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On the Existence of Nontrivial 3-folds with Vanishing Hodge Cohomology

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1. Introduction

We study the structure of algebraic manifolds Y of dimension 3 with $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$. Originally this question was raised by J.-P. Serre for complex manifolds [Se]. Since by Serre duality Y is not complete, Y is affine if it is a curve [H2, p. 68]. If Y is a surface, it has been classified by Mohan Kumar [M] (see the following theorem in this section). We are interested in the 3-dimensional case. Suppose that X is a smooth completion of Y. If there are nonconstant regular functions on Y (i.e., if $h^0(Y, \mathcal{O}_Y) > 1$), then Y contains no complete curves and the boundary is connected [Zh]. We may therefore assume that the boundary is of pure codimension 1 by suitable blowing-up of subvarieties on the boundary. Let D be an effective divisor with simple normal crossings [KM, p. 5] such that $\operatorname{supp} D = X - Y$. The condition $h^0(Y, \mathcal{O}_Y) > 1$ is equivalent to $\kappa(D, X) > 0$. Here we use the standard definition of D-dimension due to Iitaka. If for all integers $m > 0$ we have $H^0(X, \mathcal{O}_X(mD)) = 0$, then we define the D-dimension of X, denoted by $\kappa(D, X)$, to be $-\infty$. If $h^0(X, \mathcal{O}_X(mD)) \geq 1$ for some m, choose a basis $\{f_0, f_1, \dots, f_n\}$ of the linear space $H^0(X, \mathcal{O}_X(mD))$; it defines a rational map Φ_{mD} from X to the projective space $\mathbb{P}^n$ by sending a point x on X to $(f_0(x), f_1(x), \dots, f_n(x))$ in $\mathbb{P}^n$. Then we define $\kappa(D, X)$ to be the maximal dimension of the images of the rational map Φ_{mD}; that is,

$$\kappa(D, X) = \max_m \{\dim(\Phi_{mD}(X))\}.$$

Let K_X be the canonical divisor of X. Then the Kodaira dimension of X is the K_X-dimension of X, denoted by $\kappa(X)$:

$$\kappa(X) = \kappa(K_X, X).$$

Before we state our theorems, we need Mohan Kumar's result for surfaces.

Theorem (Mohan Kumar). *Let Y be a smooth algebraic surface over $\mathbb{C}$ with $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$. Then Y is one of the three types described as follows.*

1. *Y is affine.*
2. *Let C be an elliptic curve and E the unique nonsplit extension of $\mathcal{O}_C$ by itself. Let $X = \mathbb{P}_C(E)$ and let D be the canonical section; then $Y = X - D$.*

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3. *Let X be a projective rational surface with an effective divisor $D = -K$, where $D^2 = 0$. Let $\mathcal{O}(D)|_D$ be nontorsion, and let the dual graph of D be $\tilde{D}_8$ or $\tilde{E}_8$. Then $Y = X - D$.*

THEOREM 1.1. *If Y is an algebraic manifold of dimension 3 with $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$ and with $h^0(Y, \mathcal{O}_Y) > 1$, then we have a surjective morphism from Y to a smooth affine curve C such that all smooth fibres are of the same type—that is, exactly one of the three types of open surfaces in Mohan Kumar's classification. Moreover, if one fibre is not affine then X has Kodaira dimension $-\infty$ and D-dimension 1.*

It is well known that the type-2 and type-3 projective surfaces are rigid. However, the rigidity of the projective surfaces does not imply the rigidity of the open surfaces. The problem is that if a surface is affine, then its smooth completion can be any projective surface; in particular, it can be type-2 or type-3 projective surface. More precisely, assume that a type-3 projective surface X_0 deforms to a projective surface X_1; then X_0 and X_1 have the same minimal model (see [I1; B+, Chap. VI, Thm. 8.1]). If S_0 and S_1 are the open surfaces in Y contained in X_0 and X_1, respectively, then $H^i(S_0, \Omega_{S_0}^j) = H^i(S_1, \Omega_{S_1}^j) = 0$ for all $j \geq 0$ and $i > 0$ [Zh]. Since both affine surfaces and type-3 open surfaces satisfy this condition, S_0 and S_1 may not be of the same type even though X_0 is a priori isomorphic to X_1. So we need to rule out the following case: some isolated fibre is affine but general fibres are not affine. We will carefully analyze how the cohomology of the sheaves $\mathcal{O}_X(nD)$ changes when restricted to each fibre in order to obtain the deformation invariant of the open surfaces.

THEOREM 1.2. *With the same assumptions as in the previous theorem, if one smooth fibre S_0 of $f|_Y$ over $t_0 \in C$ is affine then, by removing finitely many fibres $S_1, S_2, \dots, S_m$ from Y, the new 3-fold $Y' = Y - \bigcup S_i$ is affine.*

When restricted to a fibre, if the global divisor D on X is ample then Theorem 1.2 is trivial by [KMo, Prop. 1.41]. However, if an open fibre is affine, we know only that its boundary on the corresponding projective surface is the support of an ample divisor on the surface. There is no guarantee that this ample divisor on the fibre can be extended to a global divisor on X. We will use Goodman and Hartshorne's result (Lemma 3.1) to transfer the cohomology condition on the open fibre to the closed fibre in order to apply the upper semicontinuity theorem of Gravert and Grothendieck.

Let $\bar{C}$ be a smooth projective curve containing C. Let $F_n = \Omega_X^j \otimes \mathcal{O}_X(nD)$. Now we do *not* assume $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$. We want to know whether Y satisfies this condition if every fibre S satisfies it, that is, if $H^i(S, \Omega_S^j) = 0$ for all $j \geq 0$ and $i > 0$. We know that if globally Y is such a 3-fold, then each fibre must satisfy the same vanishing condition [Zh]. The converse is very subtle. Assume that each fibre and the base satisfy some property in a fibre space; then globally the property may fail. A famous example is Skoda's counterexample [Sk] for Serre's question [Se]: Is the total space of a holomorphic fibre bundle with Stein base Z and Stein fibre F a Stein manifold? In order to prove that the vanishing Hodge cohomology holds for Y, we will first prove the local freeness of the higher direct images $R^i f_* F_n$ for $n \gg 0$. This local freeness is interesting in its own right.

THEOREM 1.3. *If f is proper and surjective in the commutative diagram*

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow {\scriptstyle f|_Y} & & \downarrow {\scriptstyle f} \\ C & \hookrightarrow & \bar{C} \end{array}$$

and if each fibre X_t over $t \in C$ is a type-2 projective surface, then $R^i f_* F_n|_C$ is locally free for all $i \geq 0$ and $n \gg 0$. Therefore, $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$.

If we also assume that every horizontal divisor D_i (i.e., $f(D_i) = \bar{C}$) intersects each smooth fibre $X_t = f^{-1}(t)$ over $t \in C$ with one prime divisor on X_t, then for type-3 fibres the theorem still holds. We add this technical condition because a prime component of D might intersect some fibre with two or more curves.

THEOREM 1.4. *In the commutative diagram of Theorem 1.3, if each fibre X_t over $t \in C$ is a type-3 projective surface then $R^i f_* F_n|_C$ is locally free for all $i \geq 0$ and $n \gg 0$. Furthermore, $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$.*

COROLLARY 1.5. *If there is a surjective morphism from a smooth 3-fold Y to a smooth affine curve C such that every fibre is smooth and the diagram of Theorem 1.3 commutes, then $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$ if and only if, for every fibre S, $H^i(S, \Omega_S^j) = 0$ for all $j \geq 0$ and $i > 0$.*

One consequence of Theorems 1.1–1.4 is the following existence result.

THEOREM 1.6. *There exist nonaffine and nonproduct 3-folds Y with $H^i(Y, \Omega_Y^j) = 0$ for all $j \geq 0$ and $i > 0$.*

We shall prove these theorems in the sections that follow. The proof of Theorem 1.4 is similar to that of Theorem 1.3 and so will be omitted.

QUESTION. Are the 3-folds Y Stein in Theorem 1.3 and Theorem 1.4?

CONVENTION. Unless otherwise explicitly mentioned, we always use Zariski topology. Thus, "open set" means a Zariski open set.

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2. Proof of Theorem 1.1

THEOREM (Iitaka). *Let X be a normal projective variety and let D be an effective divisor on X. Then there exist two positive numbers α and β such that, for all sufficiently large n,*

$$\alpha n^{\kappa(D,X)} \leq h^0(X, \mathcal{O}_X(nD)) \leq \beta n^{\kappa(D,X)}.$$

For the proof of Iitaka's theorem, see Lecture 3 in [I4] or Theorem 8.1 in [U].

The following two lemmas are known (see [M]).

LEMMA 2.1. *Let S be a smooth open surface with $H^i(S, \Omega_S^j) = 0$ for all $j \geq 0$ and $i > 0$. Let $\bar{S}$ be a smooth projective surface containing S, and let G be the divisor in Mohan Kumar's theorem. Then there are three cases.*

(1) *If S is affine, then $\kappa(G, \bar{S}) = 2$ and*

$$h^0(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) \geq cn^2$$

for some positive constant c and $n \gg 0$.

(2) *If S is of type 2, then*

$$h^0(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = h^1(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = 1 \quad \textit{and} \quad h^2(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = 0$$

for all $n \gg 0$.

(3) *If S is of type 3, then*

$$h^0(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = 1 \quad \textit{and} \quad h^1(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = h^2(\bar{S}, \mathcal{O}_{\bar{S}}(nG)) = 0$$

for all $n \gg 0$.

Proof. (1) Since S is affine, it follows from Goodman's theorem (see [H2, p. 69]) that $\bar{S} - S$ is the support of an ample divisor A. Therefore, $\kappa(A, \bar{S}) = \kappa(G, \bar{S}) = 2$ [I3; B+, Chap 14]. The estimate is obvious by Iitaka's theorem.

(2) The equalities follow from [M, Lemma 1.8] and Lemma 2.2(1) to follow.

(3) See [M, Lemma 1.8, Lemma 3.1] and Lemma 2.2(2). □

LEMMA 2.2. *With the preceding notation, we have:*

(1) *if $\bar{S}$ is of type 2, then $G^2 = 0$, $K_{\bar{S}} = -2G$, $p_g = 0$, and $q = 1$;*

(2) *if $\bar{S}$ is of type 3, then $G^2 = 0$, $K_{\bar{S}} = -G$, and $p_g = q = 0$.*

Proof. (1) This is a standard result for the ruled surface over an elliptic curve. The proof can be found in [H3, Chap. V, Sec. 2] or in [M].

(2) See Lemma 1.6 and Lemma 3.1 in [M]. □

Let $f : X \to Z$ be a morphism between varieties (schemes) with Z connected. Let $z_0 \in Z$, $k(z_0) = K$, and $X_{z_0} \cong X_0$. Then the other fibres X_z of f are called *deformations* of X_0 [H3, p. 89]. In the proof of Theorem 1.1, the deformation of a nonsingular complex surface X_0 means the following by the same notation: Both

X and Z are smooth, and f is a surjective, proper, and flat morphism (i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Z,f(x)}$-module for all $x \in X$) such that the fibre over $z_0 \in Z$ is $X_{z_0} \cong X_0$ [B+, p. 36]. By [I1] we know that the deformation of a rational surface is again rational. By [B+, Chap. VI, Thm. 8.1], the deformation of a ruled surface over a smooth curve of genus $g \geq 1$ is also of the same type—that is, it has the same minimal model.

We need Kodaira's stability of (-1)-curves, which is Theorem 5 in [Ko].

Theorem (Kodaira). *Let $f: X \to Z$ be a surjective and proper holomorphic map that is flat. If for some point $0 \in Z$ the fibre X_0 contains a (-1)-curve E_0, then there exist an open neighborhood (in complex topology) U of 0 in Z as well as a closed and connected submanifold E of $f^{-1}(U)$ such that $E \cap X_0 = E_0$ and $E \cap X_t = E_t$ is a (-1)-curve for every $t \in U$.*

Moreover, in Kodaira's theorem there is a $g: X' \to U$, which is a surjective, flat, and proper holomorphic map such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\approx} & U; \end{array}$$

here $h|_{X_t}: X_t \to X'_t$ is the blowing-down of E_t. Let us state the contraction part precisely. The proof is due to Suwa (see [I2, Apx. 1]).

Theorem (Suwa). *Let X and Z be complex manifolds, and let f be a proper, surjective, and flat holomorphic map from X to Z such that every fibre X_z is a smooth surface. If there exists a complex submanifold E of X whose restriction to X_z, $E_z = E \cap X_z$, is an irreducible exceptional curve of the first kind on X_z at any $z \in Z$, then we can construct a complex manifold X' (which is proper over Z) and a holomorphic map $h: X \to X'$ over Z such that $h|_{X_z}: X_z \to X'_z$ shrinks E_z to a point in X'_z for every point $z \in Z$ and such that $h|_{X-E}: X - E \to X' - h(E)$ is biholomorphic.*

Upper Semicontinuity Theorem (Grauert & Grothendieck). *Let $f: X \to Z$ be a proper morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X that is flat over Z.*

(1) *The i*th *direct image $R^i f_* \mathcal{F}$ is a coherent sheaf on Z for any nonnegative integer i.*
(2) *Let $\mathcal{F}_z = \mathcal{F}|_{X_z}$ (the sheaf $\mathcal{F}$ restricted to the fibre $X_z = f^{-1}(z)$); then the function*

$$d_i(z) = h^i(X_z, \mathcal{F}_z) = \dim_{k(z)} H^i(X_z, \mathcal{F}_z)$$

is upper semicontinuous on z. That is: for any $n \in \mathbb{Z}$, the set $\{z \in Z : d_i(z) \geq n\}$ is a closed set, where $k(z) = \mathcal{O}_z/\mathcal{M}_z$, the residue field at the point z.

(3) *The Euler characteristic of the restriction sheaf* $\mathcal{F}_z$,

$$\chi(\mathcal{F}_z) = \sum (-1)^i \dim_{k(z)} H^i(X_z, \mathcal{F}_z),$$

is locally constant on Z.

(4) *The following statements are equivalent*:

(i) $h^i(X_z, \mathcal{F}_z)$ *is a constant function on* Z;

(ii) $R^i f_* \mathcal{F}$ *is locally free sheaf on* Z *and, for all* $z \in Z$, *the natural map*

$$R^i f_* \mathcal{F} \otimes_{\mathcal{O}_z} k(z) \to H^i(X_z, \mathcal{F}_z)$$

is an isomorphism.

In addition, if conditions (i) *and* (ii) *are satisfied, then*

$$R^{i-1} f_* \mathcal{F} \otimes_{\mathcal{O}_z} k(z) \to H^{i-1}(X_z, \mathcal{F}_z)$$

is an isomorphism for all $z \in Z$.

For a proof, see [Mu, pp. 46–53].

In this section, from now on we assume that the condition of Theorem 1.1 holds. Theorem 1.1 is a direct consequence of the following lemmas.

LEMMA 2.3. *If one smooth fibre* $S_{t_0} = S_0$ *is a type-2 or a type-3 open surface in Mohan Kumar's classification, then there is an affine open set* U *such that* $S_t = f^{-1}(t) - D$ *over every* $t \in U$ *is of the same type.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow{\scriptstyle f|_Y} & & \downarrow{\scriptstyle f} \\ C & \hookrightarrow & \bar{C} \end{array}$$

[Zh], where f is proper and surjective and where $X_t = f^{-1}(t)$ is a smooth projective surface for all $t \in C$. The minimal model of X_t is the same as the minimal model of a type-2 or type-3 surface, but it may contain exceptional curves of the first kind.

Note that S_0 is not affine. Let $X_0 = f^{-1}(t_0)$. By Lemma 2.1 and [U, Lemma 5.3], $H^0(X_0, \mathcal{O}(nD_0)) = \mathbb{C}$ for all nonnegative integers n even though the divisor $D_0 = D|_{X_0}$ contains exceptional curves of the first kind. Let $D_t = D|_{X_t}$; then D_t is a connected curve on $X_t = f^{-1}(t)$, since X_t is smooth and since $H^i(S_t, \Omega^j_{S_t}) = 0$ [M, Lemma 1.4]. By upper semicontinuity, there is an affine open set U in C such that $H^0(X_t, \mathcal{O}(nD_t)) = \mathbb{C}$, since every D_t is effective. Hence, by Lemma 2.1, every fibre S_t over $t \in U$ is not affine.

Second, if S_0 is a type-2 open surface in Mohan Kumar's classification, then $p_g(X_0) = h^2(\mathcal{O}_{X_0}) = 0$ and $q(X_0) = h^1(\mathcal{O}_{X_0}) = 1$ (Lemma 2.2). Here the boundary divisor $D_0 = D|_{X_0}$ may contain exceptional curves and so X_0 may not be minimal. But p_g is birational invariant [B+, p. 107] and q is bimeromorphic invariant [H3, p. 181]. Since $R^i f_*(\mathcal{O}_X)$ and $R^i f_*(\mathcal{O}_X(K_X))$ are locally free for all $i \geq 0$ [K1; K2], again by upper semicontinuity it follows that $p_g(X_t) = 0$ and

$q(X_t) = 1$ for every $t \in C$. Now X_0 has the minimal model of a ruled surface over an elliptic curve by the classification theorem [B+, Chap. VI, Thm. 1.1, p. 243] and the deformation theorem [B+, Chap. VI, Thm. 8.1, p. 263]. Hence there is an affine open set U such that, for every $t \in U$, X_t has the same minimal model as X_0 in Mohan Kumar's theorem—that is, a type-2 projective surface.

Similarly, if S_0 is a type-3 open surface then there is an affine open set U such that S_t over every $t \in U$ is of the same type, since the deformation of a rational surface is still rational [I1]. □

REMARK 2.4. If S is a type-2 open surface in Mohan Kumar's theorem, then any point on S cannot be contained in any exceptional curve of $\bar{S}$, where $\bar{S}$ is a smooth completion of S. Hence, if $\bar{S}$ is not minimal then all exceptional curves are contained in the boundary $\bar{S} - S$.

LEMMA 2.5. *If there is an affine open set U in C such that, for every $t \in U$, $S_t = f^{-1}(t) - D$ is a type-2 open surface and $t \neq t_0$, where t_0 is a fixed point of U, then S_0 must be a surface of the same type.*

Proof. First, S_0 cannot be of type 3 because $X_t = f^{-1}(t)$ is not rational and the deformation of a rational surface is still rational [I1]. We know that there are three possible smooth fibres [M; Zh], so we need only prove that S_0 is not affine. For this it suffices to prove that $h^0(X_0, \mathcal{O}_{X_0}(nD_0))$ is bounded for all n; in fact, in our case it is 1. Here $X_0 = f^{-1}(t_0)$, $D_0 = D|_{X_0}$, and $S_0 = X_0 - D_0$.

By Suwa's theorem and Kodaira's stability theorem of (-1)-curves, we may assume that D_0 has no exceptional curve of the first kind. Hence there is a small open set V in C (complex topology) such that, for all points $t \in V$, $D_t = D|_{X_t}$ has no exceptional curves of the first kind. In fact, if there is a $t_1 \in V$ and if D_{t_1} has a component E_1 such that E_1 is an exceptional curve of the first kind, then $E_1^2 = E_1 \cdot K_{X_1} = -1$, where $X_1 = f^{-1}(t_1)$. There is a prime component G of D in X such that $E_1 \subset G$. Let $E_t = G|_{X_t}$ for $t \in V$; then, by upper semicontinuity, the Euler characteristic of $\mathcal{O}_{X_t}(nE_t)$ is constant for every $t \in V$ and every $n \geq 0$. As a result, for any $n \geq 0$,

$$\chi(\mathcal{O}_{X_t}(nE_t)) = \chi(\mathcal{O}_{X_0}(nE_0)).$$

By the Riemann–Roch formula, for all $n \geq 0$ we have

$$\tfrac{1}{2}n^2E_t^2 - \tfrac{1}{2}nE_t \cdot K_{X_t} = \tfrac{1}{2}n^2E_1^2 - \tfrac{1}{2}nE_1 \cdot K_{X_1}.$$

Therefore, $E_t^2 = E_t \cdot K_{X_t} = -1$ for all $t \in V$; in particular, $E_0^2 = E_0 \cdot K_0 = -1$. This is impossible because D_0 has no (-1)-curves by our assumption. Thus, for all $t \in V$ and $t \neq t_0$, X_t is a type-2 surface (i.e., a minimal ruled surface over an elliptic curve). But D_t may not be a prime divisor. Let D_t' be the elliptic curve (a section) as in Mohan Kumar's classification. Then there is a positive integer $n(t)$ depending on t and such that $D_t = n(t)D_t'$. Since the function $n(t)$ is discrete, there is a dense subset B in V such that $n(t)$ is a constant c for all $t \in B$. Let $t_1 \in V - B$ and $K_1 = K_{X_1}$. Consider the divisor $D + cK_X$ restricted to the corresponding fibre $X_1 = f^{-1}(t_1)$. By upper semicontinuity we have

$$h^0(X_{t_1}, \mathcal{O}_{X_{t_1}}(D_{t_1} + 2cK_1)) \geq h^0(X_t, \mathcal{O}_{X_t}(D_t + 2cK_t)) = 1$$

and

$$h^0(X_{t_1}, \mathcal{O}_{X_{t_1}}(-D_{t_1} - 2cK_1)) \geq h^0(X_t, \mathcal{O}_{X_t}(-D_t - 2cK_t)) = 1,$$

where $t \in B$ and $D_t + 2cK_t = cD'_t + 2cK_t = c(D'_t + 2K_t) = 0$ (Lemma 2.2). So $\mathcal{O}_{X_{t_1}}(D_{t_1} + 2cK_1)$ must be trivial; that is,

$$D_{t_1} + 2cK_1 = n(t_1)D'_1 + 2cK_1 = -2n(t_1)K_1 + 2cK_1 = 0.$$

Hence $n(t_1) = c$ for every $t_1 \in V - B$ and so $D_t = cD'_t$ for every $t \in V$. By changing coefficients locally, we may assume that $D|_{X_t} = D'_t$, where $2D'_t + K_t = 0$.

Since $2D_t + K_{X_t} = 0$ for every $t \neq t_0$, we may similarly consider the divisor $2D + K_X$ restricted to every fibre X_t and obtain

$$h^0(X_0, \mathcal{O}_{X_0}(2D_0 + K_0)) \geq h^0(X_t, \mathcal{O}_{X_t}(2D_t + K_{X_t})) = 1.$$

On the other hand,

$$h^0(X_0, \mathcal{O}_{X_0}(-2D_0 - K_0)) \geq h^0(X_t, \mathcal{O}_{X_t}(-2D_t - K_{X_t})) = 1.$$

We therefore have

$$h^0(X_0, \mathcal{O}_{X_0}(2D_0 + K_0)) = h^0(X_0, \mathcal{O}_{X_0}(-2D_0 - K_0)) = 1.$$

Again this implies that the sheaf $\mathcal{O}_{X_0}(2D_0 + K_0))$ is trivial. Hence $2D_0 + K_0 = 0$. Since S_t has vanishing Hodge cohomology and since X_0 is isomorphic to X_t, it follows that

$$\begin{aligned} h^0(X_0, \mathcal{O}_{X_0}(2nD_0)) &= h^0(X_0, \mathcal{O}_{X_0}(-nK_0)) \\ &= h^0(X_t, \mathcal{O}_{X_t}(-nK_t)) = h^0(X_t, \mathcal{O}_{X_t}(2nD'_t)) = 1. \end{aligned}$$

Consequently, S_0 is not affine. □

REMARK 2.6. Let U be covered by a set of small open discs U_i. By the foregoing argument, for each i there is a constant c_i such that $D_t = c_i D'_t$ for $t \in U_i$, where D'_i is the irreducible boundary elliptic curve on X_t. Because U is connected, all these c'_i are equal. That is, there exists a constant c such that $D|_{X_t} = D_t = cD'_t$ for all $t \in U$. Thus, by changing the coefficients of D, we have proved that the new boundary divisor D' on X satisfies $D'|_{X_t} = D'_t$.

LEMMA 2.7. *If there is an affine open set U in C such that, for every $t \in U$, S_t is a type-3 open surface and $t \neq t_0$, where t_0 is a fixed point of U, then S_0 must be of the same type.*

Proof. First, S_0 is not a type-2 open surface because X_0 is rational by Iitaka's theorem [I1]. As in Lemma 2.5, we need only prove that S_0 is not affine. For this it suffices to prove that $h^0(X_0, \mathcal{O}_{X_0}(nD_0)) < cn^2$ for all positive numbers c (Lemma 2.1).

As in Lemma 2.5, we may assume that D_t contains no exceptional curves of the first kind for every t in U. In fact, if there is an exceptional curve E_1 of the first kind in D_{t_1} for some point t_1 in U then, locally analytically, E_1 sits in an irreducible nonsingular divisor D_1 of X; that is, D_1 is a prime component of D. (We may assume that D is an effective divisor on X with simple normal crossings [Zh].) Now f is proper on D_1 and D_1 is a manifold, so we can apply Kodaira's extension theorem locally on D_1 near D_{t_1}. More precisely, in our case we can compute it directly. Since D_1 is smooth it follows that, for a small number $\varepsilon > 0$, in a neighborhood $V = \{t \in C, |t - t_1| < \varepsilon\}$ of t_1, D_1 intersects every fibre X_t with a prime divisor on X_t. Since $h^0(\mathcal{O}_{X_t}) = 1$ and $h^1(\mathcal{O}_{X_t}) = h^2(\mathcal{O}_{X_t}) = 0$ by Lemmas 2.1 and 2.2, respectively, the Riemann–Roch formula and upper semicontinuity yield

$$\chi(\mathcal{O}_{X_t}(nE_t)) = 1 + \tfrac{1}{2}n^2E_t^2 - \tfrac{1}{2}nE_t \cdot K_{X_t} = 1 + \tfrac{1}{2}n^2E_1^2 - \tfrac{1}{2}nE_1 \cdot K_{X_1},$$

where $E_1 = D_1|_{X_t}$. So E_t is again an (-1)-curve on D_1. This implies that all the extended (-1) exceptional curves near D_{t_1} sit in D_1 and do not meet Y. Thus, after contraction, Y remains the same; that is, when contracting (-1)-curves, we change only the boundary D_t while all the open surfaces S_t over $t \in U$ remain unchanged.

If D_t is the special divisor D_t' as in Mohan Kumar's theorem (i.e., if its dual graph is either $\tilde{D}_8$ or $\tilde{E}_8$), then by Lemma 2.2 we have $D_t + K_{X_t} = 0$ for every $t \in U$ and $t \neq t_0$. By inequalities similar to those in the proof of Lemma 2.5,

$$h^0(X_0, \mathcal{O}_{X_0}(D_0 + K_0)) = h^0(X_0, \mathcal{O}_{X_0}(-D_0 - K_0)) = 1.$$

Hence $D_0 + K_0 = 0$. Since X_0 is a type-3 projective surface and since also $H^i(S_0, \Omega^j_{S_0}) = 0$, we know that S_0 is not affine and must be a type-3 open surface. But we cannot guarantee that the dual graph of D_t is either $\tilde{D}_8$ or $\tilde{E}_8$. We know only that D_t has nine components and that every prime component is isomorphic to $\mathbb{P}^1$ with self-intersection -2 [M]. In Lemma 2.5 we could assume that the special divisor D_t' on X_t is the restriction of a global divisor D on X_t, because D_t' has only one component by Remark 2.6. Here the situation is more delicate.

Let D_t' be the special divisor of a type-3 projective surface as before (i.e., its dual graph is either $\tilde{D}_8$ or $\tilde{E}_8$), let $D_t' \cdot D_t' = D_t' \cdot K_t = 0$, and let $\mathcal{O}_{D_t'}(D_t')$ be nontorsion [M]. For any nonnegative integer n, there is an m such that $mD_t' - nD_t$ is effective. For example, we may choose $m = an$, where a is the maximum coefficient of D_t's components. Hence

$$0 < h^0(X_t, \mathcal{O}_{X_t}(nD_t)) \leq h^0(X_t, \mathcal{O}_{X_t}(mD_t')) = 1.$$

Therefore, $h^0(X_t, \mathcal{O}_{X_t}(nD_t)) = 1$. By Serre duality, $H^2(X_t, \mathcal{O}_{X_t}(nD_t)) = 0$ for all $n \gg 0$, since K_{X_t} and D_t have the same support by Lemma 2.2. We now consider $h^1(X_t, \mathcal{O}_{X_t}(nD_t))$, for which there are three cases [Za].

Case 1. $h^1(X_t, \mathcal{O}_{X_t}(nD_t))$ is bounded. In other words, there is a positive integer k such that, for all $n \geq 0$,

$$h^1(X_t, \mathcal{O}_{X_t}(nD_t)) \leq k < \infty.$$

By Zariski's theorem [Za, p. 611], D_t is arithmetically effective. By the Riemann–Roch formula and Lemmas 2.1 and 2.2, we have

$$h^1(X_t, \mathcal{O}_{X_t}(nD_t)) = -\tfrac{1}{2}n^2D_t^2 + \tfrac{1}{2}nD_t \cdot K_{X_t}.$$

This equality yields $D_t^2 = D_t \cdot K_t = 0$, since $h^1(X_t, \mathcal{O}_{X_t}(nD_t))$ is bounded for all n. Then, for every prime component E in D_t, we have $E \cdot D_t = 0$ because D_t is arithmetically effective. By [M, Lemma 1.7] we know that $D_t = n(t)D'_t$, where the positive integer $n(t)$ depends on the point t in U. Hence, for every $n \geq 0$, by Lemma 2.1 we have

$$h^1(X_t, \mathcal{O}_{X_t}(mD_t)) = h^1(X_t, \mathcal{O}_{X_t}(mn(t)D'_t)) = 0.$$

Now the Euler characteristic of $\mathcal{O}_{X_0}(nD_0)$ is

$$\chi(\mathcal{O}_{X_0}(nD_0)) = 1 - \tfrac{1}{2}n^2 D_0^2 + \tfrac{1}{2}nD_0 \cdot K_{X_0} = 1.$$

Thus again $D_0^2 = D_0 \cdot K_{X_0} = 0$. By the same argument as in the proof of Lemma 2.5 and Remark 2.6, there is a positive integer c such that $D_t = cD'_t$ for every $t \in U$ when $t \neq t_0$. Considering the divisor $D + cK_X$ on X when restricted to X_0, we have

$$h^0(X_0, \mathcal{O}_{X_0}(D_0 + cK_{X_0})) \geq 1, \qquad h^0(X_0, \mathcal{O}_{X_0}(-D_0 - cK_{X_0})) \geq 1.$$

Therefore, $D_0 = -cK_{X_0}$. Let $D_0 = P + N$ be the Zariski decomposition of D_0; then P is nef, N is negative definite (both are effective), and every component of N does not intersect P. Let E be a prime component of P. Locally analytically, E is contained in a prime divisor G of X. Let $G|_{X_t} = E_t$. Applying upper semicontinuity and the Riemann–Roch formula to $\mathcal{O}_{X_0}(nE)$ and $\mathcal{O}_{X_t}(nE_t)$ yields $E \cdot K_0 = E_t \cdot K_t = 0$ [M, Lemma 3.1]. Thus $E \cdot D_0 = E \cdot (-cK_0) = 0$. If $E \cdot P > 0$, then $E \cdot P = E \cdot (D_0 - N) = -E \cdot N > 0$; hence $E \cdot N < 0$. This means that E must be a component of N, which is a contradiction because no component of N intersects P. So $P^2 = 0$. By [Bă, Cor. 14.18], $\kappa(D_0, X_0) \leq 1$. By Lemma 2.1, S_0 is not affine.

Case 2. If $h^1(X_t, \mathcal{O}_{X_t}(nD_t))$ is as large as cn for some positive number c, then by Zariski's theorem [Za, p. 611] it follows that D_t is arithmetically effective and that the intersection form of D_t is negative definite. This contradicts Lemma 1.6 in [M], so this case cannot occur.

Case 3. If $h^1(X_t, \mathcal{O}_{X_t}(nD_t))$ is as large as kn^2 for some positive number k, then the Riemann–Roch formula yields $D_t^2 < 0$. Let $D_t = A + B$ be the Zariski decomposition such that A is arithmetically effective, $B \geq 0$ is negative definite, and every prime component of B does not meet A. Then there is a positive integer n_0 such that $n_0 A$ and $n_0 B$ are integral. Without loss of generality, we may assume that A and B are integral. Because there is a positive integer l such that $lD'_t - D_t$ is effective, we have the exact sequence

$$0 \to \mathcal{O}(nD_t) \to \mathcal{O}(nlD'_t) \to Q \to 0,$$

where Q is the cokernel. Hence we still have $h^0(X_t, \mathcal{O}_{X_t}(nD_t)) = 1$ even though D_t is different from D'_t. Therefore, $\kappa(D_t, X_t) = 0$ by Iitaka's theorem. This implies that $A^2 = 0$ (see [Za] or [Bă, Cor. 14.18]). Since A is arithmetically effective and since $\operatorname{supp} D_t = \operatorname{supp} A \cup \operatorname{supp} B$, for every prime component E of D_t it follows that $E \cdot A = 0$. By [M, Cor. 1.7] there exists a positive integer m_0 such that

$A = m_0 D'_t$, so $D_t^2 = B^2$ and $D_t - D'_t \geq 0$. Let $D_{0,i}$ be a prime component of $D_0 = D|_{X_0}$. Choose a small neighborhood V of t_0 such that, locally analytically in V, $D_{0,i}$ lies in a unique prime divisor D_i of $f^{-1}(V)$. Then D_i cuts every fibre X_t $(t \in V)$ with an irreducible (-2)-curve. As a result, over V there is a one-to-one correspondence between the prime divisor of D_t and the prime divisor of $f^{-1}(V)$. We may rearrange the coefficients of D_i locally (as in the proof of Lemma 2.5) such that $D_t = cD'_t$, where $t_0 \neq t \in V$; hence $h^1(X_t, \mathcal{O}_{X_t}(nD_t)) = 0$ for all such t. Then we reduce Case 3 to Case 1, proving that S_0 is not affine. □

Remark 2.8. If D_t is not the special divisor as in Mohan Kumar's theorem—that is, if D_t has different coefficients from D'_t but they have the same support—then we still have $D_t \cdot K_t = 0$ by Lemma 3.1 in [M]. Since $h^0(X_t, \mathcal{O}_{X_t}(nB)) = 1$, the Riemann–Roch formula yields $h^1(X_t, \mathcal{O}_{X_t}(nD_t)) = -\frac{1}{2}n^2 D_t^2 = -\frac{1}{2}n^2 B^2 = h^1(X_t, \mathcal{O}_{X_t}(nB)) \sim cn^2$. Thus $B^2 < 0$. Since B is negative definite, by [M, Lemma 1.6] we know that the support of B is strictly smaller than the support of D_t.

Remark 2.9. Let X_t be a type-3 projective surface and let D'_t be the special divisor as before. Let E be any prime component of D'_t. Then $E^2 = -2$ [M]. Since the canonical divisor $K_t = -D'_t$, by Riemann–Roch we have

$$h^1(X_t, \mathcal{O}_{X_t}(nD'_t + E)) = n^2.$$

This together with the argument of Remark 2.8 means that, for any divisor D_t with the same support as D'_t, either $h^1(X_t, \mathcal{O}_{X_t}(nD_t)) = 0$ or $h^1(X_t, \mathcal{O}_{X_t}(nD_t)) \sim cn^2$ for some positive integer c.

Lemma 2.10. *If S_0 is affine then there is an affine open set U in C such that S_t is affine for every $t \in U$.*

Proof. This is a direct implication of the previous lemmas, since S_0 can only be one of the three types of surfaces. □

Lemma 2.11. *If there is an affine open set U in C such that S_t is affine for every $t \neq t_0$, then S_0 is affine.*

Proof. This is an immediate consequence of Mohan Kumar's classification and the upper semicontinuity theorem. □

The first half of Theorem 1.1 is entailed by the lemmas in this sectio; the second half follows from Theorems 5.11 and 6.12 in [U]. In fact, the foregoing lemmas show that if one smooth fibre X_0 is not affine then all smooth fibres are not affine. Since X_0 is a ruled surface, $\kappa(X_0) = -\infty$; hence

$$\kappa(X) \leq \kappa(X_0) + 1 = -\infty.$$

By Lemma 2.1,

$$0 < \kappa(D, X) \leq \kappa(D_t, X_t) + 1 = 1.$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

LEMMA 3.1 (Goodman & Hartshorne). *Let V be a scheme and D an effective Cartier divisor on V. Let $U = V - \operatorname{Supp} D$ and let F be any coherent sheaf on V. Then, for every $i \geq 0$,*

$$\varinjlim_n H^i(V, F \otimes \mathcal{O}(nD)) \cong H^i(U, F|_U).$$

This lemma enables us to transfer the cohomology information from Y to its completion X.

Proof of Theorem 1.2. The idea is to prove, for any coherent sheaf F on Y', that $H^i(Y', F_{Y'}) = 0$ for all $i > 0$. Because the dimension of Y' is 3, we need only consider $i = 1, 2, 3$. We use the technique in [Zh] with some modification, presenting all the details for completeness.

Notice that $Y' \subset Y$. Let $F_{Y'}$ be any coherent sheaf on Y'; then it can be extended to a coherent sheaf F_X on X, and $F_Y|_{S_t}$ and $F_X|_{X_t}$ are coherent [H3, pp. 115, 126]. We will not distinguish between them and will simply write F. Since a general fibre X_t over $t \in C$ is smooth and irreducible [Zh] and since, for any F, there is an open set U in C such that $R^i f_* F$ is locally free on U, we may assume that $R^i f_* F$ is locally free on C and that every fibre over C is smooth and irreducible.

Step 1: Proof of $H^3(Y, F) = 0$. Since S_t is affine for every t in C, it follows that $H^i(S_t, F|_{S_t}) = 0$ for every $i > 0$. Let $F_n = F \otimes \mathcal{O}_X(nD)$ and $F_{n,t} = F \otimes \mathcal{O}_X(nD)|_{X_t}$. By Goodman and Hartshorne's lemma,

$$\varinjlim_n H^i(X_t, F_{n,t}) = 0$$

for all $i > 0$ and $t \in C$. Since each fibre has dimension 2, we have $H^3(X_t, F_{n,t}) = 0$ for all $n \geq 0$ and $t \in C$. By upper semicontinuity, $R^3 f_* F_n = 0$ for all n. Again by Goodman and Hartshorne's lemma,

$$H^3(Y, F) = \varinjlim_n H^3(f^{-1}(C), F_n) = \varinjlim_n R^3 f_* F_n(C) = 0.$$

Step 2: Proof of $H^2(Y, F) = 0$. It suffices to prove the claim for locally free sheaves. In fact, suppose $H^2(Y, L) = 0$ for any locally free sheaf L on X. For any coherent sheaf F on X, there is a locally free sheaf L on X such that we have the surjective map $L \to F$. Let K be the kernel; then we have the following short exact sequence on Y:

$$0 \to K \to L \to F \to 0.$$

By Step 1 we know that $H^3(Y, K) = 0$, since K is also coherent [H3]. Hence $H^2(Y, L) = 0$ implies $H^2(Y, F) = 0$, so we may assume that F is a locally free sheaf on X.

Let $t \in C$. Given the exact sequence

$$0 \to \mathcal{O}_X(nD) \to \mathcal{O}_X((n+1)D) \to \mathcal{O}_D((n+1)D) \to 0,$$

tensoring with F and then with $\mathcal{O}_{X_t}$ yields

$$0 \to F_{n,t} \to F_{n+1,t} \to F_{n+1,t}|_{D_t} \to 0.$$

Since D_t is a curve, it follows that $H^2(X_t, F_{n+1,t}|_{D_t}) = 0$ for all $n \geq 0$ and $t \in C$. Therefore, the map $H^2(X_t, F_{n,t}) \to H^2(X_t, F_{n+1,t})$ is surjective. Since $S_t = X_t - D_t$ is affine, by Goodman and Hartshorne's lemma we obtain

$$\varinjlim_n H^2(X_t, F_{n,t}) = H^2(S_t, F) = 0.$$

Thus for any $t \in C$ there exists a positive integer $n(t)$ depending on t and such that, for every $n \geq n(t)$, we have $H^2(X_t, F_{n,t}) = 0$.

Given any n, there is an affine open set U_n of C such that $R^2 f_* F_n$ is locally free on U_n. By the same argument as in the next paragraph, the intersection of these infinitely many open sets is not empty. Now fix some t_0 in $\bigcap U_n$ such that $H^2(X_{t_0}, F_{n,t_0}) = 0$ for every $n \geq n(t_0)$, and suppose there is an open neighborhood U_0 of t_0 in $\bar{C}$ such that $R^2 f_* F_{n(t_0)}$ is locally free on U_0. Then $H^2(X_t, F_{n(t_0),t}) = 0$ for every t in U_0, so $H^2(X_t, F_{n,t}) = 0$ for every t in U_0 and every $n \geq n(t_0)$. Let $C - U_0 = \{t_1, t_2, \ldots, t_k\}$ and choose $n_0 = \max(n(t_0), n(t_1), \ldots, n(t_k))$; then $H^2(X_t, F_{n,t}) = 0$ for every $t \in C$ and every $n \geq n_0$. By the upper semicontinuity theorem, $(R^2 f_* F_n)_t / \mathcal{P}(R^2 f_* F_n)_t = 0$ for all points t in C. By Nakayama's lemma, $R^2 f_* F_n|_C = 0$. Finally, by Goodman and Hartshorne's lemma,

$$H^2(Y, F) = \varinjlim_n H^2(f^{-1}(C), F_n) = \varinjlim_n R^2 f_* F_n(C) = 0.$$

Step 3: Proof of $H^1(Y', F) = 0$. Here Y' is an open subset of Y obtained by removing finitely many fibres from Y.

Let F_n be as described in Step 1. For any fixed n, there is an open set U_n in $\bar{C}$ such that $R^1 f_* F_n$ is locally free on U_n. Let $U_n = \bar{C} \setminus A_n$, where A_n is closed in $\bar{C}$; that is, U_n consists of only finitely many points of $\bar{C}$. Since any complete metric space is a Baire space [Bo2, Chap. 9], it follows that $B = \bar{C} \setminus \bigcup A_n = \bigcap U_n$ is a dense subset of $\bar{C}$ in complex topology. Hence, for every point t in B, all stalks $(R^1 f_* F_n)_t$ are locally free. Write B as a union of connected subsets B_m: $B = \bigcup B_m$. Then there is one B_m such that B_m is dense in $\bar{C}$ and connected in complex topology, so we may assume that B is connected. Again by the upper semicontinuity theorem, for every point t in C and every $n \geq n_0$ we have

$$(R^1 f_* F_n)_t \otimes \mathbb{C} \cong H^1(X_t, F_{n,t}),$$

since $R^2 f_* F_n|_C = 0$. For any m we know that $h^1(X_t, F_{m,t})$ is constant on B because $R^1 f_* F_m$ is locally free at every point t in B and B is connected. Thus, for $n \geq n_0$ and for all points t in B, there is an l such that the map

$$H^1(X_t, F_{n,t}) \to H^1(X_t, F_{n+l,t})$$

is zero. Moreover, for every point t in C and sufficiently large n, we have the following commutative diagram:

$$\begin{array}{ccc} R^1 f_* F_n \otimes \mathbb{C}(t) & \xrightarrow{\approx} & H^1(X_t, F_{n,t}) \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ R^1 f_* F_{n+l} \otimes \mathbb{C}(t) & \xrightarrow{\approx} & H^1(X_t, F_{n+l,t}). \end{array}$$

The map β is zero for every $t \in B$, so the map

$$\alpha\colon (R^1 f_* F_n)_t / \mathcal{P}(R^1 f_* F_n)_t \to (R^1 f_* F_{n+l})_t / \mathcal{P}(R^1 f_* F_{n+1})_t$$

is zero for all points t in B. By the local freeness this means that, for every point t in B, the stalks satisfy

$$\varinjlim_n (R^1 f_* F_n)_t = 0.$$

To see this, fix a point t_0 in B; for any sufficiently large n and for the l just described, choose an affine open set V containing t_0 such that both $R^1 f_* F_n$ and $R^1 f_* F_{n+l}$ are locally free on V. Hence there are two positive integers, m_1 and m_2, such that $R^1 f_* F_n(V) = \mathcal{O}(V)^{m_1}$ and $R^1 f_* F_{n+l}(V) = \mathcal{O}(V)^{m_2}$. Now, for infinitely many maximal ideals $\mathcal{P}$ we have commutative diagram

$$\begin{array}{ccc} \mathcal{O}(V)^{m_1} & \xrightarrow{\psi} & \mathcal{O}(V)^{m_2} \\ \Big\downarrow \pi_1 & & \Big\downarrow \pi_2 \\ \mathcal{O}(V)^{m_1} / \mathcal{P}\mathcal{O}(V)^{m_1} & \xrightarrow{\phi} & \mathcal{O}(V)^{m_2} / \mathcal{P}\mathcal{O}(V)^{m_2}. \end{array}$$

Since $\psi(\mathcal{O}(V)^{m_1}) \subset \bigcap \mathcal{P}\mathcal{O}(V)^{m_2} = 0$, where $\mathcal{P}$ runs over infinitely many maximal ideals of $\mathcal{O}(V)$, it follows that $\psi(\mathcal{O}(V)^{m_1}) = 0$. This proves

$$\varinjlim_n (R^1 f_* F_n)_t = 0.$$

Because the direct limit of $R^1 f_* F_n$ is quasi-coherent, its support is locally closed. Now B is dense and connected in complex topology, so there exists an affine open set U in $\bar{C}$ such that, on U, the direct limit

$$\varinjlim_n R^1 f_* F_n|_U = 0.$$

Let $Y' = f^{-1}(U) - D$. By Goodman and Hartshorne's lemma, we have

$$H^1(Y', F) = \varinjlim_n H^1(f^{-1}(U), F_n) = \varinjlim_n R^1 f_* F_n(U) = 0.$$

This finishes the proof of Theorem 1.2.

REMARK 3.2. In our Step 3 proof we encounter the following two questions if we do not know the local freeness of $R^1 f_* F_n$.

1. If U is a smooth affine curve, then $\mathcal{O}(U) = A$ is a Dedekind domain. Let N be a finitely generated module over A. Then, under what conditions does $\bigcap(\mathcal{P}N) = 0$? (Here $\mathcal{P}$ runs over all maximal ideals of A.) A sufficient condition is that N be a projective module, but this condition is too strong. Our N is defined by cohomology, so it is hard to see whether it is projective or not. It is definitely not

sufficient for N to be a finitely generated module. For example, let $U = \mathbb{A}^1$, $A = \mathcal{O}(U) = \mathbb{C}[x]$, and $N = \mathbb{C}[x]/(x^2)$; then $\bigcap(\mathcal{P}N) \neq 0$.

2. Given A and $\mathcal{P}$ as in the previous question, let (M_n, f_n) be a direct system of finitely generated A-modules. If

$$\varinjlim_n (M_n/\mathcal{P}M_n) = 0,$$

then under what conditions can we say that

$$\varinjlim_n M_n = 0?$$

Again, that all M_n are finitely generated is not sufficient. For example, let $A = \mathbb{C}[[t]]$, the ring of formal power series, and let $M_n = t^{-n}A$. Then

$$\varinjlim_n M_n = \mathbb{K} \neq 0,$$

where $\mathbb{K} = \mathbb{C}((t))$, but

$$\varinjlim_n M_n/\mathcal{P}M_n = 0.$$

4. Proof of Theorem 1.3

LEMMA 4.1. *$R^i f_* \mathcal{O}_X(nD)$ is locally free for all $i \geq 0$ and $n \gg 0$.*

Proof. Since each fibre has dimension 2, by the upper semicontinuity theorem it follows that $R^i f_* \mathcal{O}_X(nD) = 0$ for all $i > 2$ and $n \geq 0$. By Lemma 2.1, since each fibre X_t is of type 2, we have

$$h^0(X_t, \mathcal{O}_{X_t}(nD_t)) = h^1(X_t, \mathcal{O}_{X_t}(nD_t)) = 1 \quad \text{and} \quad h^2(X_t, \mathcal{O}_{X_t}(nD_t)) = 0$$

for all $t \in C$ and $n \gg 0$. □

LEMMA 4.2. *$R^i f_* \Omega^3_X(nD) = R^i f_* \mathcal{O}_X(K_X + nD)$ is locally free for all $i \geq 0$ and $n \gg 0$.*

Proof. Since X_t is smooth, we have $K_X + D|_{X_t} = K_{X_t} = K_t$. Hence $\Omega^3_X(nD)|_{X_t} = \mathcal{O}_X(K_X + nD)|_{X_t} = \mathcal{O}_{X_t}(K_t + (n-1)D_t)$, where $D_t = D|_{X_t}$. By Lemma 2.1,

$$h^0(X_t, \Omega^3_X(nD)|_{X_t}) = h^0(X_t, \mathcal{O}_{X_t}(K_t + (n-1)D_t) = 1,$$

$$h^1(X_t, \Omega^3_X(nD)|_{X_t}) = h^1(X_t, \mathcal{O}_{X_t}(K_t + (n-1)D_t) = 1,$$

and

$$h^2(X_t, \Omega^3_X(nD)|_{X_t}) = h^2(X_t, \mathcal{O}_{X_t}(K_t + (n-1)D_t) = 0$$

for all $n \gg 0$. This proves the local freeness. □

LEMMA 4.3. $R^i f_* \Omega^1_X(nD)$ is locally free for all $i \geq 0$ and $n \gg 0$.

Proof. From the exact sequences

$$0 \to \mathcal{O}_{X_t} \to \Omega^1_X|_{X_t} \to \Omega^1_{X_t} \to 0$$

[H3, Chap. II, Thm. 8.17; GHa, p. 157], tensoring with $\mathcal{O}_X(nD)$ yields

$$0 \to \mathcal{O}_{X_t}(nD_t) \to \Omega^1_X(nD)|_{X_t} \to \Omega^1_{X_t}(nD_t) \to 0.$$

We will prove that, for any two points $t, t' \in C$ and for all $n \gg 0$,

$$h^i(X_t, \Omega^1_X(nD)|_{X_t}) = h^i(X_{t'}, \Omega^1_X(nD)|_{X_{t'}}).$$

Then (by the upper semicontinuity theorem) we are done. By the preceding short exact sequences, for fibres X_t and $X_{t'}$ we have the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(\mathcal{O}_{X_t}(nD_t)) & \xrightarrow{\alpha_1} & H^0(\Omega^1_X(nD)|_{X_t}) & \xrightarrow{\alpha_2} & H^0(\Omega^1_{X_t}(nD_t)) & \xrightarrow{\alpha_3} & H^1(\mathcal{O}_{X_t}(nD_t)) \\
& \| & & \downarrow \phi & & \| & & \| \\
0 \longrightarrow & H^0(\mathcal{O}_{X_{t'}}(nD_{t'})) & \xrightarrow{\beta_1} & H^0(\Omega^1_X(nD)|_{X_{t'}}) & \xrightarrow{\beta_2} & H^0(\Omega^1_{X_{t'}}(nD_{t'})) & \xrightarrow{\beta_3} & H^1(\mathcal{O}_{X_{t'}}(nD_{t'})),
\end{array}$$

where the natural map ϕ is defined as follows. If $\xi \in H^0(\Omega^1_X(nD)|_{X_t})$ is contained in the image of $H^0(\mathcal{O}_{X_t}(nD_t)) = \mathbb{C}$, then there is a number $a \in \mathbb{C}$ such that $\xi = \alpha_1(a)$. Thus we define $\phi(a) = \beta_1(a)$. If ξ is not contained in the image of α_1, then $\alpha_2(\xi) \in H^0(\Omega^1_{X_t}(nD_t))$ and $\alpha_3 \circ \alpha_2(\xi) = 0$. Hence there is an $\eta \in H^0(\Omega^1_X(nD)|_{X_{t'}})$ such that $\beta_3 \circ \beta_2(\eta) = 0$. Define $\phi(\xi) = \eta$; then, by [La, Lemma 5], we have

$$H^0(\Omega^1_X(nD)|_{X_t}) = H^0(\Omega^1_X(nD)|_{X_{t'}}).$$

Similarly,

$$H^i(\Omega^1_X(nD)|_{X_t}) = H^i(\Omega^1_X(nD)|_{X_{t'}})$$

for $i > 0$. □

LEMMA 4.4. *$R^i f_* \Omega^2_X(nD)$ is locally free for all $i \geq 0$ and $n \gg 0$.*

Proof. Observe that we have the short exact sequence

$$0 \to \Omega^1_{X_t} \to \Omega^2_X|_{X_t} \to \Omega^2_{X_t} \to 0$$

[H3, Chap. II, Thm. 8.17; GHa, p. 157]. Tensoring with $\mathcal{O}_X(nD)$ then yields

$$0 \to \Omega^1_{X_t}(nD_t) \to \Omega^2_X(nD)|_{X_t} \to \Omega^2_{X_t}(nD_t) \to 0.$$

By Lemma 2.1 we know that, for every $t \in C$,

$$h^0(X_t, \Omega^2_{X_t}(nD_t)) = h^1(X_t, \mathcal{O}_{X_t}(K_t + nD_t)) = 1,$$

$$h^2(X_t, \mathcal{O}_{X_t}(K_t + nD_t)) = 0.$$

Using the same argument as in the proof of Lemma 4.3, for any two points $t, t' \in C$ and all $n \gg 0$ we may write the long exact sequences for t and t' and obtain

$$h^i(X_t, \Omega^2_X(nD)|_{X_t}) = h^i(X_{t'}, \Omega^2_X(nD)|_{X_{t'}}).$$ □

LEMMA 4.5. *For every $t \in C$ we have $H^i(S_t, \Omega^j_Y|_{S_t}) = 0$ for all $j \geq 0$ and $i > 0$, where $S_t = X_t - D_t$.*

Proof. Since S_t is a surface, we need only consider $i = 1, 2$.

The claim is obvious for $\mathcal{O}_Y$. Because S_t is smooth, we have the exact sequence

$$0 \to \phi_t/\phi_t^2 = \mathcal{O}_{S_t} \to \Omega_Y^1|_{S_t} \to \Omega_{S_t}^1 \to 0,$$

where ϕ_t is the defining sheaf of S_t. Hence the claim holds for $\Omega_Y^1|_{S_t}$. Since the normal sheaf

$$\mathcal{N}_{S_t/Y} = \mathcal{H}om(\phi_t/\phi_t^2, \mathcal{O}_{S_t}) = \mathcal{O}_{S_t},$$

we have

$$\omega_{S_t} \cong \omega_Y \otimes \mathcal{N}_{S_t/Y} \cong \omega_Y \otimes \mathcal{O}_{S_t} = \omega_Y|_{S_t}$$

and so the claim holds for $\Omega_Y^3|_{S_t}$. From the exact sequence

$$0 \to \Omega_{S_t}^1 \to \Omega_Y^2|_{S_t} \to \Omega_{S_t}^2 \to 0$$

we derive the claim for $\Omega_Y^2|_{S_t}$. □

Lemma 4.6. *For all $j \geq 0$, $H^2(Y, \Omega_Y^j) = 0$.*

Proof. For all $n \gg 0$, the sheaves $\Omega_X^j \otimes \mathcal{O}_X(nD)$ are locally free by Lemmas 4.1–4.5. By the upper semicontinuity theorem, there exists an integer n_0 such that $R^2 f_* \Omega_X^j \otimes \mathcal{O}_X(nD)|_C = 0$ for all $n \geq n_0$. By Goodman and Hartshorne's lemma [GoH], we have

$$\begin{aligned} H^2(Y, \Omega_Y^j) &= \varinjlim_n H^2(f^{-1}(C), \Omega_X^j \otimes \mathcal{O}_X(nD)) \\ &= \varinjlim_n R^2 f_* \Omega_X^j \otimes \mathcal{O}_X(nD)(C) = 0. \end{aligned}$$ □

Lemma 4.7. *For all $j \geq 0$, $H^1(Y, \Omega_Y^j) = 0$.*

Proof. By the local freeness lemmas and Goodman and Hartshorne's lemma, we have

$$\begin{aligned} H^1(Y, \Omega_Y^j) &= \varinjlim_n H^1(f^{-1}(C), \Omega_X^j \otimes \mathcal{O}_X(nD)) \\ &= \varinjlim_n R^1 f_* \Omega_X^j \otimes \mathcal{O}_X(nD)(C) = 0. \end{aligned}$$ □

5. Proof of Theorem 1.6

We will prove Theorem 1.6 by constructing an example. Let C_t be a smooth projective elliptic curve defined by $y^2 = x(x-1)(x-t)$, $t \neq 0, 1$, and let Z be the elliptic surface defined by the same equation. Then we have surjective morphism from Z to $C = \mathbb{C} - \{0, 1\}$ such that, for every $t \in C$, the fibre $f^{-1}(t) = C_t$.

Lemma 5.1. *There is a rank-2 vector bundle E on Z such that, when restricted to C_t, $E|_{C_t} = E_t$ is the unique nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$ and f is the morphism from Z to C.*

Proof. Since $f: Z \to C$ is an elliptic fibration, for every t we have

$$h^1(f^{-1}(t), \mathcal{O}_{f^{-1}(t)}) = h^1(\mathcal{O}_{C_t}) = 1.$$

Therefore,

$$R^1 f_* \mathcal{O}_Z \otimes \mathbb{C}(t) \cong H^1(C_t, \mathcal{O}_{C_t}) \cong \mathbb{C}.$$

This yields

$$(R^1 f_* \mathcal{O}_Z)_t / \mathcal{P}_t (R^1 f_* \mathcal{O}_Z)_t \cong \mathbb{C}.$$

By Nakayama's lemma, $R^1 f_* \mathcal{O}_Z$ is a line bundle on C. Since $\mathbb{C}[x, 1/x, 1/(x-1)]$ is a principal ideal domain, the Picard group of C is trivial; that is, any line bundle on C is trivial. Hence $R^1 f_* \mathcal{O}_Z \cong \mathcal{O}_C$ and

$$H^1(Z, \mathcal{O}_Z) = R^1 f_* \mathcal{O}_Z(C) = \mathcal{O}_C(C) = \mathbb{C}\left[x, \frac{1}{x}, \frac{1}{x-1}\right].$$

Given any exact sequence of vector bundles

$$0 \to \mathcal{O}_Z \to E \to \mathcal{O}_Z \to 0,$$

let ξ be the image of unit of $H^0(Z, \mathcal{O}_Z)$ in $H^1(Z, \mathcal{O}_Z)$. Thus we have an element of $H^1(Z, \mathcal{O}_Z)$. Conversely, given any element ξ in $H^1(Z, \mathcal{O}_Z)$, we can obtain an exact sequence like the one just displayed by the following procedure. Take any (degree) large ample line bundle L on the elliptic surface Z; then, for any positive integer n, we have an exact sequence

$$0 \to \mathcal{O}_Z \xrightarrow{\alpha} L^{\oplus n} \xrightarrow{\beta} G \to 0,$$

where G is the quotient that is a vector bundle. We may assume $H^1(Z, L) = 0$ by raising the degree of L because L is ample. Hence we have a surjective map $H^0(Z, G) \twoheadrightarrow H^1(Z, \mathcal{O}_Z)$ and so ξ can be lifted to an element η in $H^0(Z, G)$. This element η defines a map from $\mathcal{O}_Z$ to G ($\eta\colon \mathcal{O}_Z \to G$) sending 1 to η. Setting $E = \beta^{-1}(\eta(\mathcal{O}_Z))$ then yields the exact sequence

$$0 \to \mathcal{O}_Z \xrightarrow{\alpha} E \xrightarrow{\beta} \eta(\mathcal{O}_Z) = \mathcal{O}_Z \to 0.$$

So there is a one-to-one correspondence between the elements of $H^1(Z, \mathcal{O}_Z)$ and the foregoing exact sequences. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Z & \longrightarrow & L^{\oplus n} & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_Z & \longrightarrow & E & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0. \end{array}$$

Since $\mathbb{C} \subset H^1(Z, \mathcal{O}_Z) = \mathbb{C}[x, 1/x, 1/(x-1)]$, it follows that $1 \in H^1(Z, \mathcal{O}_Z)$. This nonzero element 1 corresponds to a rank-2 vector bundle E such that, when restricted to every fibre C_t, $E|_{C_t}$ is the nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$. In fact, in the natural restriction map

$$H^1(Z, \mathcal{O}_Z) \to H^1(C_t, \mathcal{O}_{C_t}),$$

1 goes to 1. A nonzero element of $H^1(C_t, \mathcal{O}_{C_t})$ determines a nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$. □

LEMMA 5.2. *There is a divisor D on $X = \mathbb{P}_Z(E)$ such that, when restricted to $X_t = \mathbb{P}_{C_t}(E_t)$, $D|_{X_t} = D_t$ is the canonical section of X_t.*

Proof. By Lemma 5.1 there is a surjective map from E to $\mathcal{O}_Z$. This map corresponds to a section $\sigma\colon Z \to X$. When restricted to C_t, $\sigma|_{C_t} = \sigma_t\colon C_t \to X_t$ is the unique nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$. □

Let $Y = X - D$. By Theorem 1.3, $H^i(Y, \Omega_Y^j) = 0$ for all $i > 0$ and $j \geq 0$. Thus we have constructed a nonaffine, nonproduct example of a 3-fold Y with vanishing Hodge cohomology, which proves Theorem 1.6.

In the example that we have constructed, every fibre is Stein and the base curve is Stein but we do not know whether the 3-fold is Stein. It would be interesting to also construct a 3-fold with type-3 open surfaces as fibres.

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