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On the Solid Hull of the Hardy Space H^p, $0 < p < 1$

Miroljub Jevtić & Miroslav Pavlović

1. Introduction

Finding the solid hull $S(H^p)$ of the Hardy space H^p—that is, finding the strongest growth condition the absolute value of the coefficients of H^p functions must satisfy—is an old and difficult problem. It follows from Littlewood's theorem on random power series [7, Thm. A.5, p. 228] that $S(H^p) = H^2$ for $2 < p < \infty$. Much later, Kisliakov [12] identified the solid hull of the space H^∞. A deep result of Kisliakov shows that $S(H^\infty)$ is also H^2. In this paper we identify $S(H^p)$ in the case $0 < p < 1$.

The Hardy space H^p $(0 < p \le \infty)$ is the space of all functions f holomorphic in the unit disc U $(f \in H(U))$ for which

$$\|f\|_p = \lim_{r\to 1} M_p(r, f) < \infty,$$

where, as usual,

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{0\le t<2\pi} |f(re^{it})|.$$

Throughout this paper, we identify a holomorphic function $f(z) = \sum_{n=0}^{\infty} \hat f(n) z^n$ with its sequence of Taylor coefficients $(\hat f(n))_{n=0}^{\infty}$. Hardy and Littlewood proved that if f belongs to H^p, $0 < p < 1$, then

$$\sum_{n=0}^{\infty} (n+1)^{p-2} |\hat f(n)|^p < \infty \tag{1.1}$$

and

$$|\hat f(n)| = o((n+1)^{1/p-1}), \quad n \to \infty \tag{1.2}$$

(see [7] for information and references).

In [13] it was proved that if $f \in H^p$, $0 < p < 1$, then

$$\sum_{n=1}^{\infty} 2^{-n(1-p)} \Big(\sup_{0\le k\le 2^n} |\hat f(k)|\Big)^p < \infty, \tag{1.3}$$

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which is equivalent to

$$\sum_{n=0}^{\infty}(n+1)^{p-2}\Big(\sup_{0\le k\le n}|\hat{f}(k)|\Big)^{p}<\infty.$$

It is easy to see that condition (1.3) is stronger than (1.1) and (1.2). We will show that (1.3) is the strongest condition that the moduli of the coefficients of a function $f\in H^p$ $(0<p<1)$ must satisfy. In the terminology of [2], this means that the smallest solid space containing H^p $(0<p<1)$ is the vector space of sequences satisfying (1.3).

Recall that a sequence space X is solid (cf. [2]) if $(b_n)\in X$ whenever $(a_n)\in X$ and $|b_n|\le|a_n|$. The solid hull of X is the smallest solid space containing X. Explicitly,

$$S(X)=\{(\lambda_n): \text{there exists } (a_n)\in X \text{ such that } |\lambda_n|\le|a_n|\}.$$

To state our first result in a more precise form, we need to introduce some more notation. A complex sequence (a_n) is of class $l(p,q)$, $0<p,q\le\infty$, if

$$\|(a_n)\|_{p,q}^{q}=\|(a_n)\|_{l(p,q)}^{q}=\sum_{n=0}^{\infty}\Big(\sum_{k\in I_n}|a_k|^p\Big)^{q/p}<\infty,$$

where $I_0=\{0\}$ and $I_n=\{k\in N: 2^{n-1}\le k<2^n\}$ for $n=1,2,\dots$. In the case where p or q is infinite, replace the corresponding sum by a supremum. Note that $l(p,p)=l^p$.

For $t\in R$ we write D^t for the sequence $((n+1)^t)_0^\infty$. If $\lambda=(\lambda_n)$ is a sequence and X a sequence space, we write $\lambda X=\{(\lambda_n x_n):(x_n)\in X\}$; thus, for example, $(a_n)\in D^t l^\infty$ if and only if $|a_n|=O(n^t)$.

Here is our main result.

THEOREM 1. *If* $0<p<1$, *then* $S(H^p)=D^{1/p-1}l(\infty,p)$.

We also determine the solid hull of the Bergman space A^p for $0<p\le1$.

The Bergman space A^p for $0<p<\infty$ consists of all holomorphic functions f on U such that

$$\|f\|_{A^p}=\Big(\int_U|f(z)|^p\,dm(z)\Big)^{1/p}<\infty,$$

where $dm(z)$ stands for the Lebesgue measure in the plane.

It is well known that if $0<p\le1$ then $A^p\subset D^{2/p-1}l^\infty$ and $A^p\subset D^{3/p-1}l^p$ (see [17]). We improve both these inclusions by showing the following theorem.

THEOREM 2. *If* $0<p\le1$, *then* $S(A^p)=D^{2/p-1}l(\infty,p)$.

Our results can be applied to various problems concerning multipliers. Thus they easily imply the main result in [9], for instance. Details will be given in Section 4.

Given two vector spaces A,B of sequences, we denote by (A,B) the space of multipliers from A to B. More precisely,

$$(A,B)=\{\lambda=(\lambda_n):(\lambda_n a_n)\in B \text{ for every } (a_n)\in A\}.$$

The D-dual of a sequence space A, denoted by A^D, is defined to be (A, D), the multipliers from A to D. The Köthe dual is obtained when $D = l^1$ and will be denoted A^K. As in [2], let $A(1,1)$ be the space of all $f \in H(U)$ such that $f' \in A^1$.

An easy consequence of Theorem 2 is the following statement.

COROLLARY 3. $S(A(1,1)) = l(\infty, 1)$.

Anderson and Shields [2, p. 263] showed that the second Köthe dual of $A(1,1)$ is $l(\infty,1)$ (i.e., $A(1,1)^{KK} = l(\infty,1)$) and that $S(A(1,1)) \subset A(1,1)^{KK}$. They conjectured that the inclusion is strict. Corollary 3 disproves this conjecture.

Analogously, we have $S(A^1) = (A^1)^{KK} = D^1 l(\infty,1)$. (See Theorem 4.2.)

Our method of determining the solid hulls can be applied more generally to the mixed norm spaces $H^{p,q,\alpha}$. The space $H^{p,q,\alpha}$ $(0 < p \le \infty, 0 < q, \alpha < \infty)$ consists of all $f \in H(U)$ for which

$$\|f\|_{p,q,\alpha} = \left(\int_0^1 (1-r)^{q\alpha - 1} M_p(r,f)^q \, dr \right)^{1/q} < \infty.$$

In particular, we have Bergman spaces $A^p = H^{p,p,1/p}$ for $0 < p < \infty$.

The space $H^{p,q,\alpha}$ can also be defined when $q = \infty$, in which case it is sometimes known as the weighted Hardy space $H^{p,\alpha} = H^{p,\infty,\alpha}$, and consists of all $f \in H(U)$ for which

$$\|f\|_{p,\alpha} := \|f\|_{p,\infty,\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(r,f) < \infty.$$

Instead of Theorem 2 we prove the following result.

THEOREM 4. *If* $0 < p \le 1$, *then* $S(H^{p,q,\alpha}) = D^{\alpha + 1/p - 1} l(\infty, q)$.

Observe that $S(H^{p,q,\alpha}) = H^{2,q,\alpha} = D^\alpha l(2,q)$ for $2 \le p \le \infty$ (see [1; 3; 14]). Hence the problem of determining $S(H^p)$ for $1 \le p < 2$ and $S(H^{p,q,\alpha})$ for $1 < p < 2$ remains open.

2. Solid Hull of the Hardy Space H^p, $0 < p < 1$

If $f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k$ and $g(z) = \sum_{k=0}^\infty \hat{g}(k) z^k$ are holomorphic functions in U, then the holomorphic function $f \star g$ is defined by

$$(f \star g)(z) = \sum_{k=0}^\infty \hat{f}(k) \hat{g}(k) z^k.$$

The main tool for proving our results are the polynomials W_n $(n \ge 0)$ constructed in [9]. Here we recall their construction and some of their properties.

Let $\omega \colon R \to R$ be a nonincreasing function of class C^∞ such that $\omega(t) = 1$ for $t \le 1$ and $\omega(t) = 0$ for $t \ge 2$. We define the polynomials $W_n = W_n^\omega$ $(n \ge 0)$ as follows:

$$W_0(z) = \sum_{k=0}^\infty \omega(k) z^k, \qquad W_n(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \varphi\left(\frac{k}{2^{n-1}}\right) z^k \quad \text{for } n \ge 1,$$

where $\varphi(t) = \omega(t/2) - \omega(t)$, $t \in R$.

The coefficients $\hat{W}_n(k)$ of these polynomials have the following properties:

$$\operatorname{supp}(\hat{W}_n) \subset [2^{n-1}, 2^{n+1}], \tag{2.1}$$

$$0 \le \hat{W}_n(k) \le 1 \quad \text{for all } k, \tag{2.2}$$

$$\sum_{n=0}^{\infty} \hat{W}_n(k) = 1 \quad \text{for all } k, \tag{2.3}$$

$$\hat{W}_n(k) + \hat{W}_{n+1}(k) = 1 \quad \text{for } \; 2^n \le k \le 2^{n+1}, \; n \ge 0. \tag{2.4}$$

Property (2.3) implies that

$$f(z) = \sum_{n=0}^{\infty} (W_n \star f)(z), \quad f \in H(U),$$

and the series is uniformly convergent on compact subsets of U.

Since $0 \le \hat{W}_n(k) \le 1$ for $n, k = 0, 1, 2, \dots$, we have

$$|W_n(z)| \le 2^{n+1}, \quad z \in U, \; n = 0, 1, 2, \dots. \tag{2.5}$$

Choose an integer N so that $Np > 1$. Note that $\varphi(k/2^{n-1}) = 0$ if k is an integer such that $k \le 2^{n-1}$ or $2^{n+1} \le k$. Hence,

$$\begin{aligned}
(1 - e^{it})^N W_n(e^{it}) &= \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right)(1 - e^{it})^N e^{ikt} \\
&= \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) \sum_{m=0}^{N} \binom{N}{m} (-1)^m e^{i(m+k)t} \\
&= \sum_{m=0}^{N} (-1)^m \binom{N}{m} \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) e^{i(k+m)t} \\
&= \sum_{m=0}^{N} (-1)^m \binom{N}{m} \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k-m}{2^{n-1}}\right) e^{ikt} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^{N} (-1)^m \binom{N}{m} \varphi\left(\frac{k-m}{2^{n-1}}\right)\right) e^{ikt}.
\end{aligned} \tag{2.6}$$

By the Lagrange theorem for symmetric differences, for each k there exists a $\xi_{k,N}$ such that

$$\sum_{m=0}^{N} (-1)^m \binom{N}{m} \varphi\left(\frac{k-m}{2^{n-1}}\right) = 2^{(1-n)N} \varphi^{(N)}(\xi_{k,N}). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$|W_n(e^{it})| \le C t^{-N} 2^{n(1-N)}. \tag{2.8}$$

Using (2.5) and (2.8) now yields

$$\|W_n\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |W_n(e^{it})|^p\,dt \le C2^{-n(1-p)}. \tag{2.9}$$

Observe that here we needed $Np > 1$.

In this paper we follow the custom of using the letter C to stand for a positive constant that changes its value from one appearance to another while remaining independent of the important variables.

Proof of Theorem 1. Let $f \in H^p$, $0 < p < 1$. Then, by [7, Thm. 5.11],

$$\int_0^1 (1-r)^{-p} M_1(r,f)^p\,dr < \infty.$$

Since $\sup_{k\in I_n} |\hat{f}(k)| r^k \le M_1(r,f)$ for $n \ge 0$, it follows that

$$\begin{aligned}\infty > \int_0^1 (1-r)^{-p} M_1(r,f)^p\,dr &\ge \sum_{n=1}^{\infty} \int_{1-2^{1-n}}^{1-2^{-n}} (1-r)^{-p} \Big(\sup_{k\in I_n} \hat{f}(k) r^k\Big)^p\,dr \\ &\ge C\sum_{n=1}^{\infty} 2^{-n(1-p)} \Big(\sup_{k\in I_n} |\hat{f}(k)|\Big)^p.\end{aligned}$$

Thus, $H^p \subset D^{1/p-1} l(\infty, p)$.

To show that $D^{1/p-1} l(\infty, p)$ is the solid hull of H^p, it is enough to prove that if $(a_n) \in D^{1/p-1} l(\infty, p)$ then there exists a $(b_n) \in H^p$ such that $|b_n| \ge |a_n|$ for all n.

Toward this end, let $(a_n) \in D^{1/p-1} l(\infty, p)$. Define

$$g(z) = \sum_{j=0}^{\infty} B_j(W_j(z) + W_{j+1}(z)) = \sum_{k=0}^{\infty} c_k z^k,$$

where $B_j = \sup_{2^j \le k < 2^{j+1}} |a_k|$. The function g belongs to H^p because

$$\begin{aligned}M_p^p(r,g) &\le \sum_{j=0}^{\infty} B_j^p [M_p^p(1, W_j) + M_p^p(1, W_{j+1})] \\ &\le C\sum_{j=0}^{\infty} B_j^p 2^{-j(1-p)} < \infty.\end{aligned}$$

Here we have used (2.9).

To prove that $|c_k| \ge |a_k|$ for $k = 1, 2, \ldots$, choose n so that $2^n \le k < 2^{n+1}$. It follows from (2.2) and (2.4) that

$$\begin{aligned}c_k &= \sum_{j=0}^{\infty} B_j(\hat{W}_j(k) + \hat{W}_{j+1}(k)) \ge B_n(\hat{W}_n(k) + \hat{W}_{n+1}(k)) \\ &= B_n = \sup_{2^n \le j < 2^{n+1}} |a_j| \ge |a_k|.\end{aligned}$$

Now the function $h(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_0 = a_0$ and where $b_n = c_n$ for $n \ge 1$, belongs to H^p, and $|b_n| \ge |a_n|$ for all $n \ge 0$. □

REMARK. Note that the proof of Theorem 1 shows that the solid hull of H^p, $0 < p < 1$, may also be described as the set

$$\left\{(a_n) : \sum_{n=0}^{\infty} 2^{-n(1-p)} \sup_{0\le k\le 2^n} |a_k| < \infty\right\}.$$

3. The Solid Hull of the Mixed Norm Space $H^{p,q,\alpha}$, $0 < p \le 1$

Proof of Theorem 4. Let $f \in H^{p,q,\alpha}$. In order to prove that $f \in D^{\alpha+1/p-1}l(\infty, q)$, we use the familiar inequality

$$M_p(r, f) \ge C(1-r)^{1/p-1}M_1(r^2, f), \quad 0 < p \le 1.$$

(see [7, Thm. 5.9]) to obtain

$$\begin{aligned}\infty &> \int_0^1 (1-r)^{q\alpha-1}M_p(r, f)^q\,dr \\ &\ge C\int_0^1 (1-r)^{q(\alpha+1/p-1)}M_1(r, f)^q\,dr \\ &\ge C\sum_{n=1}^{\infty}\int_{1-2^{1-n}}^{1-2^{-n}} (1-r)^{q(\alpha+1/p-1)-1}\Big(\sup_{k\in I_n}|a_k|r^k\Big)^q\,dr \\ &\ge C\sum_{n=1}^{\infty} 2^{-nq(\alpha+1/p-1)}\Big(\sup_{k\in I_n}|a_k|\Big)^q.\end{aligned}$$

Thus, $f \in D^{\alpha+1/p-1}l(\infty, q)$.

Similarly, if $q = \infty$ then

$$\begin{aligned}\infty > \sup_{0<r<1}(1-r)^{\alpha}M_p(r, f) &\ge C\sup_{0<r<1}(1-r)^{\alpha+1/p-1}M_1(r, f) \\ &\ge C\sup_{0<r<1}\sup_n\sup_{k\in I_n}(1-r)^{\alpha+1/p-1}|a_k|r^k \\ &\ge C\sup_n 2^{-n(\alpha+1/p-1)}\sup_{k\in I_n}|a_k|;\end{aligned}$$

that is, $f \in D^{\alpha+1/p-1}l(\infty, \infty)$.

Now let $(a_n) \in D^{\alpha+1/p-1}l(\infty, q)$ for $0 < q < \infty$. As before, define

$$h(z) = \sum_{j=0}^{\infty} C_j(W_j(z) + W_{j+1}(z)) = \sum_{k=0}^{\infty} d_k z^k,$$

where $C_j = \sup_{k\in I_j}|a_k|$.

The function h belongs to $H^{p,q,\alpha}$ ($0 < p \le 1$, $0 < q, \alpha < \infty$) because

$$\begin{aligned}&\int_0^1 (1-r)^{q\alpha-1}M_p(r, h)^q\,dr \\ &\quad\le \int_0^1 (1-r)^{q\alpha-1}\Big(\sum_{j=0}^{\infty} C_j^p[M_p^p(r, W_j) + M_p^p(r, W_{j+1})]\Big)^{q/p}\,dr. \quad (3.1)\end{aligned}$$

Using [9, Lemma 2.1] together with (2.1) and (2.9) yields

$$M_p^p(r, W_j) \le r^{2^{j-1}p} \|W_j\|_p^p \le C r^{2^{j-1}p} 2^{-j(1-p)}, \quad j = 1, 2, \ldots. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}\int_0^1 (1-r)^{q\alpha-1} M_p(r,h)^q \, dr &\le C \int_0^1 (1-r)^{q\alpha-1} \left(\sum_j^\infty C_j^p 2^{-j(1-p)} r^{2^{j-1}p} \right)^{q/p} dr \\ &\le C \sum_j^\infty C_j^q 2^{-jq(\alpha+1/p-1)} < \infty.\end{aligned}$$

Here we have used [14, Prop. 4.1].

As before, we have $|d_k| \ge |a_k|$, $k = 1, 2, \ldots$. The function $\psi(z) = \sum_{k=0}^\infty b_k z^k$, where $b_0 = a_0$ and where $b_k = d_k$ for $k = 1, 2, \ldots$, belongs to $H^{p,q,\alpha}$, and $|b_k| \ge |a_k|$ for all $k \ge 0$.

The case $q = \infty$ may be treated similarly. □

4. Applications to Multipliers

The next lemma is due to Kellog. He states it for exponents no smaller than 1, but it then follows for all exponents because $(\lambda_n) \in (l(a,b), l(c,d))$ if and only if $(\lambda_n^{1/t}) \in (l(at,bt), l(ct,dt))$.

LEMMA 4.1 [11]. *If* $0 < a, b, c, d \le \infty$, *then*

$$(l(a,b), l(c,d)) = l(a \ominus c, b \ominus d),$$

where $a \ominus c = \infty$ *if* $a \le c$, $b \ominus d = \infty$ *if* $b \le d$, *and*

$$\frac{1}{a \ominus c} = \frac{1}{c} - \frac{1}{a} \quad \text{for } 0 < c < a,$$
$$\frac{1}{b \ominus d} = \frac{1}{d} - \frac{1}{b} \quad \text{for } 0 < d < b.$$

In [2] it is proved that if X is any solid space and A any vector space of sequences then $(A, X) = (S(A), X)$.

Since $l(u,v)$ are solid spaces, we have $(H^p, l(u,v)) = (S(H^p), l(u,v))$ and $(H^{p,q,\alpha}, l(u,v)) = (S(H^{p,q,\alpha}), l(u,v))$. Together with Lemma 4.1 and Theorems 1 and 4, this yields our last two results.

THEOREM 4.2. *Let* $0 < p < 1$. *Then*

$$(H^p, l(u,v)) = D^{1-1/p} l(u, p \ominus v).$$

THEOREM 4.3. *If* $0 < p \le 1$, *then*

$$(H^{p,q,\alpha}, l(u,v)) = D^{1-1/p-\alpha} l(u, q \ominus v).$$

In particular, if $u = v$ then $(H^p, l^u) = D^{1-1/p} l(u, p \ominus u)$. This was proved in [9] by a different method. Similarly, from Theorem 4.3 we deduce that $(H^{p,q,\alpha}, l^u) = D^{1-1/p-\alpha} l(u, q \ominus u)$ for $0 < p \leq 1$ (see [10]).

REMARK. The referee pointed out to us that Theorem 4.3 was already known; see Theorem 5.2 in [6] and remark (2) following that result. By the same result of [6], the answer when $1 < p < 2$ cannot be of the form $D^t l(a, b)$.

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Matematički Fakultet
Studentski trg 16
11 000 Beograd
Serbia