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A Monotonicity Result for Volumes of Holomorphic Images

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Introduction

This paper studies the Euclidean $2k$-dimensional volumes of parameterized holomorphic images of certain pseudoconvex domains in complex Euclidean space $\mathbf{C}^k$. Its motivation arises from studying a more specific situation—namely, the classification problem for proper holomorphic mappings between balls in complex Euclidean spaces (usually of different dimensions). We provide two versions of a monotonicity result. The first applies for mappings from balls and eggs; its proof involves the computation of explicit integrals. The second applies for mappings from more general domains, but it requires some regularity assumptions on the mappings at the boundary.

By way of introduction, first suppose that $k = 1$, that f is holomorphic on the unit disk B_1, and that f' is square integrable. Then the area of the parameterized image of the unit disk is $\pi \|f'\|_{L^2}^2$. In particular, when $f(z) = z^m$, the area (with multiplicity taken into account) is $m\pi$. Furthermore, one can easily see that $\|(zf)'\|_{L^2}^2 \geq \|f'\|_{L^2}^2$ with strict inequality unless $f = 0$, so the area is increased when f is replaced by zf.

In this paper we generalize these results to the more complicated situation in higher dimensions and show how they fit into the classification problem. The main results are Theorem 3, its consequence Corollary 2, and Theorem 4. Theorem 3 states that parameterized volumes of images of balls and eggs increase under a tensor product operation; Corollary 2 provides a sharp upper bound for the parameterized volume of a proper polynomial mapping between balls in terms of its degree and the domain dimension. Theorem 4 provides a monotonicity result for more general domains, assuming that the mapping is continuously differentiable at the boundary so that Stokes's theorem can be applied.

Section I considers the 1-dimensional case. The tensor product $z \to z^{\otimes m}$ provides a natural generalization to higher dimensions of the map $z \to z^m$ in one dimension. We let H_m denote a concrete form of the mapping $z \to z^{\otimes m}$; see (6) in Section II. We show there how the mappings H_m combine with a *tensor product on a subspace* operation to play a crucial role in the classification of proper polynomial mappings between balls.

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Given domains in complex Euclidean spaces of possibly different dimensions, a natural question in complex analysis is to classify the proper holomorphic mappings between them. This question is particularly compelling when the domains are the unit balls $B_n \subset \mathbf{C}^n$ and $B_N \subset \mathbf{C}^N$; in this case the question has many interesting aspects (see e.g. [D3; D6; DL; Fa1; Fa2; F2; H; Ru1; W]). When a proper holomorphic mapping f has a smooth extension to the sphere, we can regard it as a solution to the Cauchy–Riemann equation $\bar{\partial} f = 0$ together with the nonlinear boundary condition $\|f(z)\|^2 = 1$ on the unit sphere.

In Section III we show how to find the $2k$-dimensional parameterized volume of a holomorphic image of the unit ball B_k. In Theorem 1 we show that the $2k$-dimensional parameterized volume of the image of a homogeneous proper mapping between balls of degree m is $m^k \pi^k / k!$. In Example 2 we provide exact answers for the volumes determined by many interesting proper mappings from B_2.

In Section IV we pause to compute a real-variables analogue of the formula in the homogeneous case; a curve in the unit sphere plays an interesting role.

We prove Theorem 3 in Section V. We prove that the $2k$-dimensional volume increases under tensor products, providing the appropriate analogue of the easy result in dimension 1. As a consequence we obtain an extremal property for the mappings H_m; they maximize the parameterized volume over proper polynomial mappings of degree m. The proof of Theorem 3 uses an inequality on determinants.

In Section VI we determine the effect of the juxtaposition operation on volume. We also compute the volumes for certain 1-parameter families of proper mappings in order to exhibit the dependence on the parameter.

We prove Theorem 4 in Section VII. Let Ω be the (pseudoconvex) domain given by $\{z : \|P(z)\|^2 < 1\}$, where P is a holomorphic mapping taking values in $\mathbf{C}^M$; we assume that Ω is bounded. Consider a holomorphic mapping $f : \Omega \to \mathbf{C}^N$ that is continuously differentiable on $b\Omega$. We prove a monotonicity result for volumes involving the tensor product of f with P. It is likely that such results hold more generally, but the main focus in this paper remains the unit ball.

We close this introduction by mentioning that the mappings H_m warrant study for several reasons. They are invariant under natural unitary representations of finite cyclic groups and thus exhibit numerous combinatorial and number-theoretic properties. Perhaps most important, these mappings provide the analogue in higher dimensions of the most celebrated proper mapping in mathematics: namely, $z \to z^m$ in one complex dimension.

I. Observations in One Dimension

Suppose that $f : B_1 \to \mathbf{C}$ is holomorphic. Then f has a power series expansion $\sum_{n=0}^{\infty} b_n z^n$ in B_1. The parameterized area of the image of f is (by definition)

$$A_f = \int_{B_1} |f'(z)|^2 \, dV_2 = \|f'\|_{L^2}^2. \tag{1}$$

The following result relates A_f to A_{zf}: multiplication by z increases the area. The conclusion requires integration; the pointwise inequality $|(zf(z))'| > |f'(z)|$ fails in general.

PROPOSITION 1. *Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be a holomorphic function with both f and f' in $L^2(B_1)$. Then*

$$\|f\|_{L^2}^2 = \pi \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}, \tag{2}$$

$$\|f'\|_{L^2}^2 = \pi \sum_{n=0}^{\infty} (n+1)|b_{n+1}|^2, \tag{3}$$

$$\|(zf)'\|_{L^2}^2 = \|f'\|_{L^2}^2 + \pi \sum_{n=0}^{\infty} |b_n|^2. \tag{4}$$

Hence $A_{zf} \geq A_f$, and equality occurs only when f vanishes identically.

Proof. The proof of (2) is an easy calculation in polar coordinates, and (3) follows from (2). To prove (4), first observe that $(zf)'(z) = \sum_{n=0}^{\infty}(n+1)b_n z^n$. By (2) we have

$$\begin{aligned}\|(zf)'\|_{L^2}^2 &= \pi \sum_{n=0}^{\infty}(n+1)|b_n|^2 = \pi \sum_{n=0}^{\infty} n|b_n|^2 + \pi \sum_{n=0}^{\infty}|b_n|^2 \\ &= \pi \sum_{n=0}^{\infty}(n+1)|b_{n+1}|^2 + \pi \sum_{n=0}^{\infty}|b_n|^2 = \|f'\|_{L^2}^2 + \pi \sum_{n=0}^{\infty}|b_n|^2.\end{aligned}$$

We have obtained (4). □

REMARK. Suppose that $f' \in L^2(B_1)$. It follows from (3) that $\sum_{n=0}^{\infty}|b_n|^2$ converges and hence f can be extended to the circle. The second expression on the right-hand side of (4) is easily seen to be the integral

$$\frac{1}{2}\int_{|z|=1} |f|^2 \, d\theta.$$

We can express Proposition 1 in another manner as follows. Let D denote differentiation, and let M denote multiplication by z. Consider the operator given by $(DM)^*DM - D^*D$ on an appropriate subspace of the holomorphic functions in L^2. By (4) it is positive definite and hence of the form T^*T for an operator T (essentially the Szegö projection) with trivial kernel:

$$\|DMf\|^2 - \|Df\|^2 = \|Tf\|^2 = \pi \sum_{n=0}^{\infty}|b_n|^2 = \frac{1}{2}\int_{|z|=1} |f(z)|^2 \, d\theta. \tag{5}$$

When considering areas and volumes in this paper, we take multiplicity into account. For example, the set-theoretic image of the unit disc under the map $z \to z^m$ is the unit disk, but the image is covered m times. For clarity we use the term *parameterized volume* (or area) to emphasize that we are computing an integral where multiplicity is considered.

COROLLARY 1. *The parameterized area of the image of the disk under the map $z \to z^m$ is $m\pi$.*

Proof. It is easy to see that the integral of $|mz^{m-1}|^2$ is πm. Alternatively, the result is trivial when $m = 1$ and follows by induction on m using (4), since the infinite sum consists of the single term unity. □

II. Proper Holomorphic Mappings between Balls and Tensor Products

In order to take advantage of the CR geometry of the unit sphere, we assume for most of this section that the domain dimension is at least 2. We begin by mentioning some rigidity results and then consider the opposite extreme, where the codimension is allowed to be arbitrarily large. See the survey [F2] and [D3; D5] for additional references.

The simplest rigidity result [A; Pi] is that a proper holomorphic self-mapping of B_n for $n \geq 2$ is necessarily an automorphism. This result holds without any regularity assumption on the mapping at the boundary. It is known that proper holomorphic mappings between balls, if they are assumed to be sufficiently differentiable at the boundary, are necessarily rational. By contrast, in each positive codimension there are proper holomorphic mappings that extend continuously to the boundary but are not rational. It is an open problem how smooth (at the boundary) a proper map between balls needs to be before it must be rational.

A proper holomorphic rational mapping from B_n to B_N for $N \leq 2n - 2$ must also be a linear fractional transformation [Fa2], and (with this restriction on the dimensions) the smoothness requirement to guarantee rationality is less than in the general case (see [H; HJ]). The author has conjectured sharp quantitative generalizations of these results: For $n \geq 2$, a rational proper mapping between B_n and B_N is of degree at most $C(n, N)$, where $C(2, N) = 2N - 3$ and $C(n, N) = \frac{N-1}{n-1}$ for $N \geq 3$. There are explicit polynomial proper mappings that achieve these suggested bounds and hence, if the conjecture is true, then the number $C(n, N)$ is sharp. The polynomials yielding the conjectured sharp bounds are not homogeneous. It is possible that information on volumes of parameterizations can be related to this conjecture, but we do not pursue the topic here. On the other hand, Corollary 2 provides an extremal result for volumes.

At the opposite extreme from rigidity we consider the collection of proper holomorphic rational mappings from B_n to B_N for all possible N. We write $(p, g) = p \oplus g$ for the direct sum of maps p and g. Virtually anything is possible in this situation; for example, we have the following result (see [D5] or [D6]). Let $p/q: \mathbf{C}^n \to \mathbf{C}^K$ be a (holomorphic) rational function that maps the closed unit ball $\overline{B_n}$ into the open unit ball B_K. Then there is an integer k and a polynomial mapping $g: \mathbf{C}^n \to \mathbf{C}^k$ such that $p/q \oplus g/q$ is a proper holomorphic rational mapping from B_n to B_{K+k}. Now take $q = 1$. Then, given a polynomial mapping p of arbitrary degree d that maps the closed unit ball $\overline{B_n}$ to the open unit ball B_K, we can find a polynomial mapping $g: \mathbf{C}^n \to \mathbf{C}^k$ such that $p \oplus g$ is a proper mapping from B_n to B_N, where $N = K + k$.

The author has classified all proper polynomial mappings between balls by combining a kind of tensor division with the determination of all homogeneous polynomial examples. We briefly recall these results, which motivate some of the computations performed in this paper. First we express a version of the mapping $z \to z^{\otimes m}$ in coordinates.

Let $N = N(n, m)$ denote the dimension of the space of homogeneous polynomials of degree m in n variables, and let $\{e_\alpha\}$ form an orthonormal basis of $\mathbf{C}^N$. We define H_m by

$$H_m(z) = \sum_{|\alpha|=m} \sqrt{\binom{m}{\alpha}} z^\alpha e_\alpha. \tag{6}$$

Then $H_m\colon B_n \to B_N$ is a proper mapping, because $\|H_m(z)\|^2 = \|z\|^{2m} = 1$ on $\|z\|^2 = 1$ and $\|H_m(z)\|^2 < 1$ on the open ball.

THEOREM [D3; Ru2]. *Let $p\colon \mathbf{C}^n \to \mathbf{C}^K$ be homogeneous of degree m. Suppose that the components of p are linearly independent and that $\|p(z)\|^2 = 1$ on $\|z\|^2 = 1$. Then $K = N(n, m)$ and there is a unitary mapping U such that $p = UH_m$.*

Next we define the tensor product and division operations with respect to a subspace. Given functions p and g with the same domain but different targets, we want to multiply them on this subspace. Let A be a subspace of $\mathbf{C}^N$ with orthogonal complement $A^\perp$, and let π denote the orthogonal projection onto A. Given a function p with values in $\mathbf{C}^N$, we naturally write

$$p = (1 - \pi)(p) \oplus \pi(p).$$

Let g be a function with the same domain as p and with values in some $\mathbf{C}^k$. We define a mapping $E_A(p, g)$ by

$$E_A(p, g) = (1 - \pi)(p) \oplus (\pi(p) \otimes g). \tag{7}$$

In case p and g are proper mappings to balls, it follows that $E_A(p, g)$ also is. To see this fact we observe that

$$\begin{aligned} \|E_A(p, g)\|^2 &= \|(1 - \pi)(p)\|^2 + \|\pi(p) \otimes g\|^2 \\ &= \|(1 - \pi)(p)\|^2 + \|\pi(p)\|^2 \|g\|^2 \\ &= \|(1 - \pi)(p)\|^2 + \|\pi(p)\|^2 + \|\pi(p)\|^2 (\|g\|^2 - 1) \\ &= \|p\|^2 + \|\pi(p)\|^2 (\|g\|^2 - 1). \end{aligned} \tag{8}$$

If we let $\|p\|$ and $\|g\|$ tend to unity, then we see from (8) that $\|E_A(p, g)\|$ does also. Hence, if p and g are proper mappings to balls, then $E_A(p, g)$ also is. Notice that the target dimension of $E_A(p, g)$ is $(N - a) + ak$, where $a = \dim(A)$ (hence $N - a = \dim(A^\perp)$) and k is the dimension of the target of g.

When g is the identity mapping z, the operation $p \to E_A(p, g)$ is known as *orthogonal homogenization*. The inverse operation sending $E_A(p, g) \to p$ is an analogue of tensor division; we sometimes call it *undoing*. When g is the identity mapping, undoing is *orthogonal dehomogenization*.

Suppose that the subspace A is 1-dimensional—say, the span of the vector e_N. The tensor product $E_A(f,z)$ is then easy to understand; it is analogous to multiplication by z in one dimension:

$$E_A(f,z) = (f_1, \dots, f_{N-1}, z_1 f_N, \dots, z_n f_N). \tag{9}$$

Undoing replaces the right-hand side of (9) with f.

We next state, in slightly changed language, a result proved by the author (see [D2; D3]) on the classification problem for polynomial mappings.

THEOREM 0. *Let $p\colon \mathbf{C}^n \to \mathbf{C}^K$ be a polynomial mapping of degree m, and suppose that $p\colon B_n \to B_K$ is proper. Then there exist an integer N', subspaces $A_0, \dots, A_m$ of $\mathbf{C}^{N'}$, tensor products $E_0, \dots, E_m$, and a linear mapping $L\colon \mathbf{C}^{N'} \to \mathbf{C}^{N(n,m)}$ such that*

$$H_m = L \prod_{j=0}^{m} E_j(p). \tag{10}$$

Each tensor product E_j is of the form $E_{A_j}(f,z)$ for the corresponding subspace A_j.

Sketch of Proof. The main idea uses a Fourier series argument resulting from replacing z by $e^{i\theta}z$. The identity $\|p(e^{i\theta}z)\|^2 = 1$ holds for z on the sphere and for all θ. When p isn't already homogeneous, this identity forces the lowest-order part of p to be orthogonal to the highest-order part of p on the sphere and hence everywhere. We then tensor on the subspace determined by the lowest-order part; doing so results in a new proper polynomial mapping of the same degree but for which the order has increased. The procedure is repeated until we obtain something homogeneous. □

The choice of the subspaces is not unique, but the proof provides a canonical way of choosing them. If we wish, by allowing more than m tensor products in (10) we may choose all the A_j to be 1-dimensional. Moreover, we can ignore the L in (10) and obtain a homogeneous proper polynomial mapping from B_n to $B_{N'}$.

We may also introduce linear mappings into the factorization. Consider a simple monomial example, where we apply various linear mappings along the way to shorten the procedure. Given the proper mapping $p\colon B_2 \to B_4$, we have

$$(z,w) \to (z, zw, zw^2, w^3) = p(z,w). \tag{11}$$

We now indicate how to obtain H_3 from (11) as in (10), leaving it to the reader to see how the linear mappings work:

$$\begin{aligned}(z, zw, zw^2, w^3) &\to ((z^2, zw), zw, zw^2, w^3) \to \left(z^2, \sqrt{2}zw, zw^2, w^3\right)\\ &\to \left((z^2 z, z^2 w), \left(\sqrt{2}zwz, \sqrt{2}zww\right), zw^2, w^3\right)\\ &\to \left(z^3, \sqrt{3}z^2 w, \sqrt{3}zw^2, w^3\right) = H_3(z,w).\end{aligned}$$

The mapping (11) also can be realized simply via tensor products, which does not require any undoing. In general, however, undoing is required. The simplest example is given by

$$(z,w) \to \left(z^3, \sqrt{3}zw, w^3\right); \tag{12}$$

it arises from a single undoing of H_3, but it cannot be realized by tensor products alone.

The homogeneous proper mappings H_m between balls thus play a crucial role in the general classification. They also exhibit many other interesting properties; for example, H_m is invariant under the diagonal representation of a cyclic group of order m given by $z \to \omega z$, where ω is a primitive mth root of unity. See [D4; DL] for more information on invariant holomorphic mappings, and see [D7] for a primality test resulting from such considerations.

We next discuss a second way to build a new proper mapping from two given ones. More precisely, suppose that $f: B_n \to B_k$ and $g: B_n \to B_l$ are proper. For each $t \in [0,1]$, consider the juxtaposition mapping $J_t(f,g)$ defined by

$$J_t(f,g) = tf \oplus \sqrt{1-t^2}g. \tag{13}$$

Since $\|J_t(f,g)\|^2 = t^2\|f\|^2 + (1-t^2)\|g\|^2$ and since f and g are proper mappings to balls, it follows that $J_t(f,g): B_n \to B_{k+l}$ also is. We may think of $J_t(f,g)$ as providing a homotopy between f and g by allowing the natural embedding of each target ball in a target ball of higher dimension. We allow composition of $J_t(f,g)$ with a linear mapping in order to make the target dimension minimal. Here is a simple example: Set $f(z,w) = (z, zw, w^2)$ and $g(z,w) = \left(z^2, \sqrt{2}zw, w^2\right)$. Let $u_t = LJ_t(f,g)$, where

$$u_t(z,w) = \left(tz, \sqrt{1-t^2}z^2, \sqrt{2-t^2}zw, w^2\right). \tag{14}$$

This yields a 1-parameter family of proper mappings from B_2 to B_4.

In Section VI we will see how the juxtaposition operation affects volume.

III. Volume Computations for Balls and Eggs

This section is primarily calculus, but some of the results use subtle multi-index notation.

Lemma 1. *The dimension of the vector space of homogeneous polynomials of degree d in n (commuting) variables is* $\binom{n+d-1}{n-1}$.

This combinatorial result is well known, so we omit the proof.

Let K_+ denote the part of the unit ball in $\mathbf{R}^n$ lying in the first orthant; that is, $K_+ = \left\{x : \sum x_j^2 \le 1 \text{ and } x_j \ge 0 \text{ for all } j\right\}$. Let α be an n-tuple of nonnegative real numbers. We denote by $|\alpha|$ the sum of the α_j (the length of the multi-index α). For $t > 0$ we let $\Gamma(t)$ denote the usual Gamma function; for example, $\Gamma(n+1) = n!$ when n is an integer ≥ 0. When each $\alpha_j > 0$, we define an n-dimensional analogue of the Euler Beta function by

$$\mathcal{B}(\alpha) = \frac{\prod \Gamma(\alpha_j)}{\Gamma(|\alpha|)}. \tag{15}$$

There is a beautiful integral formula for this $\mathcal{B}$ function. See [D5, p. 75] for more discussion and see [D1] for the place where the author first used this formula:

$$\mathcal{B}(\alpha) = 2^n |\alpha| \int_{K_+} r^{2\alpha-1}\, dV(r). \tag{16}$$

As an aside, we observe that one can find the volume $\Gamma\left(\frac{1}{2}\right)^n / \Gamma\left(\frac{n}{2}+1\right)$ of the unit ball in $\mathbf{R}^n$ instantly from (16) by setting each α_j equal to $\frac{1}{2}$. We evaluate additional integrals using (16) after describing two uses of multi-index notation:

$$\|z\|^{2d} = \left(\sum_{j=1}^{n} |z_j|^2\right)^d = \sum_{|\alpha|=d} \binom{d}{\alpha} |z|^{2\alpha}; \tag{17}$$

$$|z|^{2\alpha} = \prod_{j=1}^{n} |z_j|^{2\alpha_j}. \tag{18}$$

LEMMA 2. *Let d be a nonnegative integer and α a multi-index of nonnegative real numbers. Let B_n denote the unit ball in $\mathbf{C}^n$. Then*

$$\int_{B_n} \|z\|^{2d}\, dV = \frac{\pi^n}{(n-1)!\,(n+d)}, \tag{19}$$

$$\int_{B_n} |z|^{2\alpha}\, dV = \frac{\pi^n}{n+|\alpha|} \mathcal{B}(\alpha+1). \tag{20}$$

Let p be an n-tuple of positive integers, and put $\Omega(p) = \left\{z : \sum |z_j|^{2p_j} < 1\right\}$. Then

$$\int_{\Omega(p)} |z|^{2\alpha}\, dV = \frac{\pi^n}{\prod p_j} \frac{\mathcal{B}\left(\frac{\alpha+1}{p}\right)}{\left[\frac{\alpha+1}{p}\right]}. \tag{21}$$

Proof. In each case we use polar coordinates in each variable separately. To evaluate (19), we have

$$I = \int_{B_n} \|z\|^{2d}\, dV_{2n} = (2\pi)^n \int_{K_+} \|r\|^{2d} \prod r_j\, dV_n.$$

We then expand $\|r\|^{2d}$ using the multinomial theorem (17) and use multi-index notation to obtain (22):

$$I = \pi^n 2^n \sum_{|\gamma|=d} \binom{d}{\gamma} \int_{K_+} r^{2\gamma+1}\, dV_n. \tag{22}$$

Using formula (16) for the Beta function in (22) and then (15) yields

$$\begin{aligned} I &= \pi^n \sum_{|\gamma|=d} \binom{d}{\gamma} \frac{\mathcal{B}(\gamma+1)}{|\gamma+1|} \\ &= \pi^n \sum_{|\gamma|=d} \frac{d!}{\prod \gamma_j} \frac{\prod \gamma_j}{(d+n)\Gamma(d+n)} = \pi^n \frac{d!}{(d+n)!} \sum_{|\gamma|=d} 1. \end{aligned} \tag{23}$$

Finally, using Lemma 1, we simplify further to obtain the desired result:

$$I = \pi^n \frac{d!}{(d+n)!} \frac{(n+d-1)!}{(n-1)!\,d!} = \frac{\pi^n}{(n-1)!\,(n+d)}. \tag{24}$$

The calculation of (20) is similar but easier (since there is no summation for us to compute):

$$I = \int_{B_n} |z|^{2\alpha}\, dV_{2n} = (2\pi)^n \int_{K_+} r^{2\alpha+1}\, dV_n = \pi^n \frac{\mathcal{B}(\alpha+1)}{|\alpha|+n}.$$

The calculation of (21) is also similar. After introducing polar coordinates in each variable and integrating them, we change variables and imitate the proof of (20). □

For convenience we write (20) when $n = 2$ and a, b are integers:

$$\int_{B_2} |z|^{2a} |w|^{2b}\, dV_4 = \frac{\pi^2 a!\, b!}{(a+b+2)!}. \tag{25}$$

We next recall how to find $2k$-dimensional volumes of nice sets in $\mathbf{C}^n$ that are parameterized by holomorphic mappings. We want to find the parameterized volume rather than the volume of the image set. The distinction arises because our mappings need not be injective.

Let Ω be an open subset in $\mathbf{C}^k$, and suppose that $f : \Omega \to \mathbf{C}^N$ is holomorphic. Definition 1 provides a method for finding the $2k$-dimensional parameterized volume of $f(\Omega)$.

Notation. Let ω denote the usual (1,1)-form on $\mathbf{C}^N$ given by

$$\omega = \frac{i}{2} \sum_{j=1}^{N} dz_j \wedge d\bar{z}_j.$$

The factor $\frac{i}{2}$ arises, of course, because $dz \wedge d\bar{z} = -2i dx \wedge dy$ in $\mathbf{C}$. We let ω^k denote the k-fold wedge product of ω with itself.

Definition 1 ($2k$-dimensional volume). Let Ω be an open subset in $\mathbf{C}^k$, and suppose that $f : \Omega \to \mathbf{C}^N$ is holomorphic. We define $V_{2k}(f, \Omega)$ by

$$V_{2k}(f, \Omega) = \int_\Omega \frac{(f^*\omega)^k}{k!} = \int_\Omega r\, dV. \tag{26}$$

We call $V_{2k}(f, \Omega)$ the *$2k$-dimensional parameterized volume*. When $\Omega = B_k$, we abbreviate $V_{2k}(f, \Omega)$ by V_f.

In (26) the asterisk denotes pullback and the $k!$ arises because there are $k!$ ways to permute the indices from 1 to k. The (k, k)-form $\frac{(f^*\omega)^k}{k!}$ is $r\, dV$, where dV is the Euclidean volume form in k complex dimensions for some function r. This function is always a squared norm:

$$r = \sum |J(f_I)|^2 = \det(H(\|f\|^2)). \tag{27}$$

The sum in (27) is taken over all choices of k component functions, and $J(f_I)$ denotes the Jacobian determinant of the mapping f_I with components $f_{i_1}, \ldots, f_{i_k}$. This sum equals the determinant of the complex Hessian $H(\|f\|^2)$ of the function $\|f\|^2$. In this paper we often compute volumes using (27).

We elaborate when $k = 2$; write the variables as (z, w). In the notation of classical differential geometry, we have

$$E = \left\| \frac{\partial f}{\partial z} \right\|^2, \tag{28a}$$

$$G = \left\| \frac{\partial f}{\partial w} \right\|^2, \tag{28b}$$

$$F = \left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \right\rangle. \tag{28c}$$

Then $r = EG - |F|^2$. Notice that there is no square root; the classical formula for the surface area form in the real case is $\sqrt{EG - F^2}$, where E, G, F have analogous definitions.

We return to the homogeneous mapping $H_m(z)$ and consider $H_m \colon B_k \to \mathbf{C}^N$, where $N = \binom{k+m-1}{k-1}$ by Lemma 1. We next obtain an explicit formula for the $2k$-dimensional parameterized volume of the image of the unit ball under H_m.

LEMMA 3. *The pullback* kth *power* $(H_m^*(\omega))^k$ *satisfies*

$$(H_m^*(\omega))^k = m^{k+1} k!\, \|z\|^{2k(m-1)} dV_{2k}. \tag{29}$$

Proof. Note first that $(H_m^*(\omega))^k$ is a smooth (k,k)-form and hence a multiple r of dV_{2k}. Next observe that H_m is invariant under unitary transformations, so r must be a function of $\|z\|^2$. Since H_m is homogeneous of degree m, each first derivative is homogeneous of degree $m-1$. The $(1,1)$-form $H_m^*(\omega)$ must then have coefficients that are bihomogeneous of degree $(m-1, m-1)$. The coefficient r of its kth power must be homogeneous of degree $2k(m-1)$. Combining the homogeneity with the dependence on $\|z\|^2$ gives the desired expression, except for evaluating the constant $m^{k+1}k!$.

For this evaluation it suffices to compute the coefficient of $|z_1|^{2k(m-1)}$. To do so, we compute dH_m and then work modulo $z_2, \ldots, z_n$. Thus, in the formula for $(H_m^*(\omega))^k$ we set all variables (except the first) equal to zero; this yields

$$H_m^*(\omega) = m^2 |z_1|^{2m-2} |dz_1|^2 + m |z_1|^{2m-2} \sum_{j=2}^{k} |dz_j|^2. \tag{30}$$

From (30) it suffices to compute

$$\left(m^2 |dz_1|^2 + m \sum_{j=2}^{k} |dz_j|^2 \right)^k. \tag{31}$$

Expanding (31) gives us

$$k!\, m^{k+1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_k \wedge d\bar{z}_k,$$

and (29) follows by putting the factor $|z_1|^{(2m-2)k}$ from (30) back in. We could also prove this lemma using (27). □

Putting these results together enables us to obtain the following result.

THEOREM 1. *Let $f\colon B_k \to B_K$ be a proper holomorphic homogeneous polynomial mapping of degree m. Then the $2k$-dimensional parameterized volume V_f is given by*

$$V_f = m^k \pi^k \frac{1}{k!}. \tag{32}$$

Proof. Consider the function $\|f\|^2$. Since

$$\|f(z)\|^2 = 1 = \|z\|^{2m} = \|H_m(z)\|^2$$

on the unit sphere and since f and H_m both are homogeneous, this equality holds everywhere. Hence $\|f\|^2 = \|H_m\|^2$, and these two functions have the same complex Hessian determinant. By (27) they determine the same volume form:

$$\sum_I |J(f_I)|^2 = \sum_I |J((H_m)_I)|^2;$$

hence, by Lemma 3,

$$V_f = \int_{B_k} \frac{(H_m^*(\omega))^k}{k!} = \int_{B_k} m^{k+1} \|z\|^{2k(m-1)}\, dV_{2k}.$$

Using Lemma 2 now yields

$$V_f = m^{k+1} \frac{\pi^k}{k(m-1)+k} \frac{1}{(k-1)!} = \frac{m^k \pi^k}{k!}.$$

As a check we observe, when $m = 1$, that $V_f = \frac{\pi^k}{k!}$, which is the volume of B_k. When $k = 1$ we obtain $V_f = \pi m$; this is also the correct result, since we have covered the unit disk m times and hence obtain $m\pi$ for the area of the parameterization. □

We next describe how proper polynomial maps between balls can be expressed using proper monomial maps between balls. Let $\{E_\alpha\}$ denote the standard orthonormal basis for $\mathbf{C}^K$ and $\{e_\alpha\}$ that for $\mathbf{C}^N$. Let $L\colon \mathbf{C}^K \to \mathbf{C}^N$ be linear; its matrix with respect to the standard bases has entries $L_{\alpha\beta}$.

Now let $f\colon B_n \to B_N$ be a proper polynomial mapping. We write

$$f(z) = \sum c_\alpha z^\alpha = \sum_\beta \sum_\alpha c_{\alpha\beta} e_\beta z^\alpha$$

for vectors $c_\alpha \in \mathbf{C}^N$. Define a monomial mapping $p\colon \mathbf{C}^n \to \mathbf{C}^K$ by

$$p(z) = \sum \|c_\alpha\| z^\alpha E_\alpha = (\ldots, \|c_\alpha\| z^\alpha, \ldots)$$

and a linear mapping $L\colon \mathbf{C}^K \to \mathbf{C}^N$ by

$$L(E_\alpha) = \sum \frac{c_{\alpha\beta}}{\|c_\alpha\|} e_\beta.$$

Thus $L_{\alpha\beta} = \frac{c_{\alpha\beta}}{\|c_\alpha\|}$ and so $f = L \circ p$. It is then not hard to show that $p\colon B_n \to B_N$ is proper. We obtain the following result from [D3].

PROPOSITION 2. *Let $f\colon B_n \to B_N$ be a proper polynomial mapping. Then there exist an integer K, a proper monomial mapping $p\colon B_n \to B_K$, and a linear mapping L such that $f = L \circ p$.*

We provide an example for later use.

EXAMPLE 1. Define $f\colon B_2 \to B_3$ and $p\colon B_2 \to B_5$ by

$$f(z,w) = \frac{1}{\sqrt{2}}(z - w, z^2 + zw, zw + w^2), \tag{33a}$$

$$p(z,w) = \frac{1}{\sqrt{2}}\big(z, -w, z^2, \sqrt{2}zw, w^2\big). \tag{33b}$$

Then f and p are proper and $f = L \circ p$, where $L\colon \mathbf{C}^5 \to \mathbf{C}^3$ has the following matrix representation (with respect to the usual bases):

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & 1 \end{pmatrix}. \tag{34}$$

Next we determine the volumes of some specific polynomial proper mappings from B_2. We illustrate the computation in one case. Put $f(z,w) = (z, zw, w^2)$; then $f\colon B_2 \to B_3$ is proper. Using $EG - |F|^2$ or (27) yields

$$\begin{aligned} f^*(\omega)^2 &= ((1 + |w|^2)(4|w|^2 + |z|^2) - |zw|^2)dV_4 \\ &= (|z|^2 + 4|w|^2 + 4|w|^4)dV_4. \end{aligned} \tag{35}$$

Integrating (35) using (25) then gives

$$\begin{aligned} V = \int_{B_2} f^*(\omega)^2 &= \int_{B_2} (|z|^2 + 4|w|^2 + 4|w|^4)\, dV_4 \\ &= \pi^2\left(\frac{1}{3!} + \frac{4}{3!} + \frac{4 \cdot 2!}{4!}\right) = \frac{7\pi^2}{6}. \end{aligned}$$

Two proper holomorphic mappings f and g from B_n to B_N are *spherically equivalent* if there are automorphisms ϕ and ψ of the domain and target balls such that $f = \psi \circ g \circ \phi$. Faran [Fa] proved there are precisely four distinct spherical equivalence classes of proper rational mappings from B_2 to B_3. These four (monomial) mappings appear in the next example, where we compare their parameterized volumes; we also include a 1-parameter family of maps to B_4 and a map to B_5.

EXAMPLE 2. Here are the 4-dimensional volumes of the image of B_2 under various proper mappings to B_3, B_4, and B_5.

$$(z,w) \to (z,w,0)\colon\ V = \frac{\pi^2}{2}. \tag{36}$$

$$(z,w) \to (z,zw,w^2)\colon\ V = \frac{7\pi^2}{6}. \tag{37}$$

$$(z,w) \to \left(z^2, \sqrt{2}zw, w^2\right)\colon\ V = 2\pi^2. \tag{38}$$

$$(z,w) \to \left(z^3, \sqrt{3}zw, w^3\right)\colon\ V = \frac{63\pi^2}{20}. \tag{39}$$

$$(z,w) \to \left(tz, \sqrt{1-t^2}z^2, \sqrt{2-t^2}zw, w^2\right)\colon\ V = \frac{(t^4-6t^2+12)\pi^2}{6}. \tag{40}$$

$$(z,w) \to \frac{1}{\sqrt{2}}(z-w, z^2+zw, zw+w^2)\colon\ V = \frac{7\pi^2}{6}. \tag{41}$$

$$(z,w) \to \frac{1}{\sqrt{2}}\left(z, -w, z^2, \sqrt{2}zw, w^2\right)\colon\ V = \frac{9\pi^2}{8}. \tag{42}$$

The mappings in (37) and (41) are spherically equivalent. We include (41) in order to follow up on Example 1. Notice that the volume in (41) exceeds that in (42), even though (see Example 1) $f = L \circ p$ holds. There is no simple way to express the relationship between the two answers in terms of L, because L is not an equidimensional mapping. The Jacobian matrix of f has three rows and two columns; the Jacobian matrix of p has five rows and two columns. We find the volume form by taking the sum of squared absolute values of three minors for f but of ten minors for p. Even though $f = L \circ p$, the various Jacobians involve different submatrices of L.

IV. A Real-Variables Analogue

In this short section we pause to present an elegant version of Theorem 1 when the mapping H_m is restricted to real variables and the domain is the unit disk. Let

$$h_m(x,y) = \left(x^m, \ldots, \sqrt{\binom{m}{a}}\, x^a y^{m-a}, \ldots, y^m\right) \tag{43}$$

denote the corresponding mapping. We provide a simple geometric argument for finding the surface area of the $h_m(D)$, where D is the unit disk. One can also obtain the result along the same lines used in Theorem 1.

We begin with the following geometric result.

Proposition 3. *Let $g\colon S^1 \to S^m$ define a continuously differentiable curve of length $l(g)$ in the sphere. Suppose $f\colon D \to \mathbf{R}^{m+1}$ is defined by $f(re^{i\theta}) = r^m g(\theta)$. Then the surface area of $f(D)$ equals $\frac{1}{2}l(g)$.*

Proof. Using the notation of classical differential geometry, the surface area V_2 is the iterated integral

$$V_2 = \int_0^{2\pi} \int_0^1 \sqrt{EG - F^2}\, dr\, d\theta, \tag{44}$$

where

$$E = \left\| \frac{\partial f}{\partial r} \right\|^2 = m^2 r^{2m-2} \|g(\theta)\|^2 = m^2 r^{2m-2}, \tag{45a}$$

$$G = \left\| \frac{\partial f}{\partial \theta} \right\|^2 = r^{2m} \|g'(\theta)\|^2, \tag{45b}$$

$$F = \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta} \right\rangle = m r^{2m-1} \langle g, g' \rangle = 0. \tag{45c}$$

In (45a) we have used that $\|g(\theta)\|^2 = 1$; hence g' is orthogonal to g. We obtain

$$V_2 = \int_0^{2\pi} \int_0^1 m r^{2m-1} \|g'(\theta)\|\, dr\, d\theta = \frac{1}{2} \int_0^{2\pi} \|g'(\theta)\|\, d\theta = \frac{1}{2} l(g). \qquad \square$$

THEOREM 2. *Let h_m be given by* (43). *Then the* (*parameterized*) *surface area of $h_m(D)$ is $\pi\sqrt{m}$.*

Proof. We claim that it suffices to find the length of the curve g_m, where

$$g_m = \left(c^m, \ldots, \sqrt{\binom{m}{a}} c^a s^{m-a}, \ldots, s^m \right)$$

and where c and s stand for cosine and sine. The claim follows because g maps to the sphere:

$$\|g_m\|^2 = (c^2 + s^2)^m = 1$$

and because $h_m(x, y) = r^m g_m(\theta)$. Hence Proposition 3 applies and so, assuming that the length of g_m is $2\pi\sqrt{m}$, we obtain the result:

$$V_2(h_m(D)) = \tfrac{1}{2} l(g) = \pi\sqrt{m}.$$

To find $l(g_m)$, we show that g has constant speed $\sqrt{m}$. Compute $\|g'(\theta)\|^2$ as follows:

$$\begin{aligned} \|g'(\theta)\|^2 &= \sum_{a=0}^{m} \binom{m}{a} (-a c^{a-1} s^{m-a+1} + (m-a) c^{a+1} s^{m-a})^2 \\ &= \sum_{a=0}^{m} \binom{m}{a} c^{2(a-1)} s^{2(m-a-1)} (-a s^2 + (m-a) c^2)^2 \\ &= \sum_{a=0}^{m} \binom{m}{a} c^{2(a-1)} s^{2(m-a-1)} (m c^2 - a)^2. \end{aligned} \tag{46}$$

Evaluating the sum in (46) is a standard kind of combinatorial problem. We shall use the following identities on the three separate sums that arise from expanding $(mc^2 - a)^2$:

$$\sum_{a=0}^{m} \binom{m}{a} t^a = (1+t)^m,$$

$$\sum_{a=0}^{m} a \binom{m}{a} t^{a-1} = m(1+t)^{m-1},$$

$$\sum_{a=0}^{m} a^2 \binom{m}{a} t^{a-1} = m \frac{d}{dt} t(1+t)^{m-1}.$$

Using also that $c^2 + s^2 = 1$, we obtain from (46) that $\|g'(\theta)\|^2 = m$ and the theorem follows. □

REMARK. In fact, all derivatives of g have constant norm.

V. The Effect of Tensor Products on Volume

This section provides the main result of this paper. We begin with a few lemmas.

LEMMA 4. *Suppose that $h\colon B_n \to \mathbf{C}$ is holomorphic and square integrable, with power series expansion $h(z) = \sum c_\alpha z^\alpha$. Then*

$$\int_{B_n} |h(z)|^2 \, dV = \int_{B_n} \sum_\alpha |c_\alpha|^2 |z|^{2\alpha} \, dV = \pi^n \sum_\alpha |c_\alpha|^2 \frac{\mathcal{B}(\alpha+1)}{|\alpha|+n}. \tag{47}$$

Proof. The first step follows because distinct monomials are orthogonal in L^2. The second step follows by plugging in (20) from Lemma 2. □

LEMMA 5. *Let $f\colon \mathbf{C}^n \to \mathbf{C}^n$ be the monomial mapping $f = (m_1, \ldots, m_n)$, and let $J(f)$ denote the Jacobian determinant of f. Then*

$$J(f) = \frac{\prod_j m_j}{\prod_j z_j} \det(a_{kl}), \tag{48}$$

where a_{kl} is the exponent of z_l in m_k.

Proof. Since each m_j is a monomial, we easily see that

$$\frac{\partial m_j}{\partial z_k} = a_{jk} \frac{m_j}{z_k}.$$

Multilinearity properties of the determinant (for rows and columns both) then imply (48). □

LEMMA 6. *Let $f\colon B_n \to \mathbf{C}^n$ be a monomial mapping $f = (m_1, \ldots, m_n)$. In the notation of Lemma 5, the $2n$-dimensional parameterized volume V_f is given by*

$$V_f = \pi^n |\det(a_{kl})|^2 \frac{\prod_k (\sum_j a_{jk} - 1)!}{(\sum_j \sum_k a_{jk})!}. \tag{49}$$

Proof. By definition, the volume V_f is the integral

$$V_f = \int_{B_n} |J(f)|^2 \, dV.$$

Using Lemmas 4 and 5 now yields

$$V_f = |\det(a_{kl})|^2 \int_{B_n} \prod_j |z_j|^{2(\sum_k a_{kj} - 1)} \, dV.$$

Using Lemma 2, we evaluate the integral to obtain the result. □

Let $f\colon B_n \to \mathbf{C}^n$ be a holomorphic mapping. Then

$$f(z) = \Big(\sum c_{\alpha_1} z^{\alpha_1}, \ldots, \sum c_{\alpha_n} z^{\alpha_n} \Big),$$

where each α_j is itself a multi-index. By multilinearity, the Jacobian $J(f)$ satisfies

$$J(f) = \sum \Big(\prod_{j=1}^{n} c_{\alpha_j} \Big) J(z^{\alpha_1}, \ldots, z^{\alpha_n}).$$

Combining the previous lemmas allows us to compute $\|J(f)\|_{L^2}^2$. First,

$$\|J(f)\|_{L^2}^2 = \int_{B_n} |J(f)|^2 \, dV = \int_{B_n} \Big| \sum \Big(\prod_{j=1}^{n} c_{\alpha_j} \Big) J(z^{\alpha_1}, \ldots, z^{\alpha_n}) \Big|^2 \, dV.$$

Define new multi-indices m by $m_j = \sum_k \alpha_{jk} - 1$. Then

$$\int_{B_n} |J(f)|^2 \, dV = \int_{B_n} \Big| \sum_m A_m z^m \Big|^2 \, dV = \pi^n \sum_m |A_m|^2 \frac{\mathcal{B}(m+1)}{|m| + n}, \tag{50}$$

where $A_m = \sum \big(\prod_{j=1}^{n} c_{\alpha_j} \big) \det(\alpha_{kl})$ and the sum is restricted by the definition of m. For later convenience we rewrite the mth term in (50) as

$$\begin{aligned} \pi^n |A_m|^2 & \frac{\mathcal{B}(m+1)}{|m| + n} \\ &= \pi^n |A_m|^2 \frac{\prod_j m_j!}{(|m| + n)!} \\ &= \Big| \sum \Big(\prod_{j=1}^{n} c_{\alpha_j} \Big) \det(\alpha_{kl}) \Big|^2 \frac{\pi^n}{(\sum \sum \alpha_{jk})!} \prod_{j=1}^{n} \Big(\Big(\sum_k \alpha_{kj} - 1 \Big)! \Big). \end{aligned} \tag{51}$$

Our aim is see the effect on (51) of replacing f by a tensor product. In order to do so, we establish an inequality about determinants.

PROPOSITION 4. *Let $L = (a_{jk})$ be an $n \times n$ matrix of real numbers. Suppose that each column sum v_j of L is nonnegative. Let $D_j = (\delta_{nj})$ denote the matrix whose entires are zero except that the entry in the* jth *column and* nth *row is unity. Let C_j be the cofactor determinant* (*with sign included*) *obtained by deleting the* jth *column and* nth *row and taking the signed determinant. Then the following relations hold*:

$$\det(L) = \sum_{j=1}^{n} C_j v_j; \tag{52}$$

$$\sum_{j=1}^{n} \det(L + D_j)^2 v_j = \det(L)^2 \left(2 + \sum_j v_j\right) + \sum_j C_j^2 v_j; \tag{53}$$

$$\det(L)^2 \left(2 + \sum_j v_j\right) \leq \sum_{j=1}^{n} \det(L + D_j)^2 v_j. \tag{54}$$

Proof. The right-hand side of (52) includes the usual expansion of the determinant in terms of cofactors plus additional terms. The additional terms occur in pairs with opposite signs and hence cancel, so (52) holds. To prove (53), we first write

$$\det(L + D_j)^2 = (\det(L) + C_j)^2 = \det(L)^2 + 2\det(L)C_j + C_j^2. \tag{55}$$

Now multiply both sides of (55) by v_j and sum to obtain

$$\sum_{j=1}^{n} \det(L + D_j)^2 v_j = \det(L)^2 \sum_{j=1}^{n} v_j + 2\det(L) \sum_{j=1}^{n} C_j v_j + \sum_{j=1}^{n} C_j^2 v_j. \tag{56}$$

Using (52) in (56) yields (53), and dropping the nonnegative last term in (53) yields (54). □

We can now generalize Proposition 1; as in one dimension, the parameterized volume increases under tensor product. As in one dimension, there is generally no pointwise inequality on the volume forms; the inequality requires integration. We note one new issue in higher dimensions: Example 3 will show that the inequality fails if we allow multiplication by only a single coordinate function. Example 4 and Theorem 4 indicate that we must tensor with something closely related to the geometry of the boundary.

THEOREM 3. *Let* $f : B_n \to \mathbf{C}^N$ *be holomorphic, and suppose that the* $2n$*-dimensional parameterized volume* V_f *of the image* $f(B_n)$ *is finite. Let* $g = Ef$ *be defined by*

$$g = (f_1, \ldots, f_{N-1}, z_1 f_N, \ldots, z_n f_N).$$

Then $V_g \geq V_f$. *Equality occurs if and only if* $f_N = 0$.

More generally, let $p = (p_1, \ldots, p_n)$ *be an* n*-tuple of positive integers, let* $\Omega(p)$ *be the egg domain* $\left\{z : \sum_{j=1}^{n} |z_j|^{2p_j} < 1\right\}$, *and let* $f : \Omega(p) \to \mathbf{C}^N$ *be a holomorphic mapping. Assume that the volume of the image of* $\Omega(p)$ *is finite. Define* $g = Ef$ *by*

$$g = (f_1, \ldots, f_{N-1}, z_1^{p_1} f_N, \ldots, z_n^{p_n} f_N).$$

Write V_g^p *and* V_f^p *for the corresponding volumes of images of* $\Omega(p)$. *Then* $V_g^p \geq V_f^p$, *and equality occurs if and only if* $f_N = 0$.

Proof. We will write the details for the case of the ball, and at the end of the proof we indicate how and why the proof easily generalizes. Assume first that Ω is the ball. We use (51), applied to both f and g, and then Proposition 4. The parameterized volume V_g is the sum of the integrals of all $|J(g_I)|^2$. We separate the n-tuples g_I into three types. Type I consists of those formed from n of the functions $f_1, \dots, f_{N-1}$. The corresponding Jacobians occur also in the expression for V_f and hence can be ignored when proving the inequality between V_f and V_g. Type II consists of those n-tuples for which exactly $n-1$ of the components are chosen from among $f_1, \dots, f_{N-1}$, and type III consists of those n-tuples for which at most $n-2$ are chosen from among $f_1, \dots, f_{N-1}$. Those of type III contribute positively to V_g but do not contribute to V_f and hence can be ignored. We will therefore prove that

$$\int_{B_n} \sum |J(f_I)|^2 \, dV \le \int_{B_n} \sum |J(g_I)|^2 \, dV, \tag{57}$$

where the sums on both sides are taken over those n-tuples of type II.

To verify (57) it suffices to prove, for any choice of n components (note the relabeling) $f = (f_1, \dots, f_n)$, the following inequality:

$$\int_{B_n} |J(f)|^2 \, dV \le \int_{B_n} \sum_{\nu=1}^{n} |J(f_1, \dots, f_{n-1}, z_\nu f_n)|^2 \, dV. \tag{58}$$

The left-hand side has been computed in (50). We compute the right-hand side in the same fashion, obtaining a sum of n terms. As before, we define multi-indices m by $m_j = \sum_k \alpha_{jk} - 1$. The following sums in A_m and $K_m(\nu)$ are taken over all n-tuples of multi-indices such that this definition holds. The mth term of the left-hand side of (58) is

$$\pi^n |A_m|^2 \frac{\prod_j m_j!}{(|m|+n)!}.$$

Computing $\|J(f_1, \dots, f_{n-1}, z_\nu f_n)\|_{L^2}^2$ in the same way, we obtain

$$\pi^n |K_m(\upsilon)|^2 \frac{(m_\nu + 1) \prod_j m_j!}{(|m|+n+1)!}$$

for the mth term, where

$$K_m(\nu) = \sum \left(\prod_{j=1}^{n} c_{\alpha_j} \right) \det(\alpha_{jk} + D_\upsilon). \tag{59}$$

Using the ideas of Proposition 4, we write

$$\det(\alpha_{jk} + D_\upsilon) = \det(\alpha_{jk}) + C(\alpha, \nu);$$

here $C(\alpha, \nu)$ is an appropriate cofactor determinant. Hence

$$K_m(\nu) = A_m + \sum \left(\prod_{j=1}^{n} c_{\alpha_j} \right) C(\alpha, \nu). \tag{60}$$

We will show for each multi-index m that

$$\pi^n |A_m|^2 \frac{\prod_j m_j!}{(|m|+n)!} \leq \sum_\nu \pi^n |K_m(v)|^2 \frac{(m_\nu+1)\prod_j m_j!}{(|m|+n+1)!}. \tag{61}$$

Summing (61) over m implies (58). After canceling common factors, (61) is equivalent to

$$|A_m|^2(|m|+n+1) \leq \sum_\nu (m_\nu+1)|K_m(v)|^2. \tag{62}$$

By (60), the right-hand side of (62) becomes

$$\begin{aligned}
&\sum_\nu (m_\nu+1)|K_m(v)|^2 \\
&\quad = |A_m|^2\Big(\sum_\nu (m_\nu+1)\Big) + \sum_\nu (m_\nu+1)\Big|\sum\Big(\prod_{j=1}^n c_{\alpha_j} C(\alpha,v)\Big)\Big|^2 \\
&\qquad + 2\,\mathrm{Re}\Big(\overline{A_m}\sum\prod_{j=1}^n c_{\alpha_j}\sum_\nu C(\alpha,v)(m_\nu+1)\Big).
\end{aligned} \tag{63}$$

Recall that $|m|+n = \sum_\nu (m_\nu+1)$ by multi-index notation. Also, because $m_\nu + 1 = \sum \alpha_{\nu j}$, (52) implies that

$$A_m = \sum\prod_{j=1}^n c_{\alpha_j}\sum_\nu C(\alpha,v)(m_\nu+1).$$

Thus, the last term in (63) is $2|A_m|^2$. We may drop the nonnegative middle term in (63). The right-hand side of (63) is thus at least $|A_m|^2(|m|+n+2)$, and (62) follows.

In other words, the determinantal inequality from Proposition 4 holds for each m. Summing over m yields (61) and hence (58). Recalling that it suffices to prove the inequality for type II terms, we have proved that $V_f \leq V_g$.

If $f_N = 0$, then $V_f = V_g$. If $f_N \neq 0$ (and $n \geq 2$), then there is a Type III term of the form $|J(z_1 f_N, \ldots, z_n f_N)|^2$; it is easy to see that such a Jacobian is not identically zero. Therefore, in this case $V_f < V_g$.

Suppose next that $\Omega(p)$ is an egg domain. The monomials remain a complete orthogonal system for $L^2(\Omega(p))$, and a holomorphic mapping f has a power series expansion valid on all $\Omega(p)$. Thus the structure of the proof is identical. The calculations are quite similar, except that the various factorials used in case of the ball become values of Gamma functions at nonintegral values. By (21) we have

$$\|z^\alpha\|^2_{L^2(\Omega(p))} = \frac{\pi^n}{\prod p_j}\frac{\mathcal{B}\big(\frac{\alpha+1}{p}\big)}{\big[\frac{\alpha+1}{p}\big]}. \tag{64}$$

In forming the tensor product we multiply z^α by each $z_\nu^{p_\nu}$. The effect is to alter the multi-index $\frac{\alpha+1}{p}$ appearing in (64) by adding 1 in the νth slot. Recalling (15), we can then use the formula $\Gamma(x+1) = x\Gamma(x)$ in both the numerator and denominator of the resulting terms, rather than using $j! = j(j-1)!$ as in the proof for the ball. After these modifications, the proof goes through as before. □

The following consequence gives an extremal property of the homogeneous proper mappings between balls, and it is one of the main results of this paper.

COROLLARY 2. *Let $p\colon B_k \to B_K$ be a proper polynomial mapping of degree d. Then the parameterized volume V_p is at most $d^k \pi^k \frac{1}{k!}$. Equality occurs only if p is homogeneous.*

Proof. We saw in Theorem 1 that $V_p = V_{H_d} = d^k \pi^k \frac{1}{k!}$, so equality holds if p is homogeneous. If p is not homogeneous, then we can repeatedly tensor p on 1-dimensional subspaces as in the proof of Theorem 0 from Section II until it is homogeneous. By Theorem 3, each tensor operation increases the volume, so we obtain $V_p < V_{H_d}$. □

EXAMPLE 3. The conclusion of Theorem 3 fails in general if we multiply a component of f by only a single coordinate function. The simplest example is given by $f(z,w) = (zw, w)$. In this case $\|J(f)\|_{L^2} > 0$, whereas $\|J(g)\|_{L^2} = 0$ if $g(z,w) = (zw, zw)$.

EXAMPLE 4. The conclusion of Theorem 3 for the ball fails in general if we tensor with a mapping other than $z = (z_1, \dots, z_n)$, even when the mapping defines a 0-dimensional variety. Let $n = 2$, and put $f(z,w) = (z^4, w)$. Suppose we tensor by multiplying the second component by both z^3 and w^3, obtaining $g(z,w) = (z^4, z^3 w, w^4)$. Then the parameterized volume decreases:

$$V_f = \frac{4\pi^2}{5} > \frac{24\pi^2}{35} = V_g.$$

Similarly, on the egg $\Omega(p)$ one must tensor by $(z_1^{p_1}, \dots, z_n^{p_n})$ in order for the conclusion to be valid.

We can express Theorem 3 in an operator-theoretic manner. For convenience we work on the ball. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$; we think of the various $\mathbf{C}^N$ as subspaces of $\mathcal{H}$. Let $f\colon B_n \to \mathcal{H}$ be holomorphic, and suppose that the range of f is finite dimensional. Such an f is square integrable if $\int_{B_n} \|f\|^2 \, dV < \infty$. Let Df denote the mapping listing all the (finitely many) $J(f_I)$. We say that Df is *square integrable* if $\int_{B_n} \sum_I \|J(f_I)\|^2 \, dV < \infty$. Equivalently, Df is square integrable if the determinant of the complex Hessian of $\|f\|^2$ is square integrable. We can regard D as an unbounded operator on the space of square-integrable holomorphic mappings from B_n to $\mathcal{H}$. We also have the operator M given by taking the tensor product on a 1-dimensional subspace. Theorem 3 states that $(DM)^* DM - D^* D$ is a nonnegative operator on its domain.

VI. The Effect of Juxtaposition on Volume

Let $f\colon B_k \to B_{N_1}$ and $g\colon B_k \to B_{N_2}$ be proper holomorphic mappings. Suppose that we know the $2k$-dimensional volumes V_f and V_g of the parameterized images of B_k under f and g. The next result tells us how to find the $2k$-dimensional volume of the parameterized image of the juxtaposition $J_t(f,g)$. For simplicity we write down the answer only when $k=1$ and $k=2$.

EXAMPLE 5. Suppose that $f\colon B_1 \to B_{N_1}$ and $g\colon B_1 \to B_{N_2}$ are proper mappings, and let $J_t = J_t(f,g)$ denote their juxtaposition. Then

$$A_{J_t} = t^2 A_f + (1-t^2) A_g.$$

This equality follows immediately once we observe that

$$J_t^*(\omega) = \frac{i}{2}(t^2 df \wedge \overline{df} + (1-t^2) dg \wedge \overline{dg}) = t^2 f^*(\omega) + (1-t^2) g^*(\omega).$$

The result in general is more complicated because we must compute the kth exterior power of the $(1,1)$-form $J_t^*(\omega)$. Here is the result when $k=2$; see Example 6 for an application.

PROPOSITION 5. *Let $f\colon B_2 \to B_{N_1}$ and $g\colon B_2 \to B_{N_2}$ be proper holomorphic mappings, and let $J_t = J_t(f,g)$ denote their juxtaposition. Then*

$$V_{J_t} = t^4 V_f + (1-t^2)^2 V_g + t^2(1-t^2) \int \left\| \frac{\partial f}{\partial z} \otimes \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \otimes \frac{\partial g}{\partial z} \right\|^2 dV. \quad (65)$$

Proof. We compute $J_t^*(\omega)$ and then $J_t^*(\omega)^2$. Let

$$E = t^2 \left\| \frac{\partial f}{\partial z} \right\|^2 + (1-t^2) \left\| \frac{\partial g}{\partial z} \right\|^2, \quad (66a)$$

$$G = t^2 \left\| \frac{\partial f}{\partial w} \right\|^2 + (1-t^2) \left\| \frac{\partial g}{\partial w} \right\|^2, \quad (66b)$$

$$F = t^2 \left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \right\rangle + (1-t^2) \left\langle \frac{\partial g}{\partial z}, \frac{\partial g}{\partial w} \right\rangle. \quad (66c)$$

The coefficient of the volume form is the determinant $EG - |F|^2$. We compute it to obtain the first two terms in (65) plus various other terms. We then have two products of squared norms as well as the inner products $\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \rangle$ and $\langle \frac{\partial g}{\partial z}, \frac{\partial g}{\partial w} \rangle$. Using the result that

$$\|u_1 \otimes v_2 - u_2 \otimes v_1\|^2 = \|u_1\|^2 \|v_2\|^2 + \|u_2\|^2 \|v_1\|^2 - 2\,\mathrm{Re}\langle u_1, u_2 \rangle \langle v_1, v_2 \rangle, \quad (67)$$

we obtain the expression claimed. □

COROLLARY 3. *Under the assumptions of Proposition 5, we have*

$$V_{J_t} \geq t^4 V_f + (1-t^2)^2 V_g \geq \frac{V_f V_g}{V_g + V_f}. \quad (68)$$

Proof. The first inequality in (68) follows from dropping the nonnegative integral in (65). The second inequality follows because the middle term is at least as large as its minimum for $0 \leq t \leq 1$. The minimum value is the term on the far right in (68). □

We conclude this section by finding volumes of the parameterized images for an interesting 1-parameter family of maps from B_2 to B_5. The resulting formula is rather complicated because of the third term in (65).

EXAMPLE 6. Define $p_t\colon B_2 \to B_5$ by

$$p_t(z,w) = \left(z^3, w^3, \sqrt{3}\sqrt{1-t^2}z^2w, \sqrt{3}\sqrt{1-t^2}zw^2, t\sqrt{3}zw\right). \tag{69}$$

For each $t \in [0,1]$, p_t is a proper mapping; it defines (via the juxtaposition idea) a homotopy between H_3 and the mapping (39). Either directly or by using Proposition 5 (each involves considerable computation), we conclude that the coefficient $EG - |F|^2$ of dV_4 is given by

$$\begin{aligned} &EG - |F|^2 \\ &\quad = 27(1-t^2)(|z|^8 + 4|z|^6|w|^2 + 4|z|^2|w|^6) + 81(1+(1-t^2)^2)|z|^4|w|^4 \\ &\qquad + 27t^2(|z|^6 + |w|^6) + 9t^2(1-t^2)(|z|^4|w|^2 + |z|^2|w|^4). \end{aligned} \tag{70}$$

The corresponding volume is

$$\pi^2\frac{90 - 12t^2 - 15t^4}{20}. \tag{71}$$

Evaluating (71) at $t = 0$ gives $9\pi^2/2$, in agreement with Theorem 1 when $m = 3$, and evaluating (71) at $t = 1$ gives $63\pi^2/20$, in agreement with (39).

VII. A General Monotonicity Result

Let $P\colon \mathbf{C}^n \to \mathbf{C}^M$ be a holomorphic mapping, and let $r(z,\bar z) = \|P(z)\|^2$. We assume that the level set $r = 1$ is compact. Note that r is plurisubharmonic and hence $\frac{\partial\bar\partial r}{-2i}$ is a nonnegative (1,1)-form. The domain $\Omega = \{z : r(z,\bar z) < 1\}$ is a bounded pseudoconvex domain.

Let $f\colon \Omega \to \mathbf{C}^N$ be a holomorphic mapping and assume that f extends to be continuously differentiable on $b\Omega$. For convenience we define the tensor product operation on the first component of f (rather than on the last component as before):

$$E_P f = g = (p_1 f_1, \dots, p_M f_1, f_2, \dots, f_N). \tag{72}$$

Let V_f and $V_{E_P f}$ denote the $2n$-dimensional volumes of the images of Ω under f and $E_P f$. We have the following monotonicity result.

THEOREM 4. *Let Ω be the bounded pseudoconvex domain defined by $\|P\|^2 < 1$, with P as just described. Let $f\colon \Omega \to \mathbf{C}^N$ be a holomorphic mapping and assume that f is continuously differentiable on $b\Omega$. Then*

$$V_{E_P f} \ge V_f.$$

Proof. As in the proof of Theorem 3, we can write

$$V_{E_P f} = \sum \int_\Omega |J(g_I)|^2\, dV, \tag{73}$$

where the multi-indices I are of three types. Those terms with multi-indices of type I occur also in the formula for V_f and will be ignored; those of type III occur only in the formula for $V_{E_P f}$ and can also be ignored. Terms of type II are all of the form

$$\int_\Omega |J(p_v f_1, f_{i_2}, \dots, f_{i_n})|^2 \, dV.$$

Hence, to prove the theorem it suffices to show that

$$\sum_\nu \int_\Omega |J(p_v f_1, f_{i_2}, \dots, f_{i_n})|^2 \, dV \geq \int_\Omega |J(f_1, f_{i_2}, \dots, f_{i_n})|^2 \, dV \tag{74}$$

for all multi-indices $I' = (i_2, \dots, i_n)$. For convenience we write $I' = (2, \dots, n)$, and in proving (74) we now assume that $f = (f_1, \dots, f_n)$. Let η denote the nonnegative $(n-1, n-1)$-form given by

$$\frac{1}{(-2i)^{n-1}} df_2 \wedge \overline{df_2} \wedge \cdots \wedge df_n \wedge \overline{df_n}. \tag{75}$$

Since f is continuously differentiable on $b\Omega$, we may apply Stokes's theorem and so obtain

$$V_f = \int_\Omega |J(f_1, \dots, f_n)|^2 \, dV = \int_{b\Omega} \frac{f_1}{-2i} \wedge \overline{df_1} \wedge \eta. \tag{76}$$

We replace f_1 by $P_\nu f_1$ in (76) and use the product rule, which yields

$$\int_\Omega |J(P_\nu f_1, f_2, \dots, f_n)|^2 \, dV = \int_{b\Omega} |P_\nu|^2 \frac{f_1}{-2i} \overline{df_1} \wedge \eta + \int_{b\Omega} |f_1|^2 \frac{P_\nu}{-2i} \overline{dP_\nu} \wedge \eta. \tag{77}$$

We sum (77) over μ to find $V_{E_P f}$. Note that $r = \sum |P_\nu|^2 = 1$ on $b\Omega$ and that

$$\bar\partial r = \sum_\nu P_\nu \overline{dP_\nu}. \tag{78}$$

Putting these together and using (76), we obtain

$$V_{E_P f} = V_f + \int_{b\Omega} \frac{|f_1|^2}{-2i} \bar\partial r \wedge \eta. \tag{79}$$

Since r is plurisubharmonic and the $(n-1, n-1)$-form η is nonnegative, it follows that $\frac{\bar\partial r}{-2i} \wedge \eta$ is a nonnegative multiple of the surface area form. The surface integral in (80) is therefore nonnegative. Theorem 4 follows. □

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