

Extremal Discs and Analytic Continuation of Product CR Maps

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Introduction

One of the essentially multidimensional phenomena in complex analysis is the forced analytic continuation of a germ of a biholomorphic map $M_1 \rightarrow M_2$ between real analytic manifolds M_1 and M_2 in \mathbf{C}^n , $n > 1$. Poincaré (1907) observed that a biholomorphic map sending an open piece of the unit sphere in \mathbf{C}^2 to another such open piece must be an automorphism of the unit ball. This was proved for \mathbf{C}^n by Tanaka (1962) and then rediscovered by Alexander [A].

Pinchuk [P] proved that, if M_1 and M_2 are strictly pseudoconvex real analytic nonspherical hypersurfaces and M_2 is compact, then a germ of a biholomorphic map $M_1 \rightarrow M_2$ holomorphically extends along any path in M_1 . Ezhov, Kruzhilin, and Vitushkin [EKV] gave a different proof of that result. Webster [W] proved that a germ of a biholomorphic map $M_1 \rightarrow M_2$ between real algebraic Levi non-degenerate hypersurfaces in \mathbf{C}^n is algebraic.

There is an impressive number of publications in which M_1 and M_2 are real *algebraic* manifolds of different dimensions or higher codimension, in particular real quadratic manifolds (see [BER]). Hill and Shafikov [HS] proved the analytic continuation result in higher codimension where only one of the manifolds M_1 and M_2 is assumed to be algebraic. There are many more results on the problem that we omit here (see e.g. [BER; HS] for references).

Despite the large amount of work done on the problem, there seem to be no results in the literature where M_1 and M_2 are manifolds of higher codimension in \mathbf{C}^n and neither of them is algebraic. In this paper we consider the case in which M_1 is a real analytic strictly pseudoconvex manifold and M_2 is the Cartesian product of several compact strictly convex real analytic hypersurfaces. In particular, we give another proof of Pinchuk's [P] result for the case where M_2 is strictly convex and neither of the M_j is assumed to be nonspherical.

For the case in which M_2 is the product of two spheres, the result was obtained earlier by the first author [Sc]. In this paper we significantly simplify and generalize the proof given in [Sc]. Following [Sc], we use a new method based on extremal discs in higher codimension. As by-products, we obtain some properties of extremal discs that may be useful elsewhere.

1. Strictly Pseudoconvex Manifolds

In this section we recall basic notation and definitions concerning real manifolds in complex space.

Let M be a C^∞ -smooth real generic manifold in \mathbf{C}^N of real codimension k . Recall that M is *generic* if $T_p(M) + JT_p(M) = T_p(\mathbf{C}^N)$ for $p \in M$, where $T(M)$ denotes the tangent bundle to M and where J is the operator of multiplication by the imaginary unit in $T(\mathbf{C}^N)$. Recall that the *complex tangent space* $T_p^c(M)$ of M at $p \in M$ is defined as $T_p^c(M) = T_p(M) \cap JT_p(M)$. If M is generic then M is a CR manifold, which means that $\dim_{\mathbf{C}} T_p^c(M)$ is independent of p and that $T^c(M)$ forms a bundle. Recall that the space $T_p^{(1,0)}(M) \subset T_p(M) \otimes \mathbf{C}$ of complex $(1,0)$ -vectors is defined as $T_p^{(1,0)}(M) = \{X \in T_p(M) \otimes \mathbf{C} : X = \sum a_j \partial / \partial z_j\}$. The CR-dimension $\dim_{\text{CR}}(M)$ of M is equal to $\dim_{\mathbf{C}} T_p^c(M) = \dim_{\mathbf{C}} T_p^{(1,0)}(M)$. If $\dim_{\text{CR}}(M) = n$, then $N = n + k$.

Let $T^*(\mathbf{C}^N)$ be the real cotangent bundle of \mathbf{C}^N . Since every $(1,0)$ -form is uniquely determined by its real part, we represent $T^*(\mathbf{C}^N)$ as the space of $(1,0)$ -forms on \mathbf{C}^N . Then $T^*(\mathbf{C}^N)$ is a complex manifold. Let $N^*(M) \subset T^*(\mathbf{C}^N)$ be the real conormal bundle of $M \subset \mathbf{C}^N$. Using the representation of $T^*(\mathbf{C}^N)$ by $(1,0)$ -forms, we define the fiber $N_p^*(M)$ at $p \in M$ as

$$N_p^*(M) = \{\phi \in T_p^*(\mathbf{C}^N) : \text{Re } \phi|_{T_p(M)} = 0\}.$$

We use the angle brackets $\langle \cdot, \cdot \rangle$ to denote the natural pairing between vectors and covectors, so we write $\langle \phi, \xi \rangle = \sum \phi_j \xi_j$ for their coordinate representations. In a fixed coordinate system, we will identify $\phi = \sum \phi_j dz_j \in T^*(\mathbf{C}^N)$ with the vector $\phi = (\phi_1, \dots, \phi_N) \in \mathbf{C}^N$. Then, for $\phi \in N_p^*(M)$, the vector $\bar{\phi}$ is orthogonal to M in the real sense; that is, $\text{Re} \langle \phi, X \rangle = 0$ for all $X \in T_p(M)$.

Since M is generic, it follows that locally M can be defined as $\rho(z) = 0$, where $\rho = (\rho_1, \dots, \rho_k)$ is a smooth real vector function such that $\partial \rho_1 \wedge \dots \wedge \partial \rho_k \neq 0$. The forms $\partial \rho_j$ ($j = 1, \dots, k$) define a basis of $N_p^*(M)$, so every $\phi \in N_p^*(M)$ can be written as $\phi = \sum c_j \partial \rho_j$, $c_j \in \mathbf{R}$.

For every $\phi \in N_p^*(M)$, we define the Levi form $L(p, \phi)$ of M at $p \in M$ in the conormal direction $\phi = \sum c_j \partial \rho_j$ as

$$L(p, \phi)(X, Y) = -\sum c_j \partial \bar{\partial} \rho_j(X, \bar{Y}),$$

where $X, Y \in T_p^{1,0}(M)$. The form $L(p, \phi)$ is a hermitian form on $T_p^{1,0}(M)$. This definition is independent of the defining function. The forms $L(p, \phi)$ can be regarded as components of the $N_p(M)$ -valued Levi form $L(p)$, where $N(M) = T(\mathbf{C}^N)|_M / T(M)$ is the normal bundle of $M \subset \mathbf{C}^N$. Indeed, $L(p)(X, X) \in N_p(M)$ is such an element with

$$\text{Re} \langle \phi, L(p)(X, X) \rangle = L(p, \phi)(X, X) \quad \text{for all } \phi \in N_p^*(M).$$

The Levi cone $\Gamma_p \subset N_p(M)$ is defined as the convex span of the values of the Levi form $L(p)$; that is,

$$\Gamma_p = \text{Conv}\{L(p)(X, X) : X \in T_p^{1,0}(M), X \neq 0\}.$$

We also need the Levi cone $H_p \subset T_p(M)$. We put

$$H_p = \{\xi \in T_p(M) : [J\xi] \in \Gamma_p\},$$

where the brackets denote the class in the quotient space $N_p(M)$. If M is a strictly pseudoconvex hypersurface, then Γ_p is the half-line defined by the inner normal to M at p and H_p is a half-space of $T_p(M)$. The dual Levi cone Γ_p^* is defined as

$$\Gamma_p^* = \{\phi \in N_p^*(M) : L(p, \phi) > 0\},$$

where $L(p, \phi) > 0$ means that the form $L(p, \phi)$ is positive definite. The cones Γ_p and Γ_p^* are dual; that is, $\xi \in \Gamma_p$ if and only if $\operatorname{Re}\langle \phi, \xi \rangle > 0$ for all $\phi \in \Gamma_p^*$.

We say that M is *strictly pseudoconvex* at p if $\Gamma_p^* \neq \emptyset$. We say that M is strictly pseudoconvex if it holds at every $p \in M$. We say that the Levi form $L(p)$ is *generating* if Γ_p has nonempty interior.

Changing notation, we introduce the coordinates $(z, w) \in \mathbf{C}^N$ ($z = x + iy \in \mathbf{C}^k$, $w \in \mathbf{C}^n$) so that the defining function of M can be chosen in the form $\rho = x - h(y, w)$, where $h = (h_1, \dots, h_k)$ is a smooth real vector function and the equations of M take the following form (see e.g. [BER]):

$$x_j = h_j(y, w) = \langle A_j w, \bar{w} \rangle + O(|y|^3 + |w|^3), \quad 1 \leq j \leq k; \quad (1.1)$$

here the A_j are hermitian matrices. Then $T_0^{1,0}(M)$ is identified with the w -space \mathbf{C}^n and, for $\phi = \sum c_j dz_j \in N_0^*(M)$, the Levi form $L(0, \phi)$ has the matrix $\sum c_j A_j$. Hence, the manifold M of the form (1.1) is strictly pseudoconvex at 0 if and only if there exists a $c \in \mathbf{R}^k$ such that $\sum c_j A_j > 0$. It has a generating Levi form at 0 if and only if the matrices A_1, \dots, A_k are linearly independent.

We say that a vector-valued hermitian form B *splits into scalar forms* of dimensions (n_1, \dots, n_k) if the source and target spaces V and Z of B split into direct sums $V = \sum V_j$ and $Z = \sum Z_j$, with $\dim V_j = n_j > 0$ and $\dim Z_j = 1$, such that $B(u, v) = \sum B_j(u_j, v_j)$; here $u_j, v_j \in V_j$, $u = \sum u_j$, $v = \sum v_j$, and B_j is a Z_j -valued hermitian form on V_j . We need the following simple result.

PROPOSITION 1.1. *Let M be a connected real-analytic generic manifold in \mathbf{C}^N . Suppose that the Levi form of M splits into scalar forms on an open subset of M ; then it splits into scalar forms everywhere on M . If M is strictly pseudoconvex, then the Levi form is generating and splits into positive-definite forms.*

Proof. The set of all splittable hermitian forms is a real analytic (even algebraic) subset of the set of all hermitian forms. The map $M \ni p \mapsto L(p)$ is real analytic. Since it takes an open set of M to splittable forms and since M is connected, it follows that the whole image belongs to splittable forms. The rest of the conclusions hold automatically, so the proof is complete. \square

2. Extremal Discs

We recall some facts about the theory of extremal discs (see [L, T1]).

Let M be a smooth generic manifold in \mathbf{C}^N . An *analytic disc* in \mathbf{C}^N is a continuous mapping $f: \bar{\Delta} \rightarrow \mathbf{C}^N$ that is holomorphic in the unit disc Δ . We say that f is *attached* to M if $f(b\Delta) \subset M$.

An analytic disc f attached to M is called *stationary* if there exists a nonzero continuous holomorphic mapping $f^*: \bar{\Delta} \setminus \{0\} \rightarrow T^*(\mathbf{C}^N)$ such that $\tilde{f} = \zeta f^*$ is holomorphic in Δ and $f^*(\zeta) \in N_{f(\zeta)}^*(M)$ for all $\zeta \in b\Delta$. In other words, f^* is a punctured analytic disc with a pole of order at most 1 at 0 and attached to $N^*(M) \subset T^*(\mathbf{C}^N)$ such that the natural projection sends f^* to f . We call f^* a *lift* of f , and we always use the term “lift” in this sense.

We call a disc f *defective* if it has a nonzero lift f^* that is holomorphic in the whole unit disc including 0. For a strictly convex hypersurface, all defective discs are constant.

We call a lift f^* of a stationary disc f *supporting* if, for all $\zeta \in b\Delta$, $f^*(\zeta)$ defines a (strong) supporting real hyperplane to M at $f(\zeta)$ —that is, if

$$\operatorname{Re}\langle f^*(\zeta), p - f(\zeta) \rangle \geq \varepsilon |p - f(\zeta)|^2 \quad \text{for all } \zeta \in b\Delta \text{ and } p \in M \quad (2.1)$$

for some $\varepsilon > 0$. Stationary discs with supporting lifts have important extremal properties, but we do not need them here. Nevertheless, we call such f *extremal* and we call the pair (f, f^*) an *extremal pair*. Although f is completely determined by f^* , we prefer to use the excessive notation (f, f^*) because it allows us to describe f^* by its fiber coordinates in $T^*(\mathbf{C}^N)$. Note that (2.1) implies $f^*(\zeta) \in \Gamma_{f(\zeta)}^*$ for $\zeta \in b\Delta$.

If M is the boundary of a strictly convex domain $D \subset \mathbf{C}^N$, then the set of all extremal discs is smoothly parameterized by the correspondence $f \leftrightarrow (f(0), f(1)) \in D \times bD$. The set of all extremal pairs is parameterized by $D \times bD \times \mathbf{R}^+$ because the lift of an extremal disc is unique up to a positive constant factor; see [L].

In higher codimension there is a local parameterization of the set of extremal pairs.

THEOREM 2.1 [T1]. *Let $M \subset \mathbf{C}^N$ be a smooth (resp. real-analytic) strictly pseudoconvex manifold with generating Levi form defined by (1.1). Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that—for every $\lambda \in \mathbf{C}^k$, $c \in \mathbf{R}^k$, $w_0 \in \mathbf{C}^n$, $y_0 \in \mathbf{R}^k$, and $v \in \mathbf{C}^n$ such that*

$$\sum \operatorname{Re}(\lambda_j \zeta + c_j) A_j > \varepsilon (|\lambda| + |c|) I$$

and such that $|w_0| < \delta$, $|y_0| < \delta$, and $|v| < \delta$ —there exists a unique stationary disc $\zeta \mapsto f(\zeta) = (z(\zeta), w(\zeta))$ with $w(1) = w_0$, $w'(1) = v$, and $y(1) = y_0$ that admits a lift f^ such that $f^*|_{b\Delta} = \operatorname{Re}(\lambda \zeta + c) G \partial \bar{\rho}$ (here λ and c are handled as row vectors, and G is a $k \times k$ matrix function on $b\Delta$ that is close to the identity matrix uniquely determined by f ; see [T1]). The pair (f, f^*) depends smoothly (resp. analytically) on $\zeta \in \bar{\Delta}$ and on all the parameters λ , c , w_0 , y_0 , and v . The pair (f, f^*) is extremal in a suitable coordinate system depending on ε only.*

Let M be a generic manifold in \mathbf{C}^N defined by (1.1). Let Q be the quadratic manifold obtained from (1.1) by dropping the O -terms. We call M *defective* at 0 if all stationary discs for Q are defective. (In other words, if every stationary disc that possibly has a lift with a pole at 0 also has another lift without the pole; the authors do not know whether this situation actually can occur.) This definition is equivalent to the one given in [T2]. If the Levi form of M splits into scalar forms, then

M is not defective. Hence, for fixed ε and sufficiently small δ , all stationary discs provided by Theorem 2.1 are not defective (see [T1, Prop. 6.8] or [T2, Prop. 8.4]).

Define $\mathbb{L}f = \frac{d}{d\theta}|_{\theta=0} f(e^{i\theta})$. Note that if f is holomorphic at $1 \in \mathbb{C}$ then $\mathbb{L}f = Jf'(1)$. Let $\zeta_0 \in b\Delta$ and $\zeta_0 \neq 1$. Let \mathcal{E} denote the set of all extremal pairs (f, f^*) obtained by Theorem 2.1 such that f is not defective. If M is a strictly convex hypersurface then \mathcal{E} stands for the set of all extremal pairs, in which case \mathcal{E} is a smooth manifold by Lempert's theory (see [L]). Consider the following evaluation maps:

$$\mathcal{F}: \mathcal{E} \ni (f, f^*) \mapsto (f(1), f^*(1), \mathbb{L}f, \mathbb{L}f^*) \in TN^*(M);$$

$$\mathcal{G}: \mathcal{E} \ni (f, f^*) \mapsto (f(1), f^*(1), f(\zeta_0), f^*(\zeta_0)) \in N^*(M) \times N^*(M).$$

PROPOSITION 2.2. *The maps \mathcal{F} and \mathcal{G} are injective.*

For the map \mathcal{F} , the proposition is proved in [T1, Prop. 3.9]. The proof for \mathcal{G} is similar. We also need the following stronger version.

PROPOSITION 2.2'. *The maps \mathcal{F} and \mathcal{G} are diffeomorphisms onto their images.*

Proof. The source and target spaces of both \mathcal{F} and \mathcal{G} have the same dimension $4N$. Hence it suffices to show that \mathcal{F} and \mathcal{G} are immersions. By an *infinitesimal perturbation* (\dot{f}, \dot{f}^*) of an extremal pair (f, f^*) we mean an element of the tangent space to the finite-dimensional manifold \mathcal{E} at (f, f^*) . To show that \mathcal{F} is an immersion, we need to show that $\dot{f}(1) = 0$, $\dot{f}^*(1) = 0$, $\mathbb{L}\dot{f} = 0$, and $\mathbb{L}\dot{f}^* = 0$ imply $\dot{f} = 0$ and $\dot{f}^* = 0$.

We realize that $(\dot{f}, \dot{f}^*) = \frac{d}{dt}|_{t=0}(f_t, f_t^*)$, where (f_t, f_t^*) is a smooth 1-parameter family of extremal pairs with $(f_0, f_0^*) = (f, f^*)$. For small t , all the pairs are close to (f, f^*) ; hence we can choose ε in (2.1) the same for all small t . By (2.1), on $b\Delta$ we have

$$\operatorname{Re}\langle f_0^*, f_t - f_0 \rangle \geq \varepsilon |f_t - f_0|^2, \quad \operatorname{Re}\langle f_t^*, f_0 - f_t \rangle \geq \varepsilon |f_0 - f_t|^2.$$

Adding the two inequalities yields

$$\operatorname{Re}\langle f_t^* - f_0^*, f_t - f_0 \rangle \leq -2\varepsilon |f_t - f_0|^2.$$

Dividing by t^2 and letting $t \rightarrow 0$ yields

$$\operatorname{Re}\langle \dot{f}^*, \dot{f} \rangle \leq -2\varepsilon |\dot{f}|^2$$

for $\zeta \in b\Delta$. The hypotheses imply that $\dot{f} = O(|\zeta - 1|^2)$ and $\dot{f}^* = O(|\zeta - 1|^2)$. Hence

$$\operatorname{Re} \int_0^{2\pi} \frac{\langle \dot{f}^*, \dot{f} \rangle}{|\zeta - 1|^4} d\theta \leq -2\varepsilon \int_0^{2\pi} \frac{|\dot{f}|^2}{|\zeta - 1|^4} d\theta,$$

where $\zeta = e^{i\theta}$. Note that for $|\zeta| = 1$ we have $d\zeta = i\zeta d\theta$ and $\zeta|\zeta - 1|^2 = -(\zeta - 1)^2$. Then

$$\int_0^{2\pi} \frac{\langle \dot{f}^*, \dot{f} \rangle}{|\zeta - 1|^4} d\theta = -i \int_{b\Delta} \left\langle \frac{\zeta \dot{f}^*}{(\zeta - 1)^2}, \frac{\dot{f}}{(\zeta - 1)^2} \right\rangle d\zeta = 0,$$

since the integrand is holomorphic in Δ . Therefore,

$$\int_0^{2\pi} \frac{|\dot{f}|^2}{|\zeta - 1|^4} d\theta = 0$$

and $\dot{f} = 0$. Since $\dot{f} = 0$, we have that $\dot{f}^*|_{b\Delta}$ is tangent to the fibers of $N^*(M)$ and gives rise to a lift of f . Since $\dot{f}^* = O(|\zeta - 1|^2)$, it follows that $\tilde{f} = \zeta(\zeta - 1)^{-2}\dot{f}^*$ is a lift of f without a pole at 0. Because f is not defective, $\tilde{f} = 0$; whence $\dot{f}^* = 0$ and \mathcal{F} is an immersion. The proof that \mathcal{G} is an immersion is similar; it uses the identity $\zeta\zeta_0|\zeta - \zeta_0|^2 = -(\zeta - \zeta_0)^2$ for $|\zeta| = |\zeta_0| = 1$. The proof is now complete. \square

Define $T^+N^*(M) \subset TN^*(M)$. We put $\xi \in T_{(p,\phi)}^+N^*(M)$ if $\phi \in \Gamma_p^*$ and $\pi_*\xi \in H_p$, where $\pi: T^*(\mathbf{C}^N) \rightarrow \mathbf{C}^N$ is the natural projection; the Levi cones Γ_p^* and H_p are defined in Section 1.

PROPOSITION 2.3. *Let M be a strictly convex hypersurface in \mathbf{C}^{n+1} . Then $\mathcal{F}(\mathcal{E}) = T^+N^*(M)$.*

Proof. The inclusion $\mathcal{F}(\mathcal{E}) \subset T^+N^*(M)$ follows by the Hopf lemma. Indeed, let M bound the domain D defined by $\rho < 0$, where ρ is a strictly convex function. Let f be a nonconstant analytic but not necessarily stationary disc attached to M , and let $f(1) = p \in M$. Then the nonconstant subharmonic function $\rho \circ f$ in Δ is zero on the boundary. By the Hopf lemma, $\langle d\rho, f'(1) \rangle > 0$. This implies $-[f'(1)] \in \Gamma_p$, whence $\mathbb{L}f = Jf'(1) \in H_p$. If $(f, f^*) \in \mathcal{E}$ then $f^*(1) \in \Gamma_p^*$, and the desired inclusion follows.

The surjectivity of \mathcal{F} follows by a simple topological argument. Fix $p \in M$. Put $\mathcal{E}_p = \{(f, f^*) \in \mathcal{E} : f(1) = p\}$. Then the set \mathcal{E}_p is contractible since f is completely determined by $f(0) \in D$ and $f(1) = p$ and since, for given f , the supporting lift f^* is unique up to a positive multiplicative constant (see [L]).

Given $(f, f^*) \in \mathcal{E}_p$, we make a substitution by an automorphism of the unit disc $\zeta = (\tau - \tau_0)e^{i\theta_0}/(1 - \bar{\tau}_0\tau)$ with fixed point 1. Put

$$g(\tau) = f(\zeta), \quad g^*(\tau) = f^*(\zeta) \frac{(\tau - \tau_0)(1 - \bar{\tau}_0\tau)}{\tau|1 - \tau_0|^2},$$

where we choose the factor so that g^* has a pole at 0 and $g^*(1) = f^*(1)$. Then $(g, g^*) \in \mathcal{E}_p$, and one can further check that

$$\mathbb{L}g = \alpha\mathbb{L}f, \quad \mathbb{L}g^* = \mathbb{L}f^* - \beta f^*(1), \quad (2.2)$$

where $\alpha, \beta \in \mathbf{R}$ and $\alpha + i\beta = (1 + \tau_0)/(1 - \tau_0)$. Since $\tau_0 \in \Delta$ is arbitrary, it follows that $\alpha > 0$ and $\beta \in \mathbf{R}$ are arbitrary.

Consider the map $\Phi: \mathcal{E}_p \ni (f, f^*) \mapsto \mathbb{L}f/|\mathbb{L}f| \in S^+$, where $S^+ = S^{2n+1} \cap H_p$ is the unit hemisphere in H_p . By (2.2), the preimages of the map Φ are contractible. Since \mathcal{E}_p is contractible, so is $\Phi(\mathcal{E}_p)$. It suffices to show that $\Phi(\mathcal{E}_p) = S^+$. We will show that $\Phi(\mathcal{E}_p)$ contains an arbitrarily small perturbation of the equator of the hemisphere S^+ . Then $\Phi(\mathcal{E}_p)$ will have to be all of S^+ .

We introduce a coordinate system $(z = x + iy, w) \in \mathbf{C} \times \mathbf{C}^n$ such that $p = 0$ and M has a local equation

$$x = |w|^2 + O(|y|^3 + |w|^3).$$

Then $T_p(M)$ is defined by $x = 0$ and $H_p \subset T_p(M)$ is the half-space $y < 0$. The stationary disc f constructed by Theorem 2.1 for $\lambda = 0$, $c = 1$, $w_0 = 0$, $y_0 = 0$, and small $v \in \mathbf{C}^n$ has the following asymptotic expression (see [T1, Cor. 5.2]):

$$z(\zeta) = O(|v|^2), \quad w(\zeta) = (\zeta - 1)v + O(|v|^2).$$

Then

$$\frac{\mathbb{L}f}{|\mathbb{L}f|} = \left(0, \frac{v}{|v|}\right) + O(|v|), \quad |v| = \varepsilon$$

for small ε describes a small perturbation of the equator of the hemisphere S^+ . Hence $\Phi(\mathcal{E}_p) = S^+$ and the proof is complete. \square

If M is a product of strictly convex hypersurfaces, then $N^*(M)$, $T^+N^*(M)$, \mathcal{E} , ... are the products of the corresponding objects for the components of the product. Then we immediately derive the following.

COROLLARY 2.4. *Let M be a product of strictly convex hypersurfaces. Then $\mathcal{F}(\mathcal{E}) = T^+N^*(M)$.*

3. The Main Result

THEOREM 3.1. *Let M_1 be a real-analytic and strictly pseudoconvex generic manifold, and let M_2 be a product of several real-analytic strictly convex hypersurfaces. Then every biholomorphic map taking an open set in M_1 to M_2 continues along any path in M_1 as a locally biholomorphic map.*

REMARK. We require that M_2 be a product because we use Corollary 2.4 in the proof. It would be interesting to find out for what manifolds the conclusion of Corollary 2.4 is valid.

Proof of Theorem 3.1. The main idea of the proof is that a biholomorphism preserves extremal pairs and so extends along the extremal discs.

Let F be a biholomorphic map defined at $p_1 \in M_1$ such that $F(U) \subset M_2$ for some open set $U \subset M_1$. The map F lifts to the cotangent bundle $T^*(\mathbf{C}^N)$ in the usual way. With some abuse of notation, we use the same letter F for the lifted map. We choose a coordinate system in which $p_1 = 0$ and M_1 is given by (1.1). Since M_2 is a product, the Levi form of M_2 splits into scalar positive-definite forms. Since the biholomorphic map F preserves the Levi forms, it follows that the Levi form of M_1 at $p_1 = 0$ also splits into scalar positive-definite forms and, after a linear change of coordinates, the equation of M_1 takes the form

$$x_j = h_j(y, w) = |w_j|^2 + O(|y|^3 + |w|^3), \quad w_j \in \mathbf{C}^{n_j}, \quad n_1 + \cdots + n_k = n. \quad (3.1)$$

We note that the size of the coordinate chart for which (3.1) holds is independent of the map F . Indeed, if we know that such an F exists then, by Proposition 1.1, the Levi form of (the component of) M_1 splits into scalar forms. Then, while extending F along a path, we can always restrict to finitely many coordinate charts by the compactness argument.

We denote by \mathcal{F}_v and \mathcal{G}_v the evaluation maps for M_v ($v = 1, 2$) defined in Section 2. Although Corollary 2.4 generally fails for M_1 , there are many extremal pairs (f, f^*) such that $\mathcal{F}_1(f, f^*) \in T^+N^*(M_1)$. Indeed, let (f, f^*) be the extremal pair constructed by Theorem 2.1 for $\lambda = 0$, $c_j = 1$, $w_0 = 0$, $y_0 = 0$, and small $v \in \mathbf{C}^n$. Then the components of f admit the following asymptotic expression (see [T1, Cor. 5.2]):

$$z(\zeta) = O(|v|^2), \quad w(\zeta) = (\zeta - 1)v + O(|v|^2). \quad (3.2)$$

Furthermore, plugging (3.2) in (3.1) and using the identity $|\zeta - 1|^2 = -2 \operatorname{Re}(\zeta - 1)$ for $|\zeta| = 1$, we obtain

$$z_j(\zeta) = -2(\zeta - 1)|v_j|^2 + O(|v|^3), \quad \mathbb{L}z_j = -2i|v_j|^2 + O(|v|^3).$$

Note that the Levi cone H_0 of M_1 is defined by $x = 0$, $y_j < 0$. Thus, if all $|v_j|$ are small and comparable, then $\mathbb{L}f \in H_0$ and $\mathcal{F}_1(f, f^*) \in T^+N^*(M_1)$. The same is true for all extremal pairs constructed using parameters λ and c that are close to these values.

Consider all extremal pairs (f_1, f_1^*) for M_1 with fixed $f_1(1) = p_1 = 0$ and $f_1^*(1)$ such that $\mathcal{F}_1(f_1, f_1^*) \in T^+N^*(M_1)$. Denote the set of such pairs by \mathcal{E}_1 .

We define the desired extension of the map F by using $\mathcal{G}_2 \circ \mathcal{F}_2^{-1} \circ F_* \circ \mathcal{F}_1 \circ \mathcal{G}_1^{-1}$. More precisely, put $\xi = \mathcal{F}_1(f_1, f_1^*)$. Since F preserves the Levi forms, we have $F_*\xi \in T^+N^*(M_2)$. By Corollary 2.4 there exists a unique extremal pair (f_2, f_2^*) for M_2 such that $\mathcal{F}_2(f_2, f_2^*) = F_*\xi$. Fix $\zeta_0 \in b\Delta$, $\zeta_0 \neq 1$. We define $\tilde{F}((f_1, f_1^*)(\zeta_0)) = (f_2, f_2^*)(\zeta_0)$. By Proposition 2.2', the map \tilde{F} is a diffeomorphism on the set $\{(f_1, f_1^*)(\zeta_0) : (f_1, f_1^*) \in \mathcal{E}_1\}$. Since all the objects are real analytic, it follows that \tilde{F} is real analytic on an open set in $N^*(M)$. Note that the pair (f_1, f_1^*) shrinks into a point as $v \rightarrow 0$. This implies that the map \tilde{F} agrees with F on an open set in $N^*(M)$, since F preserves extremal pairs. The extension preserves the fibers of $N^*(M)$ because F does. Hence, \tilde{F} defines a real-analytic diffeomorphism on the set $\{(f_1(\zeta_0) : (f_1, f_1^*) \in \mathcal{E}_1\} \subset M$. By varying $\zeta_0 \in b\Delta$, we extend \tilde{F} as a real-analytic diffeomorphism on the set $V = \bigcup \{(f_1(b\Delta \setminus \{1\}) : (f_1, f_1^*) \in \mathcal{E}_1\} \subset M$. Since \tilde{F} is real analytic and satisfies the tangential Cauchy–Riemann equations on an open set in M , we know that \tilde{F} is CR on the whole set $V \subset M$ where it is defined. Then, by real analyticity, \tilde{F} further extends to a biholomorphic map in a neighborhood of V in \mathbf{C}^N .

Thus we conclude that F extends as a biholomorphic map along the boundaries of the extremal discs f_1 . By Proposition 2.2' (see also [T1, Cor. 5.6]), the directions of the boundary curves of the discs f_1 span the tangent space $T_{p_1}(M_1)$. As a consequence, all points within the same connected component can be reached by moving along the boundaries of such discs, and the theorem follows. \square

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