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Holomorphic Extension of Decomposable Distributions from a CR Submanifold of $\mathbb{C}^L$

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1. Introduction

1.1. Statement of Results

Let $\mathfrak{N}$ be generic submanifold of $\mathbb{C}^{k+m}$ with CR dimension k, and let h be a CR map from $\mathfrak{N}$ into some $\mathbb{C}^n$ verifying $dh(0) = 0$. Setting $L = k + m + n$, we construct a CR submanifold N of $\mathbb{C}^L$ near the origin as the graph of h over $\mathfrak{N}$; that is, $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$. It turns out that any nongeneric CR submanifold of $\mathbb{C}^L$ can be obtained in this fashion (see e.g. [2]). The main question we address in this paper is the possible holomorphic extension of a CR distribution of N to some wedge $\mathcal{W}$ in a complex transverse direction. Our aim is to give a proof of an extension result using only elementary tools. The CR structure of N is determined by $\mathfrak{N}$, so any CR distribution on N is a CR distribution on $\mathfrak{N}$.

Definition 1.1. (a) Let $\mathfrak{N}$ be a smooth generic submanifold of $\mathbb{C}^L$. A CR distribution u on $\mathfrak{N}$ is *decomposable* at the point $p \in \mathfrak{N}$ if, near p, $u = \sum_{j=1}^{K} U_j$; here the U_j are CR distributions extending holomorphically to wedges $\mathcal{W}_j$ in $\mathbb{C}^{k+m}$ with edges $\mathfrak{N}$. We shall say that a distribution u on N is decomposable at a point $p = (p', h(p'))$ if u is decomposable at p' on $\mathfrak{N}$.

(b) We say that v is *complex transversal* to N at p if $v \notin \operatorname{span}_{\mathbb{C}} T_p N$.

Our main result is the following theorem.

Theorem 1.2. *Let $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$ be a nongeneric smooth ($\mathcal{C}^\infty$) CR submanifold of $\mathbb{C}^{k+m+n}$ such that the map h is decomposable at some $p'_0 \in \mathfrak{N}$. Let v be a complex transversal vector to N at $p_0 = (p'_0, h(p'_0))$. If f is a decomposable CR distribution at $p_0 \in N$ then, near p_0, there exists a wedge $\mathcal{W}$ of direction v whose edge contains a neighborhood of p_0 in N as well as an $F \in \mathcal{O}(\mathcal{W})$ such that the boundary value of F is f. Furthermore, there exist $\{F_l\}_{l=1}^{n}$ with $F_l \in \mathcal{O}(\mathcal{W})$ such that $dF_1 \wedge \cdots \wedge dF_n \neq 0$ on $\mathcal{W}$ and each F_l vanishes to order 1 on N.*

The boundary value of a holomorphic function F is defined by

$$\lim_{\lambda \to 0^+} \int_{\mathfrak{N}} F(x + \lambda v)\varphi(x)\, dx,$$

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where $v \in \mathcal{W}$. It turns out that if F has slow growth in a wedge $\mathcal{W}$—that is, if there exist a constant $C > 0$ and a positive integer l such that

$$|F(z)| \leq \frac{C}{|\mathrm{dist}(z, M)|^l},$$

where $\mathrm{dist}(z, M)$ denotes the distance from a point z to M—then the boundary value of F defines a CR distribution on N (see e.g. [1]). We call this integer l the *growth degree* of F.

REMARKS ON THE SMOOTHNESS OF N. Note that N need not be smooth in order for us to define a decomposable CR distribution on N. Indeed, suppose that F is a holomorphic function of (slow) growth degree l; then one can prove, following [1, Thm. 7.2.6], the next result.

PROPOSITION 1.3. *Let F be as before and suppose that the edge of the wedge N is of regularity $l+1$. Then the boundary value of F defines a CR distribution of order $l+1$ on N.*

We thus define the growth degree for a decomposable distribution $u = \sum bvF_j$ to be the maximum of the growth degrees of the F_j. We see that if the growth degree of a decomposable distribution u is l then it makes sense to speak of a decomposable distribution on a manifold of smoothness $l+1$. Hence, in Theorem 1.2, the hypothesis of smoothness on N can be replaced by smoothness on $l+1$.

If instead of a CR distribution we wish to consider functions, then we can reduce the smoothness hypothesis in Theorem 1.2. A $\mathcal{C}^0$ function u that is decomposable near a point p is *not* the sum of $\mathcal{C}^0$ functions U_j extending holomorphically. On the other hand, for $u \in \mathcal{C}^\alpha$ ($\alpha \notin \mathbb{N}$), if u extends holomorphically at some p into a wedge $\mathcal{W}_j$ of direction w_j then the wedge $\mathcal{W}_j$ can be written as $(\mathfrak{N} \cap V_p) + i\Gamma_j$, where V_p is a neighborhood of p and Γ_j is a conical neighborhood of w_j in the normal space to N at p. The wave-front set of u (with respect to any micro-local class) at p is contained in the dual cone of Γ_j, denoted by Γ_j^0. So if $u \in \mathcal{C}^\alpha$ is decomposable at some p then, since the Γ_j^0 are pairwise disjoint, the regularity of the sum of the U_j cannot be any better than the regularity of each U_j. Therefore, if $u = \sum U_j$ is a $\mathcal{C}^\alpha$ decomposable function, then each U_j has at least the same regularity as U. This means that, if we wish to study the problem of holomorphic extension from the point of view of functions rather than distributions, we can replace the smoothness of the manifold N by $\mathcal{C}^{1+\alpha}$ and study the extension of $\mathcal{C}^{0+\alpha}$ decomposable CR functions. We obtain the following result.

THEOREM 1.4. *Let $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$ be a nongeneric $\mathcal{C}^{1+\alpha}$ CR submanifold of $\mathbb{C}^{k+m+n}$ such that the map h is decomposable at some $p_0' \in \mathfrak{N}$. Let v be a complex transversal vector to N at $p_0 = (p_0', h(p_0'))$. If f is a $\mathcal{C}^{0+\alpha}$ decomposable CR function at p_0 then, near p_0, there exists a wedge $\mathcal{W}$ of direction v whose edge contains a neighborhood of p_0 in N and also an $F \in \mathcal{O}(\mathcal{W})$ such that $F|_N = f$. Furthermore, there exist $\{F_l\}_{l=1}^n$ with $F_l \in \mathcal{O}(\mathcal{W})$ such that $dF_1 \wedge \cdots \wedge dF_n \neq 0$ on $\mathcal{W}$ and each F_l vanishes to order 1 on N.*

Theorem 1.2 also yields the following corollaries. The first one follows because any continuous CR function is decomposable on a smooth hypersurface.

COROLLARY 1.5. *Let $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$ be a smooth ($\mathcal{C}^\infty$) nongeneric CR submanifold of $\mathbb{C}^{k+1} \times \mathbb{C}^n$ such that $\mathfrak{N} \subset \mathbb{C}^{k+1}$ is a hypersurface. If f is a continuous CR function on N then, for any $p \in N$ and any v that is complex transversal to N at p, there exists a wedge $\mathcal{W}$ of direction v whose edge contains a neighborhood of p in N as well as an $F \in \mathcal{O}(\mathcal{W})$ such that the boundary value of F on N is f.*

NOTE. One can replace "continuous CR function" with "CR distribution", but the proof of the decomposability of CR distributions is harder and is not the focus of this paper (see [3] for the proof).

COROLLARY 1.6. *Let M be a $\mathcal{C}^\infty$ generic submanifold of $\mathbb{C}^L$ containing, through some $p_0 \in M$, a proper $\mathcal{C}^\infty$ CR submanifold $N = (\mathfrak{N}, h(\mathfrak{N}))$ of the same CR dimension $p_0 = (p_0', h(p_0'))$ with $p_0' \in \mathfrak{N}$. Assume that the function h is decomposable at p_0, and let $v \in T_{p_0}M \setminus [\operatorname{span}_{\mathbb{C}} T_{p_0}N]$. If f is a CR distribution on N that is decomposable at p_0, then there exists a wedge $\mathcal{W}$ in M of direction v whose edge contains a neighborhood of p_0 in N and also a $\mathcal{C}^\infty$ CR function F on $\mathcal{W}$ such that $F|_N = f$. Furthermore, there exists a collection of $\mathcal{C}^\infty$ CR functions $\{g_l\}_{l=1}^n$ vanishing to order 1 on N and such that $dg_1 \wedge \cdots \wedge dg_n \neq 0$ on $\mathcal{W}$.*

Corollary 1.6 does not hold in the abstract CR structure case, and we shall conclude this paper by constructing an abstract CR structure on which there is no CR extension. More precisely, set $L = \frac{\partial}{\partial \bar{z}} + C(z,s,t)\frac{\partial}{\partial s} + tD(z,s,t)\frac{\partial}{\partial t}$ and define $L^0 = L|_{t=0}$. The question we now address is: If $f = f(z,s)$ is such that $L^0(f) = 0$, does there exist a g such that $L(f + tg) = 0$? The answer in general is negative.

PROPOSITION 1.7. *There exist L as before and h a real analytic function such that $L^0(h) = 0$ and the equation $L(h + tg) = 0$ has no solution for $g \in \mathcal{C}^1$.*

1.2. Remarks

As pointed out in [3], this type of extension result is well known in the totally real case and is essentially due to Nagel [7]. It can be restated as follows.

THEOREM 1.8. *Let N be a nongeneric totally real smooth submanifold of $\mathbb{C}^L$, and let $v \in \mathbb{C}^L$ be complex transversal to N at p. Then, for any continuous function f on a neighborhood of p in N, there exists $\mathcal{W}_v$, a wedge of direction v whose edge contains N, such that f has a holomorphic extension to $\mathcal{W}_v$.*

This paper provides the easiest and simplest proof of this result because any continuous function on a totally real submanifold is decomposable.

We wish to point out the main differences between this paper and [3], where we obtain similar results but with entirely different techniques that yield extension results for nondecomposable CR distributions. Also, the size of the wedges obtained in [3] are much larger than that obtained here; roughly speaking, in [3] the wedges contain $(\mathfrak{N} \times \mathbb{R}^n) \cap \{t_1 > 0\}$.

As noted in [9], all CR functions are decomposable on most CR submanifolds of $\mathbb{C}^L$; hence the hypotheses of Theorem 1.2 hold in a generic sense for CR distributions. However, there are examples of CR submanifolds of $\mathbb{C}^L$ on which indecomposable CR functions exist.

Theorem 1.2 implies that the extension obtained is not unique, which differs greatly with the holomorphic extension results obtained for generic submanifolds—where the extension (if it exists) is always unique. Observe that the question of CR extension can be viewed as a Cauchy problem with Cauchy data on a characteristic set N.

1.3. Background

For a general background on CR geometry, we refer the reader to the books of Baouendi, Ebenfelt, and Rothschild [1], Boggess [2], and Jacobowitz [6].

Most of the results on holomorphic extension deal with generic submanifolds of $\mathbb{C}^n$. In a general way, these results imply a forced unique extension of CR functions under such hypotheses on the manifold M as Lewy nondegenerateness or, more generally, minimality. Under these hypotheses, one can show that it is possible to fill a wedge having edge M with analytic discs attached to M. Using the maximum principle and the fact that continuous CR functions are uniform limits of polynomials, one obtains a unique extension for continuous CR functions (see e.g. the survey paper by Trépreau [10]). The subject of decomposable CR functions has been studied by many authors, and it was believed that all such functions were decomposable. Yet in [9] Trépreau produced examples of nondecomposable CR functions (an elementary explanation of this can be found in a paper by Rosay [8]). However, one should note that any CR function is decomposable on most CR submanifolds of $\mathbb{C}^n$.

The subject of CR extension has not been studied in as much depth as the holomorphic extension has. When studying CR extension from a submanifold of lower CR dimension, the tools involving analytic discs still work (see e.g. [4; 11]), but this is not the case when the CR dimensions are equal (see [3]).

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2. Proof of the Extension Theorem

PROPOSITION 2.1. *Let $N = \{(\mathfrak{N}, 0)\}$ be a smooth nongeneric CR submanifold of $\mathbb{C}^{k+m+n}$. Let $\widetilde{\mathcal{W}}$ be a wedge in $\mathbb{C}^{k+m}$ with edge $\mathfrak{N}$ near $p_0' \in \mathfrak{N}$. Suppose F is a holomorphic function in $\widetilde{\mathcal{W}}$ that is of slow growth, and denote by f its boundary value on $\mathfrak{N}$. For any v complex transversal to N at p_0, there exists a wedge $\mathcal{W}_v$ of direction v whose edge contains $\mathfrak{N}$ such that f extends holomorphically to $\mathcal{W}_v$.*

Proof. We begin with a choice of local coordinates on $\mathfrak{N}$, which is a generic manifold in $\mathbb{C}^{k+m}$. We introduce local coordinates near p_0. We may choose a local embedding so that $p_0 = 0$ and $\mathfrak{N}$ is parameterized in $\mathbb{C}^{k+m} = \mathbb{C}^k_z \times \mathbb{C}^m_{w'}$ by

$$\mathfrak{N} = \{(z, w') \in \mathbb{C}^k \times \mathbb{C}^m : \mathrm{Im}(w') = a(z, \mathrm{Re}(w')),\ a(0) = da(0) = 0\}. \quad (2.1)$$

We set $s = \mathrm{Re}(w') \in \mathbb{R}^m$, which yields

$$\mathfrak{N} = \{(z, s + ia(z, s)\} \subset \mathbb{C}^k \times \mathbb{C}^m, \qquad T_0\mathfrak{N} = \mathbb{C}^k \times \mathbb{R}^m. \quad (2.2)$$

Define $\mathbb{C}T_p\mathfrak{N} = T_p\mathfrak{N} \otimes \mathbb{C}$ and $T_p^{0,1}\mathfrak{N} = T_p^{0,1}\mathbb{C}^{k+m} \cap \mathbb{C}T_p\mathfrak{N}$. We say that $\mathfrak{N}$ is a CR manifold if $\dim_{\mathbb{C}} T_p^{0,1}\mathfrak{N}$ does not depend on p. The CR vector fields of $\mathfrak{N}$ are vector fields L on $\mathfrak{N}$ such that, for any $p \in \mathfrak{N}$, we have $L_p \in T_p^{0,1}\mathfrak{N}$. One can choose near the origin a basis $\mathcal{L}$ of $T^{0,1}\mathfrak{N}$ consisting of vector fields L_j of the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{l=1}^{n} F_{jl} \frac{\partial}{\partial s_l}. \quad (2.3)$$

The wedge $\widetilde{\mathcal{W}}$ in $\mathbb{C}^{k+m}$ with edge $\mathfrak{N}$ on which F is defined is given in a neighborhood of the origin by

$$\widetilde{\mathcal{W}} = (\mathcal{U} + i\Gamma),$$

where $\mathcal{U}$ is a neighborhood of the origin in $\mathfrak{N}$ and Γ is a conic neighborhood of some vector μ in $\mathbb{R}^m \setminus \{0\}$. Note then that F admits (trivially) a holomorphic extension to the region $\widetilde{\mathcal{W}} \times \mathbb{C}^n \subset \mathbb{C}^{k+m+n}$. This region is much more than a wedge. Yet it clearly contains a wedge in $\mathbb{C}^{k+m+n}$ with direction u whenever u is a vector of $u = (u', u'') \in \mathbb{C}^{k+m} \times \mathbb{C}^n$ with $u' \in \widetilde{\mathcal{W}}$. By (2.1), complex transversility of $v = (v', v'')$ means that $v'' \neq 0$.

Fix a vector $u \in \widetilde{\mathcal{W}}$. Consider a $\mathbb{C}$-linear change of variables T that is the identity on $\mathbb{C}^{k+m} \times \{0\}$ and such that $T(v) = (u, v'')$. The desired extension of f to a wedge of direction v is then given by $F(T(z, w))$. We now need to show that the boundary value of $F(T(z, w))$ on $\mathfrak{N}$ is f. The boundary value of F on the wedge $\mathcal{W}$ is defined to be

$$\langle f, \varphi \rangle = \lim_{\lambda \to 0^+} \int_{\mathfrak{N}} F(x + \lambda\gamma)\varphi(x)\, dx \quad (2.4)$$

for $\varphi \in \mathcal{C}_0^\infty(\mathfrak{N})$ and $x + \lambda\gamma \in \widetilde{\mathcal{W}}$. Write $T = (T', T'') \in \mathbb{C}^{k+m} \times \mathbb{C}^n$, and let $\tau = \tau(x, \lambda, \eta)$ be defined by $T'(x + \lambda\eta) = (x + \lambda\tau(x, \lambda, \eta))$. Since T is the identity on $\mathbb{C}^{k+m} \times \{0\}$, it follows that $\lim_{\lambda \to 0^+} \lambda\tau(x, \lambda, \eta) = 0$. The boundary value of $F(T(z, w))$ on $\mathfrak{N}$ on the wedge $\mathcal{W}_v$ is then given by

$$\lim_{\lambda \to 0^+} \int_{\mathfrak{N}} F(T(x + \lambda\eta))\varphi(x)\, dx, \quad (2.5)$$

where $(x + \lambda\eta) \in \mathcal{W}_v$. We then define

$$G_\tau(\lambda) = \int_{\mathfrak{N}} F(x + \lambda\tau(x, \lambda, \eta))\varphi(x)\, dx,$$

$$F_\gamma(\lambda) = \int_{\mathfrak{N}} F(x + \lambda\gamma)\varphi(x)\, dx.$$

Now, by [1, Prop. 7.2.22, p. 189], we have

$$G_\tau(\lambda) - F_\gamma(\lambda) = O(\lambda), \quad \lambda \to 0^+.$$

Hence the boundary value defined by (2.5) is equal to the boundary value defined by (2.4). □

We immediately note that if the boundary value is at least continuous then the proof of the proposition yields the following statement.

PROPOSITION 2.2. *Let $N = \{(\mathfrak{N}, 0)\}$ be a $\mathcal{C}^1$ nongeneric CR submanifold of $\mathbb{C}^{k+m+n}$. Let $\widetilde{\mathcal{W}}$ be a wedge in $\mathbb{C}^{k+m}$ with edge $\mathfrak{N}$ near $p_0' \in \mathfrak{N}$. Suppose F is a holomorphic function in $\widetilde{\mathcal{W}}$ that has a continuous boundary value on N, and denote by f its boundary value on $\mathfrak{N}$. For any v that is complex transversal to N at p_0, there exists a wedge $\mathcal{W}_v$ of direction v whose edge contains $\mathfrak{N}$ such that f extends holomorphically to $\mathcal{W}_v$.*

Using Proposition 2.1, we obtain as follows a special case of Theorem 1.2 from which we will deduce the latter.

PROPOSITION 2.3. *Let $N = \{(\mathfrak{N}, 0)\}$ be a $\mathcal{C}^\infty$ (resp. $\mathcal{C}^{1+\alpha}$) CR submanifold of $\mathbb{C}^{k+m+n}$. Let v be a complex transversal vector to N at p_0. If f is a decomposable distribution at p_0 (resp. a $\mathcal{C}^{0+\alpha}$ decomposable function) then, near $(p_0, 0) \in N$, there exists a wedge $\mathcal{W}$ of direction v whose edge contains a neighborhood of $(p_0, 0)$ in N and also an $F \in \mathcal{O}(\mathcal{W})$ such that $bvF = f$.*

Proof. Let v be a complex transversal vector and let u be a CR distribution on $\mathfrak{N}$. By hypothesis, $u = \sum_{j=1}^K U_j$ for each U_j a boundary value of $F_j \in O(\widetilde{\mathcal{W}}_j)$, where F_j is of slow growth (or $F_j \in \mathcal{C}^{0+\alpha}(N)$) and the $\widetilde{\mathcal{W}}_j$ are wedges with edge $\mathfrak{N}$ in $\mathbb{C}^{k+m}$. We may thus apply Proposition 2.1 (or Proposition 2.2) to each U_j in order to obtain a holomorphic extension to wedges $\mathcal{W}_j'$, all in the direction v. Let $\mathcal{W} = \bigcap_{j=1}^K \mathcal{W}_j'$; we then conclude that the function $\sum_{j=1}^K U_j$ extends holomorphically to $\mathcal{W}$ and that $\sum_{j=1}^K U_j = u$ on $\mathfrak{N}$. This concludes the proof of Proposition 2.3. □

Proof of Theorem 1.2. Use (z, w', w'') to denote the coordinates in $\mathbb{C}^k_z \times \mathbb{C}^m_{w'} \times \mathbb{C}^n_{w''}$. Recall that $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$, and consider the CR map $h\colon \mathfrak{N} \to \mathbb{C}^n$. By Proposition 2.3, each h_j extends holomorphically to some wedge $\mathcal{W}_j$ in any complex transversal direction v. Set $\mathcal{W} = \bigcap_{j=1}^n \mathcal{W}_j$; then $\mathcal{W} \neq \emptyset$ since $v \in \mathcal{W}$. Define $F\colon (\mathfrak{N}, 0) \to (\mathfrak{N}, \kappa h(\mathfrak{N}))$, where $\kappa \in \mathbb{R}^*$, by

$$F(z, w', w'') = (z, w', w'' + \kappa h(z, w')).$$

Clearly there exists a $\kappa \neq 0$ and so, on $\overline{\mathcal{W}}$, the Jacobian of F is nonzero. Hence F is a biholomorphism from $\mathcal{W}$ to $F(\mathcal{W})$ that extends to a $\mathcal{C}^{1+\alpha}$ diffeomorphism from $\mathcal{W} \cup (\mathfrak{N} \times \{0\})$ to $F(\mathcal{W} \cup (\mathfrak{N} \times \{0\}))$. Observe that, since $dh(0) = 0$, it follows that F is tangent to the identity at the origin; hence there exists $\mathcal{W}'$, a wedge in $\mathbb{C}^{k+m+n}$ of direction v, such that $\mathcal{W}' \subset F(\mathcal{W})$. Thus any decomposable distribution (with respect to the $\mathcal{C}^{0+\alpha}$ function for Theorem 1.4) on N extends holomorphically to the complex transversal wedge $\mathcal{W}'$. Note then that the functions $f_j = w_j'' - h_j$ are holomorphic on a wedge $\mathcal{W}_v$ and null on N, which clearly verifies the desired conclusions. $\square$

Proof of Corollary 1.6. Let M and N be as in the hypothesis of the corollary. After a linear change of variables, we may assume that $p_0 = 0$ and that, near the origin, M is parameterized by

$$M = \{z, u + iv(z,u) : (z,u) \in \mathbb{C}^k \times \mathbb{R}^{p-k}\}.$$

By the implicit function theorem, we may assume that N is given as a subset of M by the system

$$\begin{cases} u_{p-k-n} = \mu_1(z, u_1, \dots, u_{p-k-n-1}), \dots, u_{p-k} = \mu_n(z, u_1, \dots, u_{p-k-n-1}), \\ \mu(0) = d\mu(0) = 0. \end{cases}$$

Let $s = (u_1, \dots, u_{p-k-n-1}) \in \mathbb{R}^m$ and $t = (u_{p-k-n}, \dots, u_{p-k}) \in \mathbb{R}^n$. Setting $t' = t - \mu$, in the (z,s,t') coordinates we have N given as a subset of M by $t' = 0$ and

$$N = \{z, w'(z,s), h(z,w') : (z,s) \in \mathbb{C}^k \times \mathbb{R}^m\},$$

where h is a CR map from $\mathfrak{N} := \{z, w'(z,s)\}$. We can now apply Theorem 1.2 to obtain the CR extension as the restriction of the holomorphic extension of f to $\mathcal{W} \cap M$. The second part of the corollary follows in the same manner. $\square$

3. Example with No Extension

We will now construct an example of an abstract CR structure $(M, \mathcal{V})$ in which there is no local CR extension property.

Set $L = \frac{\partial}{\partial \bar{z}} + C(z,s,t)\frac{\partial}{\partial s} + tD(z,s,t)\frac{\partial}{\partial t}$ and define $L^0 = L|_{t=0}$. Proposition 1.7 states that there exist L as just described and h a real analytic function, with $L^0(h) = 0$ and such that the equation $L(h + tg) = 0$ has no solution for $g \in \mathcal{C}^1$.

Proof of Proposition 1.7

We first construct L^0. Let $f : \mathbb{C} \to \mathbb{R}$ be a real analytic function such that there exists a $g \in \mathcal{C}_0^\infty(B_\varepsilon(0))$ (the neighborhood is taken in $\mathbb{R}^3$) where the equation

$$L^0(u) = \left[\frac{\partial}{\partial \bar{z}} - i f_{\bar{z}} \frac{\partial}{\partial s}\right](u) = g$$

is not solvable in any neighborhood of the origin in $\mathbb{R}^3$ (cf. Hörmander's theorem [5, p. 157]).

Lemma 3.1. *There exists an $\eta = \eta(z,s) \in \mathcal{C}^\omega$ such that $L^0(\eta) \neq 0$ and the equation $L^0(u) = e^\eta g$ is nowhere solvable.*

Proof. Let $\eta \in \mathcal{C}^\omega$ such that $L^0(\eta) \neq 0$. If $h \in \mathcal{C}^\omega$ is such that $L^0(h) = 0$ and if h does not vanish in some neighborhood of the origin in $\mathbb{R}^3$, then one of the following two equations is not locally solvable:

$$L^0(u) = e^\eta g;$$
$$L^0(u) = (e^\eta + h)g.$$

Indeed, if both of these equations were solvable with solutions u_1 and u_2 on some neighborhoods U_1 and U_2 of the origin, then on $U_1 \cap U_2$ we would have, by setting $u = u_2 - u_1$,

$$L^0(u) = L^0\left(h\frac{u}{h}\right) = hL^0\left(\frac{u}{h}\right) = hg.$$

Hence we would conclude that $L^0(u/h) = g$, contradicting our choice of L^0.

Now assume without loss of generality that $L^0(u) = (e^\eta + h)g$ is not locally solvable. To finish the proof of the lemma, we wish to find h such that $h \neq 0$, $L^0(h) = 0$, and $e^\eta + h \neq 0$; then we will set $e^\eta + h = e^{\tilde{\eta}}$. By a result of Cauchy–Kovalevsky we solve the equation

$$\begin{cases} L^0(v) = 0, \\ v|_{\mathrm{Re}[z]=0} = e^\eta|_{\mathrm{Re}[z]=0}. \end{cases}$$

We thus have $v = e^{\eta(0,\mathrm{Im}[z],s)} + \mathrm{Re}[z]\zeta$ and consequently $(e^\eta + v)(0,0,0) \neq 0$. Therefore, by eventually shrinking our neighborhoods, we obtain the desired function. This completes the proof of Lemma 3.1. □

We are now ready to define L from L^0. Set

$$L = L^0 + t\left(g\frac{\partial}{\partial s} + L^0(\eta)\frac{\partial}{\partial t}\right).$$

CLAIM. *The function $h = s + if(z)$ admits no CR extension to (M, L).*

Indeed, $L^0(h) = 0$. Suppose there exists a $v \in \mathcal{C}^1$ such that $L(h + tv) = 0$ has a local solution. Then we note that $L(h) = -tg$, and thus we have $L(h + tv) = -tg + tL(v) + tvL^0(\eta) = 0$. Therefore, $-g + L(v) + vL^0(\eta) = 0$. Now set $v = v_0(x,s) + tv_1(x,s,t)$; then $L(v) = L^0(v_0) + tG$. As a result, equating terms with no t and multiplying by e^η, we have

$$e^\eta(L^0(v_0) + v_0L^0(\eta)) = L^0(v_0e^\eta) = e^\eta g,$$

contradicting Lemma 3.1.

REMARKS. (a) There are plenty of nonzero CR functions on $(M, \mathcal{V})$: any holomorphic function of z is CR, and we can also find functions of z and t that are CR. Indeed, if $f = f(z,t)$ then by Cauchy–Kovalevsky one can solve $L(f) = 0$ with nonzero Cauchy data.

(b) The CR structure $(M, \mathcal{V})$ defined in this section is not realizable in $\mathbb{C}^3$, where by "realizable" we mean that there do not exist complex-valued functions Φ_1, Φ_2, Φ_3, such that $L(\Phi_j) = 0$ and $d\Phi_1 \wedge d\Phi_2 \wedge d\Phi_3 \neq 0$ in a neighborhood of the origin. If $(M, \mathcal{V})$ were realizable then any real analytic CR function on $(N, \mathcal{V}_0)$ would admit a CR extension to M, since L^0 is real analytic.

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