{n})^{n-k} |R\nu(\Phi(x) + t^i_\lambda(x, \vec{n})\vec{n})|$.

It can also be shown that H^i_λ is bounded and compactly supported; therefore, if the limit $\lim_{\lambda\to\infty} H^i_\lambda(x, \vec{n})$ exists for all $\vec{n} \in S^{n-k-1}$ and almost every x, then the Lebesgue dominated convergence theorem implies

$$\begin{aligned}
\lim_{\lambda\to\infty} \lambda m(E_\lambda) &= \lim_{\lambda\to\infty} \int_{\mathbb{R}^k} \lambda m_{n-k}(E^i_\lambda(x))\, dx \\
&= \lim_{\lambda\to\infty} \int_{\mathbb{R}^k} \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} H^i_\lambda(x, \vec{n})\, d\sigma(\vec{n})\, dx \\
&= \int_{\mathbb{R}^k} \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \lim_{\lambda\to\infty} H^i_\lambda(x, \vec{n})\, d\sigma(\vec{n})\, dx.
\end{aligned}$$

Since $t^i_\lambda(x, \vec{n})$ decreases to 0 as $\lambda \to \infty$, it suffices to show that

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} |R\nu(\Phi(x) + t\vec{n})|$$

exists and equals the appropriate limit.

It is proved in Lemmas 2.1 and 2.2 that

$$\frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \lim_{\lambda\to\infty} H^i_\lambda(x, \vec{n})\, d\sigma(\vec{n}) = C_{n,k} |f(\Phi(x))| J\Phi(x), \tag{2.1}$$

where $J\Phi(x)$ is the Jacobian of Φ at x and where $C_{n,k}$ is given in (1.6). This completes the proof of the theorem when $f \in C_c(\Gamma)$ and Γ is a Lipschitz graph, since $\int_{\mathbb{R}^k} |f(\Phi(x))| J\Phi(x)\, dx = \|\nu\|_1$.

(4) Let $f \in L^1(\Gamma)$ and $g \in C_c(\Gamma)$. Let ν_f and ν_g be the associated measures. Define

$$\tau_\lambda(\nu) = \lambda m(x \in \mathbb{R}^n : |R\nu(x)| > \lambda) \tag{2.2}$$

for any measure ν. Then, for each $c \in (0, 1)$, by the weak-type inequality for the Riesz operator (see Mattila [9, Thm. 20.26, p. 301]) it follows that

$$\begin{aligned}
\tau_\lambda(\nu_f) &\le \frac{1}{c} \tau_{c\lambda}(\nu_g) + \frac{1}{1-c} \tau_{(1-c)\lambda}(\nu_{f-g}) \\
&\le \frac{1}{c} \tau_{c\lambda}(\nu_g) + \frac{\tilde{C}}{1-c} \|f - g\|_{L^1(\Gamma)}.
\end{aligned}$$

Since $C_c(\Gamma)$ is dense in $L^1(\Gamma)$ and since $c \in (0, 1)$ is arbitrary, the proof in part (3) implies

$$\limsup_{\lambda\to\infty} \tau_\lambda(\nu_f) \le C_{n,k} \|\nu_f\|_1.$$

The other inequality $C_{n,k}\|\nu_f\|_1 \le \liminf_{\lambda\to\infty} \tau_\lambda(\nu_f)$ is obtained similarly. Therefore, as required,

$$\lim_{\lambda\to\infty} \tau_\lambda(\nu_f) = C_{n,k}\|\nu_f\|_1 \quad \text{for all } f \in L^1(\Gamma).$$

(5) The theorem is proved when Γ is a Lipschitz graph. Now consider $\Gamma = \bigcup_i \Gamma_i$ as in the statement of the theorem. Denote τ_λ as in (2.2) and define $\tau(\mu) = \lim_{\lambda\to\infty} \tau_\lambda(\mu)$ for measure μ whenever the limit exists. The proof in part (4) implies that (1.5) holds for each ν_i. Now observe that, for $c \in (0,1)$,

$$\tau_\lambda\biggl(\sum_{i=1}^{\infty} \nu_i\biggr) \le \frac{1}{c}\tau_{c\lambda}\biggl(\sum_{i=1}^{N} \nu_i\biggr) + \frac{1}{1-c}\tau_{(1-c)\lambda}\biggl(\sum_{i=N+1}^{\infty} \nu_i\biggr),$$

and similarly

$$\tau_\lambda\biggl(\sum_{i=1}^{N} \nu_i\biggr) \le \frac{1}{c}\tau_{c\lambda}\biggl(\sum_{i=1}^{\infty} \nu_i\biggr) + \frac{1}{1-c}\tau_{(1-c)\lambda}\biggl(\sum_{i=N+1}^{\infty} \nu_i\biggr).$$

It follows from the weak-type inequality satisfied by R that

$$\tau(\nu) = \lim_{N\to\infty} \tau\biggl(\sum_{i=1}^{N} \nu_i\biggr).$$

Hence it suffices to prove the theorem when $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$ is a finite union of Lipschitz graphs.

The proof proceeds by induction. Assume the theorem holds when Γ is $\bigcup^{N-1}$ graph pieces. Let $\nu = \sum_{i=1}^{N} \nu_i$ and $\eta = \sum_{i=1}^{N-1} \nu_i$. The objective is to prove

$$\tau(\nu) = \tau(\eta) + \tau(\nu_N),$$

which in turn equals $C_{n,k}(\|\eta\|_1 + \|\nu_N\|_1) = C_{n,k}\|\nu\|_1$ as required.

Let $E_k = \bigl\{x \in \Gamma_N : \operatorname{dist}\bigl(x, \bigcup_{i=1}^{N-1}\Gamma_i\bigr) > \frac{1}{k}\bigr\}$. Then

$$\bigcup_{k=1}^{\infty} E_k = \Gamma_N \Bigm\backslash \biggl(\Gamma_N \cap \bigcup_{i=1}^{N-1} \Gamma_i\biggr) \quad \text{and} \quad \nu_N\biggl(\bigcup_k E_k\biggr) = \nu_N(\Gamma_N) = \|\nu_N\|_1.$$

Let $\nu_N^k = \nu_N|_{E_k}$. Suppose $\tau(\eta + \nu_N^k)$ exists and is equal to $\tau(\eta) + \tau(\nu_N^k)$ for each k. Then

$$\begin{aligned} \tau(\nu) &= \lim_{k\to\infty} \tau(\eta + \nu_N^k) \\ &= \tau(\eta) + \lim_{k\to\infty} \tau(\nu_N^k) \\ &= \tau(\eta) + \tau(\nu_N). \end{aligned}$$

It therefore suffices to assume that $\operatorname{dist}\bigl(\Gamma_N, \bigcup_{i=1}^{N-1}\Gamma_i\bigr) > \varepsilon$ for some $\varepsilon > 0$.

Now consider

$$\Lambda_1^{\varepsilon/k} = \bigl\{x \in \mathbb{R}^n : \operatorname{dist}\bigl(x, \textstyle\bigcup_{i=1}^{N-1}\Gamma_i\bigr) < \frac{\varepsilon}{k}\bigr\}$$

and

$$\Lambda_2^{\varepsilon/k} = \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, \Gamma_N) < \tfrac{\varepsilon}{k}\right\}.$$

If $x \in \mathbb{R}^n \setminus (\Lambda_1^{\varepsilon/k} \cup \Lambda_2^{\varepsilon/k})$, then

$$|R\nu(x)| \leq \frac{c\|\nu\|_1}{(\varepsilon/k)^n}.$$

Hence, for λ large, $E_\lambda = \{x \in \mathbb{R}^n : |R\nu(x)| > \lambda\} \subset \Lambda_1^{\varepsilon/k} \cup \Lambda_2^{\varepsilon/k}$.

Let $C_\varepsilon = c\|\nu\|_1/((1-1/k)\varepsilon)^n$. If $x \in \Lambda_1^{\varepsilon/k}$ then

$$|R\nu(x)| - C_\varepsilon \leq |R\eta(x)| \leq |R\nu(x)| + C_\varepsilon,$$

and if $x \in \Lambda_2^{\varepsilon/k}$ then

$$|R\nu(x)| - C_\varepsilon \leq |R\nu_N(x)| \leq |R\nu(x)| + C_\varepsilon.$$

This shows that, for λ large,

$$E_\lambda \approx \{x \in \mathbb{R}^n : |R\eta(x)| > \lambda\} \sqcup \{x \in \mathbb{R}^n : |R\nu_N(x)| > \lambda\},$$

from which it follows that $\tau(\nu)$ exists and equals $\tau(\eta) + \tau(\nu_N)$ as required. Thus $\tau(\nu) = C_{n,k}\|\nu\|_1$ when ν is supported on a countable union of Lipschitz surfaces and is absolutely continuous with respect to the Hausdorff k-measure $\mathcal{H}^k$. This completes the proof of the theorem.

The next step is to prove the lemmas required to confirm (2.1). Let T be a singular integral operator with kernel $K(x,y) = \Omega(x-y)/|x-y|^n$. Assume that $\Omega(x)$ as defined on $\mathbb{R}^n \setminus \{\vec{0}\}$ is homogeneous of degree 0—that is, $\Omega(y) = \Omega(y/|y|)$—and satisfies $\int_{S^{n-1}} \Omega(e)\, d\sigma(e) = 0$. Assume also that Ω is bounded and continuous on S^{n-1}.

Given $x_0 \in \mathbb{R}^k \times \{0\}^{n-k}$ and unit vector $\vec{n} \in \{0\}^k \times \mathbb{R}^{n-k}$, let Θ be the smallest angle that $\vec{n}$ makes with the tangent space $\Pi_{\Phi(x_0)}$. Let $\tilde{\Pi}_{\Phi(x_0)} = \Pi_{\Phi(x_0)} - \Phi(x_0)$ be the tangent plane parallel to $\Pi_{\Phi(x_0)}$ and passing through the origin. With a mild abuse of notation, let μ_Γ and μ_{Π_Φ} denote the k-dimensional Hausdorff measure restricted to Γ and $\Pi_{\Phi(x_0)}$ respectively.

LEMMA 2.1. *Let $x_0 \in \mathbb{R}^k$ be a point where $D\Phi(x_0)$ exists and has rank k. Let Θ be the smallest angle that $\vec{n}$ makes with the tangent space $\tilde{\Pi}_{\Phi(x_0)}$. Then the following statements hold.*

(1)

$$\lim_{t\to 0} t^{n-k}|T\nu(\Phi(x_0) + t\vec{n})| = \frac{|T\mu_{\tilde{\Pi}_\Phi}(\vec{a})|}{\sin(\Theta)^{n-k}}|f(\Phi(x_0))|, \tag{2.3}$$

where

$$\vec{a} = \frac{\vec{n} - \operatorname{Proj}_{\tilde{\Pi}_\Phi}(\vec{n})}{|\vec{n} - \operatorname{Proj}_{\tilde{\Pi}_\Phi}(\vec{n})|}. \tag{2.4}$$

(2) *If* $T = R_j$, *the* jth *Riesz transform, then*

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} |R_j \nu(\Phi(x_0) + t\vec{n})| = \frac{C_{n,k} a_j}{\sin(\Theta)^{n-k}} |f(\Phi(x_0))|, \tag{2.5}$$

where a_j *is the* jth *component of the vector* $\vec{a}$ *defined in* (2.4) *and* $C_{n,k}$ *is the constant defined in* (1.6).

(3) *If* R *is the vector Riesz transform, then*

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} |R\nu(\Phi(x_0) + t\vec{n})| = \frac{C_{n,k}}{\sin(\Theta)^{n-k}} |f(\Phi(x_0))|. \tag{2.6}$$

Proof. Without loss of generality, the following assumptions are made.

- Since the final result is independent of the position of Γ in $\mathbb{R}^n$, it suffices to assume $x_0 = \Phi(x_0) = \vec{0}$ and $\tilde{\Pi}_{\Phi(x_0)} = \Pi_{\Phi(x_0)} = \mathbb{R}^k \times \{0\}^{n-k} := \Pi$; therefore, $\vec{n}$ is any unit vector not on Π.
- f is continuous and compactly supported on Γ and $\mathcal{H}^k(\Gamma) < \infty$. The approximation arguments in the proof of Theorem 1.2 extend this to the general setting.

Using these and the continuity assumption on f, analysis shows that (2.3) is reduced to proving

$$\lim_{t\to 0} t^{n-k} |T\mu_\Gamma(t\vec{n})| = \lim_{t\to 0} t^{n-k} |T\mu_\Pi(t\vec{n})|. \tag{2.7}$$

Given $\varepsilon > 0$, let $A_\varepsilon = \Gamma \cap B(0, \varepsilon)$ and $\Lambda_\varepsilon = \mathrm{Proj}_\Pi(A_\varepsilon)$, the orthogonal projection onto Π. Observe that

$$\lim_{t\to 0} t^{n-k} |T\mu_\Gamma(t\vec{n})| = \lim_{t\to 0} t^{n-k} \left| \int_{A_\varepsilon} \frac{\Omega(t\vec{n} - y)}{|t\vec{n} - y|^n} \, d\mu_\Gamma(y) \right|,$$

since $|\Omega(t\vec{n} - y)|/|t\vec{n} - y|^n < c\varepsilon^{-n}$ for $y \in \Gamma \setminus A_\varepsilon$ and $\|\mu_{A_\varepsilon}\|_1 < \infty$. If Φ is the Lipschitz graph of A_ε on Λ_ε, define $\Phi_t \colon \frac{1}{t}\Lambda_\varepsilon \to \frac{1}{t}A_\varepsilon$ by $\Phi_t(x) = (x, t^{-1}\phi(tx))$ as the Lipschitz graph of $t^{-1}A_\varepsilon$. Then the Jacobian satisfies $J\Phi_t(x) = J\Phi(tx)$:

$$\begin{aligned} t^{n-k} T\mu_{A_\varepsilon}(t\vec{n})| &= t^{n-k} \int_{\Lambda_\varepsilon} \frac{\Omega(t\vec{n} - \Phi(x))}{|t\vec{n} - \Phi(x)|^n} J\Phi(x) \, dx \\ &= \int_{t^{-1}\Lambda_\varepsilon} \frac{\Omega(\vec{n} - \Phi_t(x))}{|\vec{n} - \Phi_t(x)|^n} J\Phi(tx) \, dx \\ &= \int_{t^{-1}\Lambda_\varepsilon} [K(\vec{n}, \Phi_t(x)) - K(\vec{n}, x)] J\Phi(tx) \, dx \qquad (2.8) \\ &\quad + \int_{t^{-1}\Lambda_\varepsilon} K(\vec{n}, x) J\Phi(tx) \, dx. \qquad (2.9) \end{aligned}$$

Because Φ is Lipschitz, $J\Phi$ is bounded in Λ_ε; also, $J\Phi(0) = 1$ by construction. Consider the second term (2.9):

$$\begin{aligned} \int_{t^{-1}\Lambda_\varepsilon} K(\vec{n}, x) J\Phi(tx) \, dx &= \int_{t^{-1}\Lambda_\varepsilon} K(\vec{n}, x)(J\Phi(tx) - J\Phi(0)) \, dx \\ &\quad + J\Phi(0) \int_{t^{-1}\Lambda_\varepsilon} K(\vec{n}, x) \, dx, \end{aligned}$$

which converges to $\int_{\mathbb{R}^k} K(\vec{n}, x)\, dx = T\mu_\Pi(\vec{n})$ as $t \to 0$. To prove this, let $B_k(0, R) = \Pi \cap B(0, R)$. Given $\tilde{\varepsilon} > 0$, there exists large R such that

$$\int_{\mathbb{R}^k \setminus B_k(0,R)} K(\vec{n}, x)\, dx < \tilde{\varepsilon}.$$

Thus, for t small enough,

$$\begin{aligned}
&\int_{t^{-1}\Lambda_\varepsilon} |K(\vec{n}, x)||J\Phi(tx) - J\Phi(0)|\, dx \\
&\qquad < 2\|J\Phi\|_\infty \tilde{\varepsilon} + \|K(\vec{n}, \cdot)\|_\infty \int_{B_k(0,R)} |J\Phi(tx) - J\Phi(0)|\, dx \\
&\qquad \to 2\|J\Phi\|_\infty \tilde{\varepsilon} \quad \text{as } t \to 0 \ \text{(since } J\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^k)).
\end{aligned}$$

Since $\tilde{\varepsilon}$ is arbitrary, it follows that

$$\lim_{t \to 0} \int_{t^{-1}\Lambda_\varepsilon} K(\vec{n}, x) J\Phi(tx)\, dx = T\mu_\Pi(\vec{n}). \tag{2.10}$$

The other term (2.8),

$$\int_{t^{-1}\Lambda_\varepsilon} [K(\vec{n}, \Phi_t(x)) - K(\vec{n}, x)] J\Phi(tx)\, dx,$$

is bounded up to constants by

$$\begin{aligned}
&\int_{t^{-1}\Lambda_\varepsilon} |K(\vec{n}, \Phi_t(x)) - K(\vec{n}, x)|\, dx \\
&\qquad \le \int_{t^{-1}\Lambda_\varepsilon} \frac{|\Omega(\vec{n} - \Phi_t(x)) - \Omega(\vec{n} - x)|}{|\vec{n} - \Phi_t(x)|^n}\, dx \qquad (2.11) \\
&\qquad\quad + \|\Omega\|_\infty \int_{t^{-1}\Lambda_\varepsilon} \left| \frac{1}{|\vec{n} - \Phi_t(x)|^n} - \frac{1}{|\vec{n} - x|^n} \right| dx. \qquad (2.12)
\end{aligned}$$

Both (2.11) and (2.12) go to 0. To see this, let $S_\varphi = \{x \in \mathbb{R}^n :$ angle x makes with Π is less than $\varphi\}$. Because of tangency at $\vec{0}$, for each $\varphi > 0$ there exists an $\varepsilon > 0$ such that $A_\varepsilon = \Gamma \cap B(0, \varepsilon) \subset S_\varphi$; hence $t^{-1}A_\varepsilon \subset S_\varphi$ for all $t > 0$. This means that $\phi(y)/|y| \to 0$ as $|y| \to 0$. Observe that, for $x \in \Pi$,

$$|\vec{n} - x| - |t^{-1}\phi(tx)| \le |\vec{n} - \Phi_t(x)| \le |\vec{n} - x| + |t^{-1}\phi(tx)|.$$

Since $\vec{n}$ is a fixed vector not on Π, there exists a $c > 0$ such that $|x| \le c|\vec{n} - x|$ for all $x \in \Pi$. Hence, for $x \in t^{-1}\Lambda_\varepsilon$,

$$|\vec{n} - x|(1 - c\tan(\varphi)) \le |\vec{n} - \Phi_t(x)| \le |\vec{n} - x|(1 + c\tan(\varphi)),$$

where $\varphi \to 0$ as $\varepsilon \to 0$. Choose ε small so that $\frac{1}{2} < c\tan(\varphi) < 1$. Then

$$\frac{|\vec{n} - x|}{2} \le |\vec{n} - \Phi_t(x)| \le \frac{|\vec{n} - x|}{2}$$

for all $x \in t^{-1}\Lambda_\varepsilon$. In short, $|\vec{n} - x| \approx |\vec{n} - \Phi_t(x)|$ for x in the domain of Φ_t. An application of this fact and the Lebesgue dominated convergence theorem shows that the difference term (2.11) goes to 0 as t goes to 0; and (2.12) is proved similarly. In conclusion, $\lim_{t\to 0} t^{n-k} T\mu_\Gamma(t\vec{n}) = T\mu_\Pi(\vec{n})$.

Recall that $\vec{a} = (\vec{n} - \mathrm{Proj}_\Pi(\vec{n}))/|\vec{n} - \mathrm{Proj}_\Pi(\vec{n})|$. Then

$$\begin{aligned} T\mu_\Pi(\vec{n}) &= \int_\Pi K(\vec{n} - \mathrm{Proj}_\Pi(\vec{n}), y)\, d\mu_\Pi(y) \\ &= \frac{1}{|\vec{n} - \mathrm{Proj}_\Pi(\vec{n})|^{n-k}} \int_\Pi K(\vec{a}, y)\, d\mu_\Pi(y) \\ &= \frac{1}{\sin(\Theta)^{n-k}} T\mu_\Pi(\vec{a}). \end{aligned}$$

This completes the proof of part (1) of the lemma.

Now consider the Riesz transform R_j with kernel $K(x,y) = C_n(x_j - y_j)/|x-y|^{n+1}$. By part (1),

$$\begin{aligned} \lim_{t\to 0} t^{n-k} R_j\mu_\Gamma(t\vec{n}) &= \frac{C_n}{\sin(\Theta)^{n-k}} \int_\Pi \frac{a_j - y_j}{|a-y|^{n+1}}\, d\mu_\Pi(y) \\ &= \frac{C_n a_j}{\sin(\Theta)^{n-k}} \int_{\mathbb{R}^k} \frac{1}{(1+|y|^2)^{(n+1)/2}}\, dy, \end{aligned}$$

since $\vec{a}$ is perpendicular to Π and y_j is an odd function when restricted to Π. To obtain the constant $C_{n,k}$ in (1.6), note first (see Jones [8, p. 314]) that

$$\int_{\mathbb{R}^k} \frac{1}{(1+|y|^2)^{(n+1)/2}}\, dy = \sqrt{\pi}^{\,k} \frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)},$$

and then sort this with corresponding representations for C_n and γ_{n-k}.

For part (3), consider the vector Riesz transform R. Then

$$\begin{aligned} \lim_{t\to 0} t^{n-k} |R\mu_\Gamma(t\vec{n})| &= \left(\sum_{j=1}^n \left(\lim_{t\to 0} t^{n-k} |R_j\mu_\Gamma(t\vec{n})| \right)^2 \right)^{1/2} \\ &= \frac{C_{n,k}}{\sin(\Theta)^{n-k}}. \end{aligned}$$

This completes the proof of the lemma. $\square$

It follows from Lemma 2.1 and what was proved before it that

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda m(E_\lambda) &= \int_{\mathbb{R}^k} \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \lim_{\lambda\to\infty} H^i_\lambda(x, \vec{n})\, d\sigma(\vec{n})\, dx \\ &= C_{n,k} \int_{\mathbb{R}^k} f(\Phi(x)) \left(\frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \frac{1}{\sin(\Theta)^{n-k}}\, d\sigma(\vec{n}) \right) dx. \end{aligned}$$

The following lemma completes the proof of Theorem 1.2. Recall that

- Φ is the Lipschitz map from $\mathbb{R}^k \times \{0\}^{n-k} \to \Gamma$,
- $\Pi_{\Phi(x)}$ is the tangent k-space to Γ at $\Phi(x)$,
- Θ is the smallest angle that the fixed vector $\vec{n} \in \{0\}^k \times \mathbb{R}^{n-k}$ makes with $\Pi_{\Phi(x)}$, and
- $J\Phi(x)$ is the Jacobian of Φ at x.

LEMMA 2.2.
$$J\Phi(x) = \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \frac{1}{\sin(\Theta)^{n-k}}\, d\sigma(\vec{n}).$$

Proof. If $\tilde{\Theta} = \frac{\pi}{2} - \Theta$, then $\frac{1}{\sin\Theta} = \sqrt{1+\tan^2\tilde{\Theta}}$. The claim is that $\tan\tilde{\Theta} = |\nabla(\phi \cdot \vec{n})(x)|$, which implies
$$\frac{1}{\sin\Theta} = \sqrt{1+|\nabla(\phi\cdot\vec{n})(x)|^2}. \tag{2.13}$$
To prove this claim, let P_1 be the orthogonal projection of $\mathbb{R}^n$ onto $\Pi_{\Phi(x)}$ and let P_2 be the orthogonal projection onto $\mathbb{R}^k \times \{0\}^{n-k}$. It is readily observed that $\Theta = \angle[\vec{n}, P_1(\vec{n})]$ and $\tilde{\Theta} = \angle[P_1(\vec{n}), P_2 \circ P_1(\vec{n})]$, where $\angle[v,w]$ denotes the angle between the vectors v and w. Since $D\Phi(x) = (x, D\phi(x))$, it follows that the minimal angle Θ occurs when
$$(\vec{e},\vec{n}) \cdot (D\Phi(x)\cdot\vec{e}) = 1 + \vec{n}\cdot D\phi(x)\cdot\vec{e}$$
is maximized over $\vec{e} \in S^{k-1} \times \{0\}^{n-k}$. But $\vec{n}\cdot D\phi(x)\cdot\vec{e} = \nabla(\phi\cdot\vec{n})(x)\cdot\vec{e}$ is maximized when $\vec{e} = \nabla(\phi\cdot\vec{n})(x)/|\nabla(\phi\cdot\vec{n})(x)|$. The triangle determined by the vectors $\vec{e}$ and $D\Phi(x)\cdot\vec{e} = (\vec{e}, |\nabla(\phi\cdot\vec{n})(x)|\vec{n})$ has the acute angle $\tilde{\Theta}$, and $\tan\tilde{\Theta} = |\nabla(\phi\cdot\vec{n})(x)|$ as required. Notice further that
$$\nabla(\phi\cdot\vec{n})(x) = [\vec{n}^T \cdot D\phi(x_0)]^T = D\phi(x)^T\cdot\vec{n},$$
where T denotes the transpose of the matrix. Therefore, Lemma 2.2 can be rewritten as
$$J\Phi(x) = \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \sqrt{1+|D\phi(x)^T\cdot\vec{n}|^2}^{\,n-k}\, d\sigma(\vec{n}). \tag{2.14}$$
The usual formulation of the Jacobian of a graph of a function agrees with (2.14) when $k = n-1$. The lemma is proved by induction, as follows.

Let $L\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ be a linear map and let $\tilde{L}\colon \mathbb{R}^k \to \mathbb{R}^n$ be the graph of L defined by $\tilde{L}\vec{v} = (\vec{v}, L\vec{v})$. Let I_k and I_{n-k} be the $k\times k$ and $(n-k)\times(n-k)$ identity matrices, respectively. There are two equivalent ways of formulating the Jacobian $[|\tilde{L}|]$ of $\tilde{L}$. The first is as the stretch factor of the action of $\tilde{L}$ on the unit ball $B_k(0,1) \subset \mathbb{R}^k$; that is,
$$[|\tilde{L}|] = \frac{\mathcal{H}^k(\tilde{L}(B_k(0,1)))}{\gamma_k}. \tag{2.15}$$
By an orthogonal transformation $A\colon \mathbb{R}^n \to \mathbb{R}^n$ with determinant 1, the image of $\tilde{L}$ can be rotated to $\mathbb{R}^k \times \{0\}^{n-k}$ in such manner that
$$A\left(\frac{\tilde{L}\cdot\vec{e}}{|\tilde{L}\cdot\vec{e}|}\right) = \vec{e}$$
for all $\vec{e} \in S^{k-1} \times \{0\}^{n-k}$. Let $G = A \circ \tilde{L}(B_k(0,1))$. Then
$$G = \left\{ t\vec{e} \in \mathbb{R}^k \times \{0\}^{n-k} : \vec{e} \in S^{k-1} \text{ and } 0 \le t < \sqrt{1+|L\cdot\vec{e}|^2} \right\}.$$
Since the $\mathcal{H}^k$ measure is invariant under A, definition (2.15) is equivalent to

$$[|\tilde{L}|] = \frac{1}{\sigma_k} \int_{S^{k-1}} \sqrt{1 + |L \cdot \vec{e}|^2}^{\,k} \, d\sigma(\vec{e}). \tag{2.16}$$

Observe that (2.14) is the *dual* version of (2.16).

The second formulation of the Jacobian $[|\tilde{L}|]$ is as the determinant of the symmetric component of $\tilde{L}$; that is,

$$[|\tilde{L}|] = \det[\tilde{L}^* \circ \tilde{L}] = \det[I_k + L^* \circ L], \tag{2.17}$$

where L^* is the adjoint of L. The dual version of this determinant corresponding to (2.14) is $\det[I_{n-k} + L \circ L^*]$. Hence the lemma is proved if

$$\det[I_k + L^* \circ L] = \det[I_{n-k} + L \circ L^*]. \tag{2.18}$$

The proof is by induction on dimension n. The case $n = 2$ follows because (2.18) is already known to hold when $k = n - 1$ for all n. Assume the identity holds for n and all $k < n$. For dimension $n + 1$, if $2k < n + 1$ then (following an orthogonal transformation on L) L may be assumed to map into $\mathbb{R}^k \times \{0\}^{n-2k}$. This allows reduction of the problem to a smaller-dimensional case for which the induction hypothesis applies. The case $2k > n+1$ is also implicit because it is the dual case ($\tilde{L}^*\colon \mathbb{R}^{n+1-k} \to \mathbb{R}^{n+1}$ whenever $\tilde{L}\colon \mathbb{R}^k \to \mathbb{R}^{n+1}$). Finally, in the special case $2k = n + 1$, equality follows from a simple algebraic proof. □

Note as a special case of (2.18) that $\det[I_n + \vec{b}^* \cdot \vec{b}] = 1 + |\vec{b}|^2$ for $\vec{b} \in \mathbb{R}^n$. The identity should have generalizations and applications elsewhere.

3. Proof of Theorem 1.4

It suffices to prove the theorem for Γ a Lipschitz graph above $\mathbb{R}^k$. The general case follows by an approximation that uses the fact that M is a weak-type $(1,1)$ operator over finite Borel measures.

Let $\Phi\colon D \subset \mathbb{R}^k \to \Gamma \subset \mathbb{R}^n$ be such that $\Phi(x) = (x, \phi(x))$, where $\phi\colon D \to \mathbb{R}^{n-k}$ is Lipschitz. Without loss of generality, assume that $D = \mathbb{R}^k$. Observe that D may be considered as $\mathbb{R}^k \times \{0\}^{n-k}$. Let $E_\lambda = \{x \in \mathbb{R}^n : |M\nu(x)| > \lambda\}$. Then repeating the arguments of the proof of Theorem 1.2 leads to

$$\lim_{\lambda\to\infty} \lambda m(E_\lambda) = \int_{\mathbb{R}^k} \frac{1}{\sigma_{n-k}} \int_{S^{n-k-1}} \lim_{t\to 0} H(t, x, \vec{n}) \, d\sigma(\vec{n}) \, dx,$$

where $H(t, x, \vec{n}) = \gamma_{n-k} t^{n-k} |M\nu(\Phi(x) + t\vec{n})|$. The next lemma shows that this $\lim_{t\to 0} H(t, x, \vec{n}) = \tilde{C}_{n,k} f(\Phi(x))/\sin(\Theta)^{n-k}$, where Θ is the angle $\vec{n}$ makes with $\Pi_{\Phi(x)}$ and $\tilde{C}_{n,k}$ is given in (1.13). This fact and an application of Lemma 2.2 yield the final result.

LEMMA 3.1. *For almost every $x \in \mathbb{R}^k$ and for each $\vec{n} \in \{0\}^k \times S^{n-k-1}$, the limit*

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} M\nu(\Phi(x) + t\vec{n}) = \frac{\tilde{C}_{n,k} f(\Phi(x))}{\sin(\Theta)^{n-k}}, \tag{3.1}$$

where Θ is the smallest angle that the vector $\vec{n}$ makes with the tangent plane $\Pi_{\Phi(x)}$ and where $\tilde{C}_{n,k}$ is the constant given in (1.13).

Proof. Suppose $x \in \mathbb{R}^k \setminus \operatorname{supp}(f \circ \Phi)$. Then $M\nu$ is bounded in a neighborhood of $\Phi(x)$ and so the limit is 0 as required. Since the boundary of the support has zero k-Hausdorff measure, assume that $f(\Phi(x)) > 0$. Let μ_Γ be the k-Hausdorff measure restricted to Γ; that is, let $\mu_\Gamma = \mathcal{H}^k_\Gamma$. The continuity of f can be used to show that it suffices to assume $f \equiv 1$ on Γ. Hence the problem reduces to proving

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(\Phi(x) + t\vec{n}) = \frac{\tilde{C}_{n,k}}{\sin(\Theta)^{n-k}}. \tag{3.2}$$

The following notation is used in the proof.

- The Lipschitz map Φ is not considered for the rest of the lemma's proof. Therefore, assume without loss of generality that $x = \Phi(x) = \underset{\sim}{0}$ and the tangent space $\Pi_{\Phi(x)} = \mathbb{R}^k \times \{0\}^{n-k} := \Pi$. This means that $\vec{n}$ is some unit vector that makes a minimal angle Θ with $\mathbb{R}^k \times \{0\}^{n-k}$. Furthermore, let $x_t = \Phi(x) + t\vec{n} = t\vec{n}$ and $t' = \operatorname{dist}(x_t, \Pi) = t \sin\Theta$.
- $S(\varphi, x) = \{y \in \mathbb{R}^n :$ the angle that $y - x$ makes with $\Pi_{\Phi(x)}$ is less than $\varphi\}$.
- $r_t = \inf\{r > 0 : M\mu_\Gamma(\Phi(x) + t\vec{n}) = \mu_\Gamma(B(\Phi(x) + t\vec{n}, r_t))/\gamma_n r_t^n\}$.

The approach behavior of r_t as $t \to 0$ can be split into three distinct cases.

Case 1: $\liminf_{t\to 0} r_t/t' > 1$.
Case 2: $\liminf_{t\to 0} r_t/t' = 1$.
Case 3: $\liminf_{t\to 0} r_t/t' < 1$.

Case 1 leads to the proof of the lemma. Case 2 probably never happens; however, the present proof shows only that if Case 2 does occur then the result agrees with the claim of this lemma. Case 3 never happens.

Case 1: $\liminf_{t\to 0} r_t/t' = \alpha > 1$. Let $\alpha' = \alpha \sin\Theta$ so that $\liminf_{t\to 0} r_t/t' = \alpha' > \sin\Theta$. Assume the limit exists and equals α'. Observe that

$$\mathcal{H}^k(\Gamma \cap B(x_t, r_t)) = \mathcal{H}^k\left(t\left(\frac{1}{t}\Gamma \cap B\left(\frac{x_t}{t}, \frac{r_t}{t}\right)\right)\right) = t^k \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B\left(x_1, \frac{r_t}{t}\right)\right).$$

Similarly, $\mathcal{H}^k(\Pi \cap B(x_t, r_t)) = t^k \mathcal{H}^k(\Pi \cap B(x_1, r_t/t))$.

Since $r_t/t \to \alpha'$, it follows that

$$\mathcal{H}^k\left(\Pi \cap B\left(x_1, \frac{r_t}{t}\right)\right) \to \mathcal{H}^k(\Pi \cap B(x_1, \alpha')) \text{ as } t \to 0.$$

Given $\varepsilon > 0$, for t small it follows that

$$\begin{aligned}\mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1, \alpha' - \varepsilon)\right) &\le \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B\left(x_1, \frac{r_t}{t}\right)\right)\\ &\le \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1, \alpha' + \varepsilon)\right).\end{aligned}$$

Therefore,

$$\lim_{t\to 0} \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B\left(x_1, \frac{r_t}{t}\right)\right) = \mathcal{H}^k(\Pi \cap B(x_1, \alpha'))$$

if and only if

$$\lim_{t\to 0} \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1,\alpha')\right) = \mathcal{H}^k(\Pi \cap B(x_1,\alpha')). \tag{3.3}$$

Equality (3.3) is proved next.

Since Π is the tangent space of Γ at 0, it follows that Γ is locally a graph over Π. There exists a $\tilde{\Phi}\colon B_k(0,R) \subset \Pi \to \Gamma$ where $\tilde{\Phi}(x) = (x, \tilde{\phi}(x))$ and $\tilde{\phi}\colon B_k(0,R) \to \mathbb{R}^{n-k}$ is Lipschitz.

Define

$$\tilde{\Phi}_t \colon \frac{1}{t}B_k(0,R) \to \frac{1}{t}\Gamma \quad \text{by} \quad \tilde{\Phi}_t(y) = \frac{1}{t}\tilde{\Phi}(ty) = \left(y, \frac{1}{t}\tilde{\phi}(ty)\right).$$

For small $t > 0$,

$$\mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1,\alpha')\right) = \int_{\mathrm{Proj}_\Pi((1/t)\Gamma\cap B(x_1,\alpha'))} J\tilde{\Phi}_t(y)\,dy.$$

Computation gives $J\tilde{\Phi}_t(y) = J\tilde{\Phi}(ty)$. Therefore,

$$\mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1,\alpha')\right) = \int_{\mathrm{Proj}_\Pi((1/t)\Gamma\cap B(x_1,\alpha'))} J\tilde{\Phi}(ty)\,dy.$$

Some basic analysis shows that $\mathrm{Proj}_\Pi\left(\frac{1}{t}\Gamma \cap B(x_1,\alpha')\right)$ converges in measure to $\Pi \cap B(x_1,\alpha')$. Hence

$$\begin{aligned}
\lim_{t\to 0} \mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B(x_1,\alpha')\right) &= \lim_{t\to 0}\int_{\Pi\cap B(x_1,\alpha')} J\tilde{\Phi}(ty)\,dy \\
&= \lim_{t\to 0}\int_{\Pi\cap B(x_1,\alpha')} (J\tilde{\Phi}(ty) - J\tilde{\Phi}(0))\,dy \\
&\quad + J\tilde{\Phi}(0)\mathcal{H}^k(\Pi \cap B(x_1,\alpha')) \\
&= \mathcal{H}^k(\Pi \cap B(x_1,\alpha')).
\end{aligned}$$

The last equality follows because $J\tilde{\Phi} \in L^1_{\mathrm{loc}}(B(0,r))$ and $J\tilde{\Phi}(0) = 1$. This proves (3.3). Then

$$\begin{aligned}
t^{n-k}M\mu_\Gamma(x_t) &= t^{n-k}\frac{\mathcal{H}^k(\Gamma \cap B(x_t,r_t))}{\gamma_n r_t^n} \\
&= \left(\frac{t}{r_t}\right)^{n-k}\frac{t^k}{\gamma_n r_t^k}\mathcal{H}^k\left(\frac{1}{t}\Gamma \cap B\left(x_1, \frac{r_t}{t}\right)\right) \\
&\to \frac{\mathcal{H}^k(\Pi \cap B(x_1,\alpha')}{\gamma_n \alpha'^n} \\
&\le M\mu_\Pi(x_1).
\end{aligned}$$

Therefore,

$$\limsup_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(x_t) \le \gamma_{n-k} M\mu_\Pi(x_1). \tag{3.4}$$

On the other hand, if $\tilde{r}$ satisfies

$$M\mu_\Pi(x_1) = \frac{\mathcal{H}^k(\Pi \cap B(x_1,\tilde{r}))}{\gamma_n \tilde{r}^n},$$

then

$$t^{n-k} M\mu_\Gamma(x_t) \geq t^{n-k} \frac{\mathcal{H}^k(\Gamma \cap B(x_t, t\tilde{r}))}{\gamma_n t^n \tilde{r}^n}$$
$$\to M\mu_\Pi(x_1).$$

Hence

$$\gamma_{n-k} M\mu_\Pi(x_1) \leq \liminf_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(x_t). \tag{3.5}$$

It follows from (3.4) and (3.5) that if $\liminf_{t\to 0} r_t/t' > 1$ then

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(x_t) = \gamma_{n-k} M\mu_\Pi(x_1).$$

Next observe that, since $\operatorname{dist}(x_1, \Pi) = \sin\Theta$,

$$M\mu_\Pi(x_1) = \sup_{r>\sin\Theta} \frac{\mu_\Pi(B(x_1, r) \cap \Pi)}{\gamma_n r^n}$$
$$= \sup_{r>\sin\Theta} \frac{\gamma_k \sqrt{r^2 - \sin^2\Theta}^{\,k}}{\gamma_n r^n}.$$

Some basic calculation shows that this is equal to $\tilde{C}_{n,k}/\gamma_{n-k} \sin\Theta^{n-k}$, where $\tilde{C}_{n,k}$ is given in (1.6). This proves the lemma in Case 1. It remains to consider Case 2 and Case 3.

Case 2: $\liminf_{t\to 0} r_t/t' = 1$. Assume the limit exists and equals 1. The proof given in Case 1 can be used here to show that

$$\liminf_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(x_t) \geq M\mu_\Pi(x_1).$$

Suppose $t_i \to 0$ and

$$\lim_{i\to\infty} \gamma_{n-k} t_i^{n-k} \frac{\mathcal{H}^k(\Gamma \cap B(x_{t_i}, r_{t_i}))}{\gamma_n r_{t_i}^n} > (1+\varepsilon) M\mu_\Pi(x_1). \tag{3.6}$$

Choose $1 < \alpha < (1+\varepsilon)^{1/n}$. Then the left term in (3.6) is

$$\leq \lim_{i\to\infty} \gamma_{n-k} \left(\frac{t_i}{r_{t_i}}\right)^n t_i^{n-k} \alpha^n \frac{\mathcal{H}^k(\Gamma \cap B(x_{t_i}, \alpha t_i))}{\gamma_n (\alpha t_i)^n}$$
$$\leq \alpha^n M\mu_\Pi(x_1) < (1+\varepsilon) M\mu_\Pi(x_1),$$

which contradicts (3.6). (The middle inequality is a consequence of Case 1.) Therefore,

$$\lim_{t\to 0} \gamma_{n-k} t^{n-k} M\mu_\Gamma(x_t) = M\mu_\Pi(x_1) = \frac{\tilde{C}_{n,k}}{(\sin\Theta)^{n-k}},$$

as required.

Although it is not proved here, it is expected that Case 2 does not occur.

Case 3: $\liminf_{t\to 0} r_t/t' < 1$. Let $S(\varphi) := S(\varphi, 0)$ with $S(\varphi, x)$ as defined near the beginning of the proof. Choose φ small enough so that, for t small,

$B(x_t, r_t) \cap S(\varphi) = \emptyset$. It is also easy to prove that, for ε small, $\Gamma \cap B(0, \varepsilon) = \Gamma \cap S(\varphi) \cap B(0, \varepsilon)$. Since $B(x_t, r_t) \subset B(0, \varepsilon)$ for t small enough, it follows that $\Gamma \cap B(x_t, r_t) = \emptyset$. This would imply by the definition of r_t that $M\mu_\Gamma(x_t) = 0$, a contradiction. Hence Case 3 cannot happen, and this completes the proof of the lemma. □

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References

[1] A. Baernstein II, *Some sharp inequalities for conjugate functions,* Indiana Univ. Math. J. 27 (1978), 833–852.

[2] B. Davis, *On the weak type* (1,1) *inequality for conjugate functions,* Proc. Amer. Math. Soc. 44 (1974), 307–311.

[3] J. Duoandikoetxea, *Fourier analysis,* Grad. Stud. Math., 29, Amer. Math. Soc., Providence, RI, 2001.

[4] L. Grafakos, *Classical and modern Fourier analysis,* Pearson, Upper Saddle River, NJ, 2004.

[5] T. Iwaniec and G. Martin, *Riesz transforms and related singular integrals,* J. Reine Angew. Math. 473 (1996), 25–57.

[6] P. Janakiraman, *Weak-type estimates for singular integrals and the Riesz transform*, Indiana Univ. Math. J. 53 (2004), 533–555.

[7] ———, *Limiting weak-type behavior for singular integral and maximal operators*, Trans. Amer. Math. Soc. 358 (2006), 1937–1952.

[8] F. Jones, *Lebesgue integration on Euclidean plane,* Jones & Bartlett, Boston, 1993.

[9] P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability,* Cambridge Stud. Adv. Math., 44, Cambridge Univ. Press, Cambridge, 1995.

[10] A. Melas, *The best constant for the centered Hardy–Littlewood maximal inequality,* Ann. of Math. (2) 157 (2003), 647–688.

[11] S. K. Pichorides, *On the best value of the constants in the theorems of M. Riesz, Zygmund, and Kolmogorov,* Studia Math. 44 (1972), 165–179.

[12] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton Univ. Press, Princeton, NJ, 1970.

[13] E. M. Stein and J.-O. Strömberg, *Behavior of maximal functions in* $\mathbb{R}^n$ *for large* n, Ark. Mat. 21 (1983), 259–269.

[14] N. T. Varopoulos, *A theorem on weak type estimates for Riesz transforms and martingale transforms,* Ann. Inst. Fourier (Grenoble) 31 (1981), 257–264.

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