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Representations of $N = 1$ ADE Quivers via Reflection Functors

Xinyun Zhu

1. Introduction

In 1972, Gabriel [4] published his celebrated theorem about the finite representation type of ADE quivers without relations. Since then, the study of quiver representations has been an important topic because it provides a successful way to solve problems in the representation theory of algebras and Lie groups. This study has recently attracted the attention of physicists (see [2; 3; 6]) owing to its close relation with the study of D-branes. A special type of quiver arising from string theory, which we will call the "$N = 1$ ADE quiver", was introduced in [2] (see Definition 2.1 herein). This quiver has a close relation with the usual ADE quiver. The representations of $N = 1$ ADE quivers will satisfy the relations

$$\sum_i s_{ij} Q_{ji} Q_{ij} + P_j(\Phi_j) = 0, \qquad Q_{ij}\Phi_j = \Phi_i Q_{ij},$$

where Q_{ij} is a linear map attached to an edge and Φ_i is a linear map attached to a vertex.

The purpose of this paper is to construct, under certain conditions, a finite-to-one correspondence between the simple representations of an $N = 1$ ADE quiver and the positive roots of the usual ADE quiver; this matches the physicists' predictions.

The reflection functors used in [1] to reprove Gabriel's theorem provide us with a way to attack this problem. In this paper we first modify the reflection functors of [1] in Definition 2.5, define new functors F_k in Definition 3.6, and then apply our modified reflection functors F_k to obtain our Main Theorem in Section 3.2. Some related results using different methods were given in [8].

This paper is organized as follows. In Section 2, we give the definition of $N = 1$ ADE quivers and their representations, state the Main Theorem, and introduce our modified reflection functors. In Section 3, we apply our modified reflection functors to prove the Main Theorem. In Section 4, we give a correspondence between simple representations and an ADE configuration of curves.

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2. Description of $N = 1$ ADE Quivers, Statement of the Main Theorem, and Definition of Reflection Functors

2.1. Describing $N = 1$ ADE Quivers

To make our presentation intelligible to nonexperts, we briefly recall some definitions and established facts. (Here all vector spaces are over a field k.)

A *quiver* $\Gamma = (V_\Gamma, E_\Gamma)$—without relations—is a directed graph. A *representation* (V, f) of a quiver Γ is an assignment to each vertex $i \in V_\Gamma$ of a vector space $V(i)$ and to each directed edge $ij \in E_\Gamma$ of a linear transformation $f_{ji} \colon V(i) \to V(j)$.

A *morphism* $h \colon (V, f) \to (V', f')$ between representations of Γ over k is a collection $\{h_i \colon V(i) \to V'(i)\}_{i \in V_\Gamma}$ of k-linear maps such that, for each edge $ij \in E_\Gamma$, the obvious diagram commutes. Compositions of morphisms are defined in the usual way. For a path $p \colon i_1 \to i_2 \to \cdots \to i_r$ in Γ and a representation (V, f), we let f_p be the composition of the linear transformations $f_{i_{k+1} i_k} \colon V(i_k) \to V(i_{k+1})$, $1 \le k < r$. Given vertices i, j in V_Γ and paths $p_1, \dots, p_n$ from i to j, a *relation σ* on quiver Γ is a linear combination $\sigma = a_1 p_1 + \cdots + a_n p_n$, $a_i \in k$. If (V, f) is a representation of Γ, then we extend the f-notation by setting $f_\sigma = a_1 f_{p_1} + \cdots + a_n f_{p_n} \colon V(i) \to V(j)$. A *quiver with relations* is a pair (Γ, ρ), where $\rho = (\sigma_t)_{t \in T}$ is a set of relations on Γ; and a *representation* (V, f) of (Γ, ρ) is a representation (V, f) of Γ for which $f_\sigma = 0$ for all relations $\sigma \in \rho$. We then define, in the obvious way, *subrepresentations* (V', f') of (V, f), the *sum* of representations, and when a representation (V, f) of (Γ, ρ) is indecomposable or simple.

DEFINITION 2.1. Given an ADE Dynkin diagram $\mathcal{D} = (V_\mathcal{D}, E_\mathcal{D})$—an undirected graph—we let the associated quiver $\Gamma_\mathcal{D}$ be $\Gamma_\mathcal{D} = (V_{\Gamma_\mathcal{D}}, E_{\Gamma_\mathcal{D}})$ with $V_{\Gamma_\mathcal{D}} := V_\mathcal{D}$ and

$$E_{\Gamma_\mathcal{D}} = \{(i, j), (j, i) \mid \{i, j\} \in E_\mathcal{D}\} \bigcup \{(i, i) \mid i \in V_\mathcal{D}\}.$$

In other words, this is the standard digraph associated to the graph $\mathcal{D}$, except that we add a loop at each vertex. To illustrate this, we take the E_n case as an example. Recall that the Dynkin diagram for E_n is

$$\begin{tikzcd}[arrows=dash] & & n \arrow[d] & & \\ E_n \colon 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (n-1); \end{tikzcd}$$

thus, the associated quivers for E_n $(n = 6, 7, 8)$ are

$$E_n\colon\quad \begin{tikzcd} 1 \arrow[loop above, "e_1"] \arrow[r, bend left, "e_{21}"] & 2 \arrow[loop above, "e_2"] \arrow[l, bend left, "e_{12}"] \arrow[r, bend left, "e_{32}"] & 3 \arrow[loop above, "e_3"] \arrow[l, bend left, "e_{23}"] \arrow[r, bend left, "e_{43}"] \arrow[d, bend left, "e_{n3}"] & 4 \arrow[loop above, "e_4"] \arrow[l, bend left, "e_{34}"] & \cdots & n-2 \arrow[loop above, "e_{n-2}"] \arrow[r, bend left, "e_{n-1,n-2}"] & n-1 \arrow[loop above, "e_{n-1}"] \arrow[l, bend left, "e_{n-2,n-1}"] \\ & & n \arrow[u, bend left, "e_{3n}"] \arrow[loop below, "e_n"] & & & & \end{tikzcd}$$

The $N = 1$ ADE quivers are just the quivers associated to the displayed graphs but with the following relations. For all vertices i, j with $i \neq j$, the relations have the form

$$\sum_i s_{ij} e_{ji} e_{ij} + P_j(e_j) = 0, \qquad e_{ij} e_j = e_i e_{ij}, \tag{2.1}$$

where

$$s_{ij} = \begin{cases} 0 & \text{if } i \text{ and } j \text{ are not adjacent}, \\ 1 & \text{if } i \text{ and } j \text{ are adjacent and } i > j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent and } i < j, \end{cases}$$

and where $P_j(x)$ is a certain fixed polynomial associated with vertex j for each j.

If (V, f) is a representation of an $N = 1$ ADE quiver, then the corresponding structures are

$$E_n\colon\quad \begin{tikzcd} V(1) \arrow[loop above, "\Phi_1"] \arrow[r, shift left, "Q_{21}"] & V(2) \arrow[loop above, "\Phi_2"] \arrow[l, shift left, "Q_{12}"] \arrow[r, shift left, "Q_{32}"] & V(3) \arrow[loop above, "\Phi_3"] \arrow[l, shift left, "Q_{23}"] \arrow[r, shift left, "Q_{43}"] \arrow[d, shift left, "Q_{n3}"] & V(4) \arrow[loop above, "\Phi_4"] \arrow[l, shift left, "Q_{34}"] & \cdots & V(n-2) \arrow[loop above, "\Phi_{n-2}"] \arrow[r, shift left, "Q_{n-1,n-2}"] & V(n-1) \arrow[loop above, "\Phi_{n-1}"] \arrow[l, shift left, "Q_{n-2,n-1}"] \\ & & V(n) \arrow[u, shift left, "Q_{3n}"] \arrow[loop below, "\Phi_n"] & & & & \end{tikzcd}$$

where we have written $Q_{ij} = f_{e_{ij}}$ and $\Phi_j = f_{e_j}$. Then, for all vertices i, j with $i \neq j$, the relations (2.1) become

$$\sum_i s_{ij} Q_{ji} Q_{ij} + P_j(\Phi_j) = 0, \qquad Q_{ij}\Phi_j = \Phi_i Q_{ij}. \tag{2.2}$$

REMARK 2.2. From now on, we use $(\Gamma, \{P_j\})$ to denote an $N = 1$ ADE quiver satisfying relations (2.1).

The following formulas define an action of the Weyl group on the space of polynomials P_j.

DEFINITION 2.3. Let $\mathfrak{W}_k$ be the Weyl group of the Dynkin diagram Γ, and let $r_i \in \mathfrak{W}_k$ $(1 \le i \le n)$ be a set of generators of reflections. If j is distinct from i and not adjacent to i, then $r_i(P_j(x)) = P_j(x)$. If j is adjacent to i and $j \neq i$, then $r_i(P_j(x)) = P_j(x) + P_i(x)$. Finally, $r_i(P_i(x)) = -P_i(x)$.

Let $(\Gamma, \{P_j\})$ be an $N = 1$ ADE quiver. Let

$$\mathcal{A}_\Gamma = \left\{ \sum_i n_i P_i \mid n_i \in \mathbb{Z}, \text{ not all } n_i \text{ zero} \right\},$$

where the P_i $(1 \le i \le n)$ are the polynomials in relations (2.2).

2.2. Statement of the Main Theorem

$(*)$ No two elements $\sum n_i P_i$ and $\sum m_i P_i$ of the set $\mathcal{A}_\Gamma$ have a common root unless there is a constant c with $m_i = c n_i$ for all i.

LEMMA 2.4. *$(*)$ holds for any very general collection of polynomials P_i of positive degree.*

Proof. Left to the reader. □

We prove the following Main Theorem in Section 3.2.

MAIN THEOREM. *Let $(\Gamma, \{P_j\})$ be an $N = 1$ ADE quiver. Let $\mathcal{B}_\Gamma = \{r_i(P_j(x))\}$, where $r_i \in \mathfrak{W}_\Gamma$ and where P_j $(j \in V_\Gamma)$ are the polynomials defined in relation* (2.2). *Assume that no element in $\mathcal{B}_\Gamma$ has a multiple root. If $(*)$ holds, then $(\Gamma, \{P_j\})$ has only finitely many nonisomorphic simple representations.*

2.3. Reflection Functors

Suppose we are given an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ and $k \in V_\Gamma$. Then we denote by Γ_k^+ the quiver defined by deleting all arrows starting from k and by Γ_k^- the quiver defined by deleting all arrows ending at k.

Given a representation V of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, we can define a representation of Γ_k^+, which we still denote as V, by forgetting all maps that have domain $V(k)$. Similarly, we define a representation of Γ_k^-, which we still denote by V, by forgetting all maps that have range $V(k)$.

The following definition is a modification of that of [1]. Let $(\Gamma, \{P_j\})$ be an $N = 1$ ADE quiver and let k be a vertex of Γ. Let

$$\Gamma^k = \{i \mid i \text{ adjacent to } k\}.$$

DEFINITION 2.5. For a quiver representation W of Γ_k^+, define a representation $F_k^+(W)$ of Γ_k^- by

$$F_k^+(W)(i) = \begin{cases} W(i) & \text{if } i \ne k, \\ \ker h & \text{if } i = k, \end{cases} \tag{2.3}$$

where

$$h \colon \bigoplus_{i \in \Gamma^k} W(i) \to W(k)$$

is defined by

$$h((x_i)_{i \in \Gamma^k}) = \sum_{i \in \Gamma^k} s_{ik} Q_{ki} x_i.$$

If $i, j \ne k$, we define

$$Q'_{ij} = Q_{ij} \colon W(j) \to W(i).$$

If $i \in \Gamma^k$, define $Q'_{ik} \colon F_k^+(W)(k) \to W(i)$ by

$$Q'_{ik}(x_j)_{j \in \Gamma^k} = -s_{ki} x_i. \tag{2.4}$$

For a quiver representation U of Γ_k^-, define a representation $F_k^-(U)$ of Γ_k^+ by

$$F_k^-(U)(i) = \begin{cases} U(i) & \text{if } i \neq k, \\ \operatorname{coker} g & \text{if } i = k, \end{cases} \tag{2.5}$$

where

$$g \colon U(k) \to \bigoplus_{i \in \Gamma^k} U(i)$$

is defined by

$$g(x) = (Q_{ik} x)_{i \in \Gamma^k}.$$

Define $Q'_{ki} \colon U(i) \to F_k^-(U)(k)$ by the natural composition of

$$U(i) \to \bigoplus_{j \in \Gamma^k} U(j) \to F_k^-(U)(k). \tag{2.6}$$

REMARK 2.6. Notice that, in Definition 2.5, there is no loop associated to the vertex k of the quiver Γ_k^+ or the quiver Γ_k^-.

REMARK 2.7. The definitions of the $F_k^+(W)$ and Q'_{ik} in Definition 2.5 are different from the corresponding definitions in [1], whereas $F_k^-(U)$ and Q'_{ki} in Definition 2.5 are the same as the corresponding definitions in [1].

3. Finite Representations of an $N = 1$ ADE Quiver

In this section we give a proof that, in the case of simple and distinct roots, an $N = 1$ ADE quiver has finitely many nonisomorphic simple representations.

3.1. Applying the Reflection Functors to $N = 1$ ADE Quiver Representations

LEMMA 3.1. *Let V be a representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, and let v_j be a λ-eigenvector of Φ_j. Then $Q_{ij}\Phi_j v_j$ either is a λ-eigenvector of Φ_i or is* 0.

Proof. If v_j is an eigenvector of Φ_j corresponding to eigenvalue λ, then by (2.2) we have

$$Q_{ij}\Phi_j v_j = \Phi_i Q_{ij} v_j,$$

which implies that

$$\lambda Q_{ij} v_j = \Phi_i Q_{ij} v_j.$$

Hence, $Q_{ij} v_j$ is either an eigenvector of Φ_i corresponding to eigenvalue λ or a 0-vector. □

LEMMA 3.2. *Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$. Then there exists a λ such that, if $v_i \in V(i) \neq 0$, then $\Phi_i v_i = \lambda v_i$.*

Proof. Let $\mathcal{A} = \{d \mid V(d) \neq 0\}$. Then $\mathcal{A}$ is connected. Otherwise, V is not simple. Let $a = \min \mathcal{A}$; then Φ_a has an eigenvector v_a with eigenvalue λ. For $l \in \mathcal{A}$, let $U(l)$ be the λ-eigenvector space of Φ_l. By Lemma 3.1, it is easy to see that $W = \{U(l) : l \in \mathcal{A}\}$ is a subrepresentation of V. Since V is simple it follows that $W = V$, which proves the result. □

Therefore, to show that we have only finitely many simple representations, it suffices to consider representations V for which there exists a λ such that, if $0 \neq v_d \in V(d)$, then $\Phi_d v_d = \lambda v_d$. A representation V with this property will be called a *type*-$(**)$ representation.

LEMMA 3.3. *Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$. Suppose V is not concentrated at vertex k. Then*

$$\dim(F_k^+(V))_k = \sum_{i \in \Gamma^k} \dim(V(i)) - \dim(V(k)).$$

Proof. We know that $(F_k^+(V))(k) = \ker h$, where $h \colon \bigoplus_{i \in \Gamma^k} V(i) \to V(k)$ is defined by

$$h(x_i)_{i \in \Gamma^k} = \sum_{i \in \Gamma^k} s_{ik} Q_{ki} x_i.$$

Proving the lemma is equivalent to proving that h is surjective.

Suppose $V(k) \neq 0$. If h is not surjective and $h \neq 0$, then we can replace $V(k)$ by $h\big(\bigoplus_{i \in \Gamma^k} V(i)\big)$ and obtain a subrepresentation of V. But this contradicts the simplicity of V. If $V(k) = 0$, then h is surjective because $h \equiv 0$ in this case. □

LEMMA 3.4. *Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, and suppose that V is not concentrated at vertex k. Then*

$$\dim(F_k^-(V))_k = \sum_{i \in \Gamma^k} \dim(V(i)) - \dim(V(k)).$$

Proof. We know that $(F_k^-(V))(k) = \operatorname{coker} g$, where $g \colon V(k) \to \bigoplus_{i \in \Gamma^k} V(i)$ is defined by $g(x) = (Q_{ik} x)_{i \in \Gamma^k}$. Proving the lemma is equivalent to proving that g is injective.

Suppose $V(k) \neq 0$. If $\ker g \neq 0$, then we can define a simple subrepresentation that is concentrated at vertex k. This contradicts the simplicity of V.

If $V(k) = 0$, then g is injective because $g \equiv 0$ in this case. □

LEMMA 3.5. *Let V be a type-$(**)$ representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, and suppose that V is not concentrated at the vertex k. If $P_k(\lambda) \neq 0$, then there is a natural isomorphism φ between $F_k^+(V)(k)$ and $F_k^-(V)(k)$.*

Proof. We have

$$F_k^-(V)(k) = \operatorname{coker} g$$

and

$$F_k^+(V)(k) = \ker h.$$

Since V is a type-($**$) representation, the relation $\sum_{i\in\Gamma^k} s_{ik} Q_{ki} Q_{ik} + P_k(\Phi_k) = 0$ becomes $h \circ g + P_k(\lambda) I = 0$. Since $P_k(\lambda) \neq 0$, it follows that g is injective and h is surjective. Since V is of type ($**$) and is not concentrated at k, we obtain

$$\dim F_k^+(V)(k) = \dim F_k^-(V)(k) = \sum_{i\in\Gamma^k} \dim V(i) - \dim V(k)$$

and

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & F_k^+(V)(k) \arrow[d, "h'"] \arrow[dr, "\varphi"] & & \\
0 \arrow[r] & V(k) \arrow[r, "g"] \arrow[dr, "-P_k(\lambda)"'] & \bigoplus_{i\in\Gamma^k} V(i) \arrow[r, "g'"] \arrow[d, "h"] & F_k^-(V)(k) \arrow[r] & 0. \\
 & & V(k) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Since $P_k(\lambda) \neq 0$, we have $\operatorname{im} g \cap F_k^+(V)(k) = \{0\}$. Let $g' \colon \bigoplus_{i\in\Gamma^k} V(i) \to F_k^-(V)(k)$ be the natural surjective map induced by g, and let $h' \colon F_k^+(V)(k) \to \bigoplus_{i\in\Gamma^k} V(i)$ be the natural inclusion map induced by h. Then

$$\varphi = g' \circ h' \colon F_k^+(V)(k) \to F_k^-(V)(k)$$

is a natural isomorphism because $\dim F_k^+(V)(k) = \dim F_k^-(V)(k)$ and φ is injective by $\operatorname{im} g \cap F_k^+(V)(k) = \{0\}$.

Definition 3.6. By Lemma 3.5, if $P_k(\lambda) \neq 0$ then, for a type-($**$) representation V, we can construct a new representation $F_k(V)$ of $(\Gamma, \{P_j\})$ as

$$F_k(V)(i) = \begin{cases} V(i) & \text{if } i \neq k, \\ F_k^+(V)(k) & \text{if } i = k. \end{cases}$$

Here we define Q'_{lk} as it is defined for $F_k^+(V)$ and define Q'_{km} as the composition map $P_k(\lambda) \cdot \varphi^{-1} \circ \underline{Q'_{km}} \colon V(m) \to F_k^+(V)(k)$, where $\underline{Q'_{km}} \colon V(m) \to F_k^-(V)(k)$ is the natural map defined in $F_k^-(V)$ and $\varphi \colon F_k^+(V)(k) \to F_k^-(V)(k)$ is the isomorphism defined in Lemma 3.5.

Now define

$$\Phi'_i \colon F_k(V)(i) \to F_k(V)(i)$$

by $\Phi'_i(x) = \lambda x$, where λ is the eigenvalue of Φ_i on $V(i)$ that appeared in the representation V of Γ. Abusing notation, we allow Φ_i to stand for Φ'_i.

REMARK 3.7. The reflection functor F_k is acting only on the type-$(**)$ representations, not on all representations on the $N = 1$ ADE quiver $(\Gamma, \{P_j\})$. How to define a reflection functor that acts on all representations of the $N = 1$ ADE quiver is still an open problem.

LEMMA 3.8. *If V is a representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, then*

$$\sum_i \operatorname{Tr} P_i(\Phi_i) = 0.$$

As a result, if V is a type-$(**)$ representation of an $N = 1$ ADE quiver then

$$\sum_i \dim(V(i)) \cdot P_i(\lambda) = 0.$$

Proof. This follows because $\operatorname{Tr} Q_{ij}Q_{ji} = \operatorname{Tr} Q_{ji}Q_{ij}$ for every pair i and j.

Now take the trace operation to relations (2.2) and then sum the resulting equations. The result follows. □

PROPOSITION 3.9. *Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ that is not concentrated at vertex k. If $P_k(\lambda) \neq 0$, then $F_k(V)$ satisfies the following new relations*:

$$\sum_i s_{ij} Q'_{ji} Q'_{ij} + r_k(P_j(\Phi_j)) = 0, \qquad Q'_{ij}\Phi_j = \Phi_i Q'_{ij}. \tag{3.1}$$

Thus, $F_k(V)$ is a representation of $(\Gamma, \{r_k(P_j)\})$.

Proof. If $i \notin \Gamma^k$ and $i \neq k$, where i is a vertex of $(\Gamma, \{P_j\})$ such that $V(i) \neq 0$, then there is nothing to prove. For $j \in \Gamma^k \cup \{k\}$ and $b \in (F_k(V))(j)$, we have

$$Q'_{ij}\Phi_j b = \lambda Q'_{ij} b = \Phi_i Q'_{ij} b.$$

For $i \in \Gamma^k$ and $x \in V(i)$, by Definition 3.6 we know that

$$Q'_{ki}x = P_k(\lambda) \cdot \varphi^{-1} \circ \underline{Q'_{ki}}x,$$

where $\underline{Q'_{ki}}x = [(x_j)_{j\in\Gamma^k}] \in F_k^-(V)(k)$ for

$$x_j = \begin{cases} 0 & \text{if } j \neq i, \\ x & \text{if } j = i. \end{cases}$$

After a short computation we see that

$$Q'_{ki}x = (y_j)_{j\in\Gamma^k},$$

where

$$y_j = \begin{cases} P_k(\lambda)x + s_{ik}Q_{ik}Q_{ki}x & \text{if } j = i, \\ Q_{jk}s_{ik}Q_{ki}x & \text{if } j \neq i. \end{cases}$$

It follows that

$$s_{ki}Q'_{ik}Q'_{ki}x = s_{ki}Q'_{ik}(y_j)_{j\in\Gamma^k} = -P_k(\lambda)x - Q_{ik}s_{ik}Q_{ki}x.$$

Hence, for $i \in \Gamma^k$,

$$\begin{aligned}
&\sum_j s_{ji} Q'_{ij} Q'_{ji} x + r_k(P_i(\lambda))x \\
&\quad = \sum_j s_{ji} Q'_{ij} Q'_{ji} x + P_i(\lambda)x + P_k(\lambda)x \\
&\quad = \sum_{j \neq k} s_{ji} Q_{ij} Q_{ji} x + s_{ki} Q'_{ik} Q'_{ki} x + P_i(\lambda)x + P_k(\lambda)x \\
&\quad = \sum_{j \neq k} s_{ji} Q_{ij} Q_{ji} x - P_k(\lambda)x - Q_{ik} s_{ik} Q_{ki} x + P_i(\lambda)x + P_k(\lambda)x \\
&\quad = 0.
\end{aligned}$$

Let $(x_i)_{i \in \Gamma^k} \in F_k^+(V)(k)$. Then

$$s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} = Q'_{ki} x_i = (x_{i_j})_{j \in \Gamma^k},$$

where

$$x_{i_j} = \begin{cases} P_k(\lambda)x_i + Q_{ik} s_{ik} Q_{ki} x_i & \text{if } j = i, \\ Q_{jk} s_{ik} Q_{ki} x_i & \text{if } j \neq i. \end{cases}$$

Therefore,

$$\begin{aligned}
&\sum_{i \in \Gamma^k} s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} + r_k(P_k(\lambda))(x_i)_{i \in \Gamma^k} \\
&\quad = \sum_{i \in \Gamma^k} s_{ik} Q'_{ki} Q'_{ik} (x_i)_{i \in \Gamma^k} - P_k(\lambda)(x_i)_{i \in \Gamma^k} \\
&\quad = \sum_{i \in \Gamma^k} (x_{i_j})_{j \in \Gamma^k} - P_k(\lambda)(x_i)_{i \in \Gamma^k} \\
&\quad = 0. \qquad \square
\end{aligned}$$

By Definition 2.3 and Proposition 3.9, we have the following result.

COROLLARY 3.10. *Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ that is not concentrated at vertex k. Then*

$$\sum \dim(F_k(V)(i)) r_k(P_i(x)) = \sum \dim V(i) P_i(x).$$

LEMMA 3.11. *If V is a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ that is not concentrated at vertex k and if $P_k(\lambda) \neq 0$, then $F_k F_k(V) \cong V$. Consequently, $F_k(V)$ is a simple representation.*

Proof. We know that $Q'_{ki} \colon V(i) \to F_k(V)(k)$ is defined by

$$Q'_{ki} x_i = P_k(\lambda) \varphi^{-1} \underline{Q_{ki}} x_i,$$

where $\underline{Q_{ki}} \colon V(i) \to F_k^-(V)(k)$ is the composition of $V(i) \to \bigoplus_{j \in \Gamma^k} V(j)$ and $\bigoplus_{j \in \Gamma^k} V(j) \to F_k^-(V)(k)$ (see Definition 3.6). We also know that

$$F_k F_k(V)(k) = \left\{ (x_j) \in \bigoplus_{j \in \Gamma^k} V(j) \;\middle|\; \sum_{j \in \Gamma^k} s_{jk} Q'_{kj} x_j = 0 \right\}.$$

Hence

$$\sum_{j\in\Gamma^k} s_{jk} Q'_{kj} x_j = P_k(\lambda)\varphi^{-1} \sum_{j\in\Gamma^k} s_{jk} \underline{Q_{kj}} x_j .$$

Since $P_k(\lambda) \neq 0$ and φ is an isomorphism, we have

$$F_k F_k(V)(k) = \{(-s_{kj} Q_{jk} x)_{j\in\Gamma^k} \mid x \in V(k)\}.$$

Let $g\colon V \to F_k F_k(V)$ be defined as follows:

$$g_i = \begin{cases} i\colon V(i) \to F_k F_k(V)(i) = V(i) & \text{if } i \neq k, \\ (-s_{kj} Q_{jk})_{j\in\Gamma^k} & \text{if } i = k; \end{cases}$$

here $i\colon V(i) \to F_k F_k(V)(i) = V(i)$ is the identity map. Then it is clear that (3.2) is commutative:

$$\begin{tikzcd} V(k) \arrow[r, "Q_{ik}"] \arrow[d, "g_k"] & V(i) \arrow[d, "g_i"] \\ F_k F_k(V)(k) \arrow[r, "Q''_{ik}"] & V(i). \end{tikzcd} \tag{3.2}$$

Let's check the commutativity of (3.3):

$$\begin{tikzcd} V(i) \arrow[r, "Q_{ki}"] \arrow[d, "g_i"] & V(k) \arrow[d, "g_k"] \\ V(i) \arrow[r, "Q''_{ki}"] & F_k F_k(V)(k). \end{tikzcd} \tag{3.3}$$

Let $(Q''_{ki} x_i)_j$ (resp. $(Q'_{ki} x_i)_j$) denote the jth coordinate of $Q''_{ki} x_i$ (resp. $Q'_{ki} x_i$). We know that

$$(Q''_{ki} x_i)_j = \begin{cases} -P_k(\lambda) x_i + Q'_{ik} s_{ik} Q'_{ki} x_i & \text{if } j = i, \\ Q'_{jk} s_{ik} Q'_{ki} x_i & \text{if } j \neq i \end{cases}$$

and

$$(Q'_{ki} x_i)_j = \begin{cases} P_k(\lambda) x_i + Q_{ik} s_{ik} Q_{ki} x_i & \text{if } j = i, \\ Q_{jk} s_{ik} Q_{ki} x_i & \text{if } j \neq i. \end{cases}$$

Let $i > k$. Then we have

$$\begin{aligned} (Q''_{ki} x_i)_i &= -P_k(\lambda) x_i + Q'_{ik} s_{ik} Q'_{ki} x_i \\ &= -P_k(\lambda) x_i + P_k(\lambda) x_i + Q_{ik} s_{ik} Q_{ki} x_i \\ &= Q_{ik} s_{ik} Q_{ki} x_i = Q_{ik} Q_{ki} x_i \\ &= -s_{ki} Q_{ik} Q_{ki} x_i . \end{aligned}$$

If $i > k$ and $j > k$, then

$$(Q''_{ki} x_i)_j = Q'_{jk} s_{ik} Q'_{ki} x_i = Q_{jk} s_{ik} Q_{ki} x_i = Q_{jk} Q_{ki} x_i = -s_{kj} Q_{jk} Q_{ki} x_i ;$$

if $i > k$ and $j < k$, then

$$(Q''_{ki} x_i)_j = Q'_{jk} s_{ik} Q'_{ki} x_i = Q'_{jk} Q'_{ki} x_i = -Q_{jk} Q_{ki} x_i = -s_{kj} Q_{jk} Q_{ki} x_i .$$

Now let $i < k$. Then we have

$$\begin{aligned}(Q''_{ki}x_i)_i &= -P_k(\lambda)x_i + Q'_{ik}s_{ik}Q'_{ki}x_i \\ &= -P_k(\lambda)x_i - Q'_{ik}Q'_{ki}x_i \\ &= -P_k(\lambda)x_i + (P_k(\lambda)x_i + Q_{ik}s_{ik}Q_{ki}x_i) \\ &= Q_{ik}s_{ik}Q_{ki}x_i = -Q_{ik}Q_{ki}x_i \\ &= -s_{ki}Q_{ik}Q_{ki}x_i.\end{aligned}$$

If $i < k$ and $j > k$, then

$$\begin{aligned}(Q''_{ki}x_i)_j &= Q'_{jk}s_{ik}Q'_{ki}x_i = -Q'_{jk}Q'_{ki}x_i \\ &= -Q_{jk}s_{ik}Q_{ki}x_i = Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i;\end{aligned}$$

if $i < k$ and $j < k$, then

$$\begin{aligned}(Q''_{ki}x_i)_j &= Q'_{jk}s_{ik}Q'_{ki}x_i = -Q'_{jk}Q'_{ki}x_i \\ &= Q_{jk}s_{ik}Q_{ki}x_i = -Q_{jk}Q_{ki}x_i = -s_{kj}Q_{jk}Q_{ki}x_i.\end{aligned}$$

Therefore, diagram (3.3) is commutative.

Diagram (3.4) is commutative because λ is a common eigenvalue of $V(k)$ and $F_kF_k(V)(k)$:

$$\begin{tikzcd} V(k) \arrow[r, "\lambda"] \arrow[d, "g_k"] & V(k) \arrow[d, "g_k"] \\ F_kF_k(V)(k) \arrow[r, "\lambda"] & F_kF_k(V)(k). \end{tikzcd} \tag{3.4}$$

If $g_k(x) = 0$ for an $x \in V(k)$, then $Q_{jk}x = 0$ for all $j \in \Gamma^k$. By the proof of Lemma 3.4, it follows that $x = 0$. Hence g_k is injective. Since $V(k)$ and $F_kF_k(V)(k)$ have the same dimension, g_k must be an isomorphism. We know that $g_i = \mathrm{id}$ whenever $i \neq k$, so $g\colon V \to F_kF_k(V)$ is also an isomorphism.

We next prove the latter part of the lemma. We claim that there is no simple subrepresentation of $F_k(V)$ that is concentrated at vertex k. By way of contradiction, suppose there does exist such a simple representation $W \subset F_k(V)$ concentrated at vertex k. Then, for $(x_i)_{i\in\Gamma^k} \in W$, since $Q'_{jk}(x_i)_{i\in\Gamma^k} = -s_{kj}x_j = 0$, it would follow that $x_j = 0$ for all $j \in \Gamma^k$ and hence $(x_i)_{i\in\Gamma^k} = 0$.

Since the natural map $\bigoplus_{i\in\Gamma^k} V(i) \to F_k^-(V)(k)$ is surjective, $\varphi\colon F_k(V)(k) = F_k^+(V)(k) \to F_k^-(V)(k)$ is an isomorphism, and $P_k(\lambda) \neq 0$, we conclude that $\{Q'_{ki}V(i)\}_{i\in\Gamma^k}$ generates $F_k(V)(k)$. As a result, there is no subrepresentation $W \subset F_k(V)$ with $\dim W(i) = \dim V(i)$ for all $i \in \Gamma^k$ and $i \neq k$ and with $\dim W(k) < \dim F_k(V)(k)$.

Suppose there exists a simple subrepresentation $W \subset F_k(V)$ that is not concentrated at vertex k. We thus obtain a proper subrepresentation $F_k(W) \subset F_kF_k(V) \cong V$. Since V is a simple representation, this cannot occur. $\square$

COROLLARY 3.12. *Assume that* $(*)$ *holds. If V is a simple representation, then either $F_k(V)$ is simple or $V \cong L_k$, where L_k is a simple representation concentrated at vertex k.*

Proof. Assume that V is not concentrated at vertex k. Since V is simple it follows that, by Lemma 3.3 and Lemma 3.4, we can apply F_k to V. Then $F_k(V)$ is simple by the latter part of Lemma 3.11. $\square$

3.2. A Proof of the Main Theorem

Let Γ be a quiver. Following [1], for a representation V we define $\dim(V) = (\dim V(i))_{i \in V_\Gamma}$. Let $\mathscr{C}_\Gamma = \{x = (x_\alpha) \mid x_\alpha \in \mathbb{Q}, \alpha \in V_\Gamma\}$, where $\mathbb{Q}$ denotes the set of rational numbers. We call a vector $x = (x_\alpha)$ *positive* (written $x > 0$) if $x \neq 0$ and $x_\alpha \geq 0$ for all $\alpha \in V_\Gamma$. For each $\beta \in V_\Gamma$, denote by σ_β the linear transformation in $\mathscr{C}_\Gamma$ defined by the formulas $(\sigma_\beta x)_\gamma = x_\gamma$ for $\gamma \neq \beta$ and $(\sigma_\beta x)_\beta = -x_\beta + \sum_{l \in \Gamma^\beta} x_l$, where $l \in \Gamma^\beta$ is the set of vertices adjacent to β.

For each vertex $\alpha \in V_\Gamma$ we denoted by Γ_α the set of edges containing α. Let Λ be an orientation of the graph Γ. We denote by $\sigma_\alpha \Lambda$ the orientation obtained from Λ by changing the directions of all edges $l \in \Gamma_\alpha$. Following [1], we say that a vertex i of a quiver $(\Gamma, \{P_j\})$ with orientation Λ is $(-)$-accessible (resp. $(+)$-accessible) if, for any edge e having i as a vertex, the final vertex $f(e)$ of e satisfies $f(e) \neq i$ (resp. the initial vertex $i(e)$ of e satisfies $i(e) \neq i$). We say that a sequence of vertices $\alpha_1, \alpha_2, \ldots, \alpha_k$ is $(+)$-accessible with respect to Λ if α_1 is $(+)$-accessible with respect to Λ, α_2 is $(+)$-accessible with respect to $\sigma_{\alpha_1}\Lambda$, α_3 is $(+)$-accessible with respect to $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, and so on. We define a $(-)$-accessible sequence similarly.

DEFINITION 3.13. Let Γ be a graph without loops. We denote by B the quadratic form on the space $\mathscr{C}_\Gamma$ defined by the formula $B(x) = \sum x_\alpha^2 - \sum_{l \in \mathcal{E}_\Gamma} x_{r_1(l)} x_{r_2(l)}$, where $r_1(l)$ and $r_2(l)$ are the ends of the edge l. We denote by $\langle \cdot, \cdot \rangle$ the corresponding symmetric bilinear form.

LEMMA 3.14 [1, Lemma 2.3]. *Suppose that the form B for the graph Γ is positive definite. Let $c = \sigma_n \cdots \sigma_2 \sigma_1$. If $x \in \mathscr{C}_\Gamma$ and $x \neq 0$ then, for some i, the vector $c^i x$ is not positive.*

We are now ready to give a proof of our Main Theorem as follows.

Proof of Main Theorem. Let V be a simple representation of an $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, and let $\mathcal{A} = \{i \mid V(i) \neq 0\}$. We can assume that $\mathcal{A}$ is connected, since otherwise V would be decomposable. We apply the forgetful functors to V and obtain the following $(+)$-accessible (resp. $(-)$-accessible) diagram (no loop):

$$\begin{tikzcd} V(1) & \cdots \arrow[l] & V(k) \arrow[l] & \cdots \arrow[l] & V(n) \arrow[l] \\ & & V(l) \arrow[u] & & \end{tikzcd}$$

$$\begin{tikzcd} V(1) \arrow[r] & \cdots \arrow[r] & V(k) \arrow[r] \arrow[d] & \cdots \arrow[r] & V(n) \\ & & V(l). & & \end{tikzcd}$$

(For the type-A Dynkin diagram case, $V(l) = 0$.)

Let $c = \sigma_n \cdots \sigma_2\sigma_1$. By [1], there exists a k such that $c^k(\dim V) \not> 0$. By $(*)$ and Corollary 3.10, we know that $\sum_i \dim V(i) \cdot P_i(x)$ is the only element in $\mathcal{A}_\Gamma$ that vanishes at λ. By Corollary 3.12 and Proposition 3.9, this implies the existence of vertices $\beta_1, \ldots, \beta_l$ and a simple representation $L_{\beta_{k+1}}$ of $(\Gamma, \{Q_j\})$ that satisfies the new relations described in Proposition 3.9 and is concentrated at a vertex of Γ such that

$$V = F_{\beta_1} \cdots F_{\beta_k}(L_{\beta_{k+1}}).$$

Here V corresponds to the positive root

$$\dim V = \sigma_{\beta_1} \cdots \sigma_{\beta_k}(\overline{\beta_{k+1}}),$$

where $\overline{\beta_{k+1}} = (\overline{\beta_{k+1}}(i))$ and

$$\overline{\beta_{k+1}}(i) = \begin{cases} 0 & \text{if } i \neq k+1, \\ 1 & \text{if } i = k+1. \end{cases}$$

From this it follows that $\sum_i \dim V(i) \cdot P_i(x) \in \mathcal{B}_\Gamma$. Because the usual ADE quiver has only finitely many positive roots, $N = 1$ ADE quivers have finite many simple representations. This finishes the proof of the theorem. $\square$

The Main Theorem implies the following corollary.

COROLLARY 3.15. *Let $(\Gamma, \{P_j\})$ be an $N = 1$ ADE quiver. Let $\mathcal{B}_\Gamma = \{r_i(P_j(x)) \mid r_i \in \mathfrak{W}_\Gamma\}$, where $\mathfrak{W}_\Gamma$ is the Weyl group of Γ and P_j is the polynomial defined on relation* (2.2). *Assume that each element in $\mathcal{B}_\Gamma$ has simple roots. If* $(*)$ *holds, then there is a finite-to-one correspondence between simple representations of the $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ and the positive roots of an ADE Dynkin diagram.*

Proof. We know that $\mathcal{B}_\Gamma$ has only finitely many elements. Each element of $\mathcal{B}_\Gamma$ that is actually a polynomial has only finitely many simple roots. By our Main Theorem, each root of an element in $\mathcal{B}_\Gamma$ corresponds with a simple representation. Hence, the desired result follows. $\square$

3.3. Further Discussions

In [9] the author proved the following theorem without using reflection functors.

THEOREM 3.16. *Let $\mathcal{A} = \{rP_i(x) \mid r \in \mathfrak{W}_{A_n}\}$, where the $P_i(x)$ are the polynomials in relation* (2.2) *and $\mathfrak{W}_{A_n}$ is the Weyl group of A_n. If no two positive elements in $\mathcal{A}$ have a common root and if none of the polynomials in $\mathcal{A}$ are identically zero, then the $N = 1$ A_n quiver is of finite representation type, which means that there are only finitely many indecomposable representations.*

One consequence of this theorem is the following result.

COROLLARY 3.17. *Let $\mathcal{A}$ be defined as in Theorem 3.16. If no two positive elements in $\mathcal{A}$ have a common root and if none of the polynomials in $\mathcal{A}$ have multiple roots, then any indecomposable representations of the $N = 1$ A_n quiver are simple representations.*

I do not know whether Theorem 3.16 and Corollary 3.17 are still correct if $\mathfrak{W}_{A_n}$ is replaced by $\mathfrak{W}_{D_n}$ or $\mathfrak{W}_{E_n}$.

4. A Correspondence between Simple Representations and an ADE Configuration of Curves

Let X be an ADE fibration with base $\mathbb{C}$ and let Y be the small resolution of X. Let $\pi : Y \to X$ be the blowup map. An *ADE configuration of curves* in Y is a 1-dimensional connected projective scheme $C \subset Y$ such that

(1) there exists a surface $\bar{S} \subset Y$, $C \subset \bar{S}$;
(2) letting $S = \pi(\bar{S})$, then $\bar{S} \to S$ is a resolution of ADE singularities with exceptional scheme C.

We need the following proposition, which is essentially part 3 of Theorem 1 in [5].

PROPOSITION 4.1. *Let $\{e_i\}$ be the simple roots of Γ. The irreducible components of the discriminant divisor $\mathfrak{D} \subset \mathrm{Res}(\Gamma)$ are in one-to-one correspondence with the positive roots of Γ. Under the identification of $\mathrm{Res}(\Gamma)$ with the complex root space U, the component $\mathfrak{D}_v$ corresponding to the positive root $v = \sum_{i=1}^n a_i e_i$ is $v^\perp \subset U$, that is, the hyperplane perpendicular to v.*

Moreover, $\mathfrak{D}_v$ corresponds exactly to those deformations of Z_0 in $\mathcal{Z}$ to which the curve

$$C_v := \bigcup_{i=1}^n a_i C_{e_i}$$

lifts. For a generic point $t \in \mathfrak{D}_v$, the corresponding surface $\mathcal{Z}_t$ has a single smooth -2-curve in the class $\sum_{i=1}^n a_i [C_{e_i}]$. Hence there is a small neighborhood B of t such that the restriction of $\mathcal{Z}$ to B is isomorphic to a product of $\mathbb{C}^{n-1}$ with the semi-universal family over $\mathrm{Res}(A_1)$.

The following example gives a concrete correspondence between the ADE configuration of curves in Y and the simple representations of an $N = 1$ ADE quiver in the A_2 case.

EXAMPLE 4.2. Let X be defined by

$$A_2 : xy + (z + t_1(t))(z + t_2(t))(z + t_3(t)) = 0$$

with

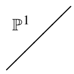
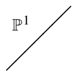
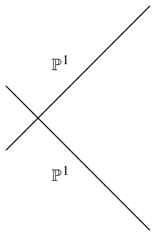
$$t_1(t) + t_2(t) + t_3(t) = 0.$$

In Table 1, "Curve" means an ADE configuration of curves in the exceptional set of the fibration. We use "$\dim V$" to denote the dimension vector of an indecomposable representation of the $N = 1$ ADE quiver.

Table 1 can be explained in the following way. If $t_1(\lambda) = t_2(\lambda) \neq t_3(\lambda)$ for some λ, then by Proposition 4.1 there exists an ADE configuration of curves $C \subset Y$. By [2], we know that $P_1(\lambda) = t_1(\lambda) - t_2(\lambda) = 0$. Thus, by our Main Theorem, there exists a simple representation V of an $N = 1$ ADE quiver with $\dim V = (1, 0)$ that corresponds to C.

The other cases are similar, so we omit them.

Table 1

Condition	Singularity	Curve	dim V
$t_1(\lambda) = t_2(\lambda) \neq t_3(\lambda)$	A_1		$\overset{1}{\bullet}$ —— $\overset{0}{\bullet}$
$t_1(\lambda) \neq t_2(\lambda) = t_3(\lambda)$	A_1		$\overset{0}{\bullet}$ —— $\overset{1}{\bullet}$
$t_1(\lambda) = t_3(\lambda) \neq t_2(\lambda)$	A_2		$\overset{1}{\bullet}$ —— $\overset{1}{\bullet}$
$t_1(\lambda) \neq t_2(\lambda) \neq t_3(\lambda)$	—	—	—

This example is generalized in the following theorem.

THEOREM 4.3. *Let X be an ADE fibration corresponding to Γ with base $\mathbb{C}$, and let Y be a small resolution of X. Let $\mathcal{B}_\Gamma = \{r_i(P_j(x)) \mid r_i \in \mathfrak{W}_\Gamma\}$, where $\mathfrak{W}_\Gamma$ is the Weyl group of Γ and P_j is the polynomial defined in relation* (2.2). *Assume that no element in $\mathcal{B}_\Gamma$ has multiple roots and assume that* $(*)$ *holds. Then there exists a one-to-one correspondence between the simple representations of the $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ and the ADE configuration of curves in Y.*

Proof. By [7] and [5], we have the commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"] & \mathcal{Y} \arrow[d] \\ X \arrow[r] \arrow[d, "\varphi"] & \mathcal{X} \arrow[d] \\ \mathbb{C} \arrow[r, "f"] & \mathbb{C}^n, \end{tikzcd}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\mathcal{Y}$ denotes the $\mathbb{C}^*$-equivariant simultaneous resolution $\mathcal{Y} \to \mathcal{X}$ inducing $Y_0 \to X_0$. For a simple representation V of the $N = 1$ ADE quiver $(\Gamma, \{P_j\})$, we have

$$\sum \dim V(i) \cdot P_i(\lambda) = 0 \tag{4.1}$$

for some λ. The dimension vector $(\dim V(i))_{i\in V_\Gamma}$ will correspond to a positive root ρ. By [2], we can express $P_i(x)$, $i = 1, \ldots, n$, in terms of t_i, $i = 1, \ldots, n$. By Proposition 3.1 or part 3 of Theorem 1 in [5, p. 467], (4.1) will give an equation for $\rho^\perp$. Hence $f(\lambda) = (t_i(\lambda))_{i\in V_\Gamma} \in \rho^\perp$. It follows from Proposition 4.1 that there exists an ADE configuration of curves $C_\rho \subset \pi^{-1}(\lambda) \subset Y$.

Conversely, for an ADE configuration of curves $C \subset Y$ we have $\varphi \circ \pi(C) = \lambda \in \mathbb{C}$. (Since π is projective, $\varphi \circ \pi(C)$ is projective in $\mathbb{C}$ and it follows that $\varphi \circ \pi(C)$ is a finite subset of $\mathbb{C}$. Since C is connected, $\varphi \circ \pi(C)$ is connected in $\mathbb{C}$; hence $\varphi \circ \pi$ is a point in $\mathbb{C}$.) Moreover, $\pi(C)$ is a point in X. (By [5], we know that $\mathcal{X}$ is affine; hence $\pi(C)$ is a point in X.) By Proposition 4.1, $f(\lambda) \in \rho^\perp$ for some positive root ρ. Since we assume that each element in $\mathcal{B}_\Gamma$ has simple roots and that $(*)$ holds, C corresponds to a unique positive root ρ. We can express ρ as $\rho = \sum a_i \cdot \rho_i$, where ρ_i is a simple positive root. From our Main Theorem, we can apply the reflection functors to construct a simple representation V of the $N = 1$ ADE quiver $(\Gamma, \{P_j\})$ that corresponds to the positive root ρ. This finishes the proof of Theorem 4.3. $\square$

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Department of Mathematics
Central Michigan University
Mount Pleasant, MI 48859

zhu1x@cmich.edu