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A Conformal Invariant for Nonvanishing Analytic Functions and Its Applications

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Dedicated to Professor Yukio Kusunoki on the occasion of his 80th birthday

1. Introduction

Conformal invariants play a central role in the modern theory of functions of a complex variable. One of the most important invariants is the hyperbolic metric $\rho_D(z)|dz|$ of a hyperbolic plane domain D. Recall that a subdomain D of $\mathbb{C}$ is called *hyperbolic* if D admits an analytic universal covering projection p of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto D. Then the density $\rho_D(z)$ of a hyperbolic metric is defined by the equation $\rho_D(z)|p'(\zeta)| = 1/(1-|\zeta|^2)$ for $\zeta \in p^{-1}(z)$. Note that the value of $\rho_D(z)$ does not depend on the particular choice of ζ or p. The Poincaré–Koebe uniformization theorem tells us that $D \subset \mathbb{C}$ is hyperbolic if and only if D is neither the whole plane $\mathbb{C}$ nor the punctured plane $\mathbb{C} \setminus \{a\}$ for any $a \in \mathbb{C}$. The hyperbolic metric is conformally invariant in the sense that the pulled-back density $f^*\rho_{D'}(z) = \rho_{D'}(f(z))|f'(z)|$ of $\rho_{D'}(w)|dw|$ under a conformal map $f : D \to D'$ is equal to $\rho_D(z)$. Throughout the paper, a conformal map means a conformal homeomorphism.

In this paper we propose a sort of conformal invariant associated with a nonvanishing analytic function. This quantity proves its usefulness in estimating the hyperbolic sup-norm of the pre-Schwarzian derivative of a locally univalent function in various situations (cf. [15]). Let φ be a nonvanishing analytic function on a hyperbolic domain D; namely, $\varphi : D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z \in D} \rho_D(z)^{-1} \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$

This quantity measures the rate of growth of φ compared with the hyperbolic metric. Note also that $V_D(\varphi)$ can be thought of as the Bloch seminorm of the (possibly multivalued) function $\log \varphi$. The quantity $V_D(\varphi)$ does not depend on the source domain D; more precisely, $V_{D_0}(\varphi \circ f) = V_D(\varphi)$ for a conformal map $f : D_0 \to D$ (see Theorem 2.2). On the other hand, $V_D(\varphi)$ may depend on the target domain.

One merit of this quantity is monotonicity in several respects. For instance, if ω is a holomorphic map of D_0 into D then $V_D(\varphi) \le V_{D_0}(\varphi \circ \omega)$ holds (see Theorem 2.2). Many more properties will be discussed in Section 2.

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Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$. Then Ω admits an analytic universal covering projection p of a simply connected proper subdomain D of $\mathbb{C}$ onto Ω. The quantity $W(\Omega) = V_D(p)$ is thus independent of the particular choice of $p\colon D \to \Omega$ and will be called the *circular width* of Ω (about the origin). An important property to note is that $W(\Omega) \le W(\Omega_1)$ if $\Omega \subset \Omega_1 \subset \mathbb{C}^*$. For example, the sector $\{w \in \mathbb{C} : |\arg w| < \pi\alpha/2\}$ has circular width 2α for $0 < \alpha \le 2$ (see Section 5). Fundamental properties and a geometric meaning of the circular width will be given in Section 3. Also, exact values of $W(\Omega)$ for some specific domains Ω are given in Section 5.

The circular widths (about boundary points) of a plane domain are closely related to uniform perfectness of the boundary; this will be explained in Section 4. As an application, we will give a proof of Osgood's theorem [21, Thm. 2] in a quantitative way: *∂D is uniformly perfect if and only if the hyperbolic sup-norm of univalent analytic functions on D is bounded.*

The information on $W(\Omega)$ is useful regarding univalence and boundedness criteria. Consider, for instance, an analytic function f in the unit disk with $\operatorname{Re} f' > 0$. In general, the function f may not be bounded (e.g., $f(z) = -\log(1 - z)$). As a consequence of our results, we obtain the boundedness criterion stating that, if $f'(\mathbb{D}) \subset \Omega$ for a subdomain Ω of the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ with $W(\Omega) < 2$, then f must be bounded. Observe that $W(\Omega) \le 2$ holds always for $\Omega \subset \mathbb{H}$.

Applying this to the function $f = \log F$ for a nonvanishing locally univalent function F on $\mathbb{D}$ yields another approach to the problem considered in [17] by MacGregor and Rønning. In Section 6 we give some sufficient conditions for a domain $\Omega \subset \mathbb{C}^*$ to have circular width less than 2.

We have already used some facts about $W(\Omega)$ implicitly in [15]. Moreover, some results in Section 5 were used by Ponnusamy and the second author [23] in order to deduce univalence criteria for meromorphic functions outside the unit disk. See Section 6 for more details about applications of circular width.

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2. Basic Properties of the Quantity $V_D(\varphi)$

In this section, basic properties of the quantity $V_D(\varphi)$ and more refined results are given. We first see how $V_D(\varphi)$ measures the rate of growth of φ with respect to the hyperbolic metric. Denote by $d_D(z_0, z_1)$ the hyperbolic distance between two points z_0 and z_1 in D:

$$d_D(z_0, z_1) = \inf_{\gamma} \int_{\gamma} \rho_D(z)|dz|,$$

where the infimum is taken over all piecewise smooth curves joining z_0 and z_1 in D. With this notation, we have the following result.

PROPOSITION 2.1. *Let φ be a nonvanishing analytic function on a hyperbolic domain D, and let c be a positive constant. Then $V_D(\varphi) \le c$ if and only if the inequality*

$$\exp\{-cd_D(z_0, z_1)\} \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\{cd_D(z_0, z_1)\} \tag{2.1}$$

holds for every pair of points z_0, z_1 in D.

Proof. We first assume that $V_D(\varphi) \le c$; that is, $|\varphi'/\varphi| \le c\rho_D$. Since

$$\log \frac{|\varphi(z_1)|}{|\varphi(z_0)|} = \operatorname{Re} \int_\gamma \frac{\varphi'(z)}{\varphi(z)}\, dz$$

for γ joining z_0 and z_1 in D, we obtain the inequalities

$$\left| \log \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \right| \le \int_\gamma \left| \frac{\varphi'(z)}{\varphi(z)} \right| |dz| \le c \int_\gamma \rho_D(z)|dz|.$$

Thus we can now see (2.1).

We next prove the converse. Set $u(z) = \log|\varphi(z)|$ for $z \in D$. Then condition (2.1) means that

$$|u(z') - u(z)| \le cd_D(z, z'), \quad z, z' \in D.$$

Now dividing both sides by $|z' - z|$ and taking upper limits as $z' \to z$ yields the inequality $|\nabla u(z)| \le c\rho_D(z)$, where ∇ denotes the gradient. Since u is the real part of the analytic function $f = \log \varphi$ (at least locally), it follows that $|\nabla u| = |f'| = |\varphi'/\varphi|$. We have thus proved the inequality $|\varphi'/\varphi| \le c\rho_D$. □

We next show certain fundamental properties of the quantity $V_D(\varphi)$. The following properties are obvious: for nonvanishing analytic functions φ and ψ on a hyperbolic domain D, the inequality

$$V_D(\varphi \cdot \psi) \le V_D(\varphi) + V_D(\psi)$$

holds and also the relation

$$V_D(\varphi^\alpha) = |\alpha| V_D(\varphi) \tag{2.2}$$

holds for $\alpha \in \mathbb{C}$ provided the power φ^α is defined as a single-valued analytic function on D. Note that φ^α is always taken to be single-valued if α is an integer or if D is simply connected.

Apart from these, we have the following important invariance properties.

THEOREM 2.2. *Let D be a hyperbolic domain and let φ be a nonvanishing analytic function on D.*

(a) *Let $p \colon D_0 \to D$ be an analytic (unbranched and unlimited) covering projection; then $V_{D_0}(\varphi \circ p) = V_D(\varphi)$. In particular, $V_D(\varphi)$ is conformally invariant in the sense that this does not depend on the source domain.*
(b) *$V_D(L \circ \varphi) = V_D(\varphi)$ holds for any conformal automorphism L of $\mathbb{C}^*$. In particular, $V_D(1/\varphi) = V_D(\varphi) = V_D(c\varphi)$ for any constant $c \in \mathbb{C}^*$.*

(c) *Let* $\omega\colon D_0 \to D$ *be a holomorphic map*; *then* $V_{D_0}(\varphi\circ\omega) \le V_D(\varphi)$.
(d) *If* $\psi\colon D \to \mathbb{C}^*$ *is univalent and if* $\varphi(D) \subset \psi(D)$, *then* $V_D(\varphi) \le V_D(\psi)$.

Proof. Assertion (a) follows from the invariance property $\rho_D(p(z))|p'(z)| = \rho_{D_0}(z)$ of hyperbolic density, and (b) is easily deduced by a straightforward computation. Next we prove property (c) when $D = D_0 = \mathbb{D}$ by the Schwarz–Pick lemma: $(1-|z|^2)|\omega'(z)| \le 1-|\omega(z)|^2$, $|z|<1$, for a holomorphic map $\omega\colon \mathbb{D} \to \mathbb{D}$. We now have the inequality

$$(1-|z|^2)\left|\frac{(\varphi\circ\omega)'(z)}{(\varphi\circ\omega)(z)}\right| = (1-|z|^2)|\omega'(z)|\left|\frac{\varphi'(\omega(z))}{\varphi(\omega(z))}\right| \le (1-|\omega(z)|^2)\left|\frac{\varphi'(\omega(z))}{\varphi(\omega(z))}\right|.$$

We note that equality holds in this expression for some (and thus all) points $z \in \mathbb{D}$ if and only if ω is an automorphism of $\mathbb{D}$; thus (c) has been proved for this special case. We proceed to the general case. Let $p\colon \mathbb{D} \to D_0$ and $q\colon \mathbb{D} \to D$ be holomorphic universal covering projections of $\mathbb{D}$ onto D_0 and D, respectively. We take a lift $\tilde{\omega}$ of $\omega \circ p$ via the projection q. That is, a holomorphic map $\tilde{\omega}\colon \mathbb{D} \to \mathbb{D}$ satisfies $\omega\circ p = q\circ\tilde{\omega}$. Then, using (a) and the special case of (c), we obtain

$$V_{D_0}(\varphi\circ\omega) = V_{\mathbb{D}}(\varphi\circ\omega\circ p) = V_{\mathbb{D}}(\varphi\circ q\circ\tilde{\omega}) \le V_{\mathbb{D}}(\varphi\circ q) = V_D(\varphi).$$

Property (d) is shown by applying (c) to the function $\omega = \psi^{-1}\circ\varphi\colon D \to D$. □

The following result ensures that the inequality $V_{\mathbb{D}}(\varphi) \le 4$ holds for any nonvanishing *univalent* function φ on $\mathbb{D}$.

PROPOSITION 2.3. *Let* φ *be a nonvanishing univalent function in the unit disk. Then*

$$(1-|z|^2)\left|\frac{\varphi'(z)}{\varphi(z)}\right| \le 4,$$

where equality holds at $z = z_0$ *if and only if* (i) $\mathbb{C}\setminus\varphi(\mathbb{D})$ *is a ray emanating from the origin and* (ii) *the value* $\varphi(z_0)$ *lies in the line containing the ray.*

Proof. By the conformal invariance of the quantity $(1-|z|^2)|\varphi'(z)/\varphi(z)|$ (see the proof of Theorem 2.2(c)), it suffices to show the claimed inequality at the origin: $|\varphi'(0)/\varphi(0)| \le 4$. Then $f(z) = (\varphi(z)-\varphi(0))/\varphi'(0)$ is a normalized univalent function in $|z|<1$. The Koebe one-quarter theorem now implies that $f(\mathbb{D})$ contains the disk $\{|w| < 1/4\}$. On the other hand, by assumption the function f omits the value $-\varphi(0)/\varphi'(0)$, so $|\varphi(0)/\varphi'(0)| \ge 1/4$ and equality holds if and only if f is a rotation of the Koebe function $K(z) = z/(1-z)^2$ (see [7, p. 31]). Now the assertion follows. □

REMARKS. (1) Proposition 2.3 can also be deduced directly from Macintyre's inequality [18] (see also [30, pp. 102, 112]). This was pointed out to the authors by Shinji Yamashita.

(2) On the other hand, the proof given here is the same as that of the well-known estimate $\rho_D(z)\delta_D(z) \ge 1/4$ for a simply connected domain D, where $\delta_D(z) = \operatorname{dist}(z, \partial D)$. Actually, the quantity V_D has the following geometric meaning. Let

$\hat{\rho}_D(z)|dz|$ denote the Hahn metric of the domain D. Minda [19] describes some fundamental properties of the Hahn metric as follows.

(i) If $f\colon D \to D'$ is holomorphic and injective, then $\hat{\rho}_{D'}(f(z))|f'(z)| \le \hat{\rho}_D(z)$.
(ii) If D is simply connected, then $\hat{\rho}_D = \rho_D$.
(iii) For the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we know that $\hat{\rho}_{\mathbb{C}^*}(z) = 1/(4|z|)$; in particular, the quantity $V_D(\varphi)$ for $\varphi\colon D \to \mathbb{C}^*$ has the expression

$$V_D(\varphi) = 4 \sup_{z\in D} \frac{\hat{\rho}_{\mathbb{C}^*}(\varphi(z))|\varphi'(z)|}{\rho_D(z)} = 4 \sup_D \frac{\varphi^*(\hat{\rho}_{\mathbb{C}^*})}{\rho_D}.$$

Proposition 2.3 is therefore nothing but an expression of the decreasing property of the Hahn metric under univalent maps, $\varphi^*(\hat{\rho}_{\mathbb{C}^*}) \le \hat{\rho}_D = \rho_D$; hence it may be viewed as a corollary of the aforementioned results due to Minda [19].

A holomorphic function $g\colon \mathbb{D} \to \mathbb{C}$ is called *Gelfer* if $g(z) + g(w) \ne 0$ for any pair of points $z, w \in \mathbb{D}$. In particular, a Gelfer function is always nonvanishing. (Note that here we drop the usual normalization condition $g(0) = 1$.)

As a corollary of Proposition 2.3, we can show the following result on Gelfer functions; this result was first shown by Gelfer in [9] by means of a known result on Bieberbach–Eilenberg functions, and it was effectively used by Yamashita in [31]. We could not find a short account of its proof in the literature other than the original article by Gelfer and thus could not attribute the conditions for equality to anyone. We therefore include a simple proof of this for the convenience of the reader.

THEOREM 2.4 (Gelfer [9]; see also [29]). *For a Gelfer function g,*

$$(1 - |z|^2)\left|\frac{g'(z)}{g(z)}\right| \le 2$$

for $|z| < 1$, where equality holds at $z = z_0$ precisely when (a) *g maps the unit disk univalently onto a half-plane H whose boundary contains the origin and* (b) *the orthogonal projection of the point $g(z_0)$ to ∂H is equal to the origin.*

Proof. Let g be a Gelfer function. Without loss of generality, we may assume that $g(0) = 1$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We set $f(z) = g(z)^2$; then the unique unbounded component C of $\hat{\mathbb{C}} \setminus f(\mathbb{D})$ connects the origin and the point at infinity. Thus $D = \hat{\mathbb{C}} \setminus C$ is a simply connected domain in $\mathbb{C}^*$ with $1 \in D$. (For this part, see also [8, Thm. 8] or [24, Lemma].) Now let $\varphi\colon \mathbb{D} \to D$ be a conformal map. By the Schwarz–Pick lemma and Proposition 2.3, we can see that

$$(1 - |z|^2)\left|\frac{f'(z)}{f(z)}\right| = (1 - |z|^2)|\omega'|\left|\frac{\varphi'(\omega)}{\varphi(\omega)}\right| \le (1 - |\omega|^2)\left|\frac{\varphi'(\omega)}{\varphi(\omega)}\right| \le 4,$$

where $\omega = \varphi^{-1} \circ f$. Here $(1 - |z_0|^2)|f'(z_0)/f(z_0)| = 4$ holds at the point z_0 if and only if f maps $\mathbb{D}$ univalently onto the complex plane minus a ray emanating from the origin and $f(z_0)$ lies in the line containing this ray. Since $f'(z)/f(z) = 2g'(z)/g(z)$, the desired statement now follows. □

In view of this proof, we also have the next result, which is a generalization of Proposition 2.3.

PROPOSITION 2.5. *Let f be a nonvanishing holomorphic function on the unit disk $\mathbb{D}$ such that the image $f(\mathbb{D})$ does not separate the origin from the point at infinity. Then the inequality $(1-|z|^2)|f'(z)/f(z)| \le 4$ follows, and equality holds at some point if and only if f maps $\mathbb{D}$ conformally onto the complex plane minus a ray emanating from the origin.*

3. Circular Width of a Proper Subdomain of $\mathbb{C}^*$

Let Ω be a hyperbolic plane domain with $0 \in \mathbb{C} \setminus \Omega$. The quantity

$$W(\Omega) = \Big(\inf_{w\in\Omega} |w|\rho_\Omega(w)\Big)^{-1}$$

will be called the *circular width* of Ω (about the origin). In general, it is not easy to compute the values of the density $\rho_\Omega(w)$ of the hyperbolic metric of Ω. Therefore, another expression of $W(\Omega)$ is often useful.

LEMMA 3.1. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$, and let p be an analytic (unbranched) covering projection of a domain D onto Ω. Then $W(\Omega) = V_D(p)$.*

Proof. First we note that the circular width of Ω can be written in the form $W(\Omega) = V_\Omega(\mathrm{id})$. Theorem 2.2(a) then implies the relation $V_\Omega(\mathrm{id}) = V_D(p)$. $\square$

We now collect basic properties of the circular width. Before doing so, we recall the notion of circular symmetrization. For a subdomain Ω of $\mathbb{C}^*$ we define the circular symmetrization Ω^* (about the origin) by

$$\Omega^* = \{re^{i\theta} : \theta \in I(r,\Omega),\ 0 < r < \infty\},$$

where $I(r,\Omega)$ denotes the interval of the form $(-t/2, t/2)$ with the same length as $I_r = \{\theta \in [-\pi,\pi] : re^{i\theta} \in \Omega\}$ if $I_r \ne [-\pi,\pi]$; otherwise, $I(r,\Omega) = [-\pi,\pi]$.

THEOREM 3.2. *Let Ω and Ω' be proper subdomains of the punctured plane $\mathbb{C}^*$.*

(i) *$W(\Omega) = W(L(\Omega))$ for any conformal automorphism L of $\mathbb{C}^*$.*
(ii) *If $\Omega \subset \Omega'$ then $W(\Omega) \le W(\Omega')$.*
(iii) *Circular symmetrization does not decrease circular width; $W(\Omega) \le W(\Omega^*)$.*
(iv) *If Ω is simply connected, then $W(\Omega) \le 4$.*

Proof. In view of the formula $W(\Omega) = V_\Omega(\mathrm{id})$, we can deduce (i) and (ii) from parts (b) and (d), respectively, of Theorem 2.2. Part (iii) lies much deeper. We will employ Weitsman's theorem [28]: $\rho_\Omega(w) \ge \rho_{\Omega^*}(|w|)$. Then

$$\frac{1}{W(\Omega)} = \inf_{w\in\Omega} |w|\rho_\Omega(w) \ge \inf_{w\in\Omega} |w|\rho_{\Omega^*}(|w|) \ge \inf_{w\in\Omega^*} |w|\rho_{\Omega^*}(w) = \frac{1}{W(\Omega^*)},$$

which proves (iii). Part (iv) follows from the Koebe one-quarter theorem and this also follows from (ii) and (iii). Indeed, if Ω is simply connected then the symmetrized domain Ω^* is contained in the slit domain $\Omega_1 = \mathbb{C} \setminus (-\infty, 0]$. A simple computation gives $W(\Omega_1) = 4$ (see Example 5.1). Thus (ii) and (iii) now yield $W(\Omega) \le W(\Omega^*) \le W(\Omega_1) = 4$. □

In general, the circular width may not be finite. We give here a characterization of domains with infinite circular width. In particular, if the origin is an isolated boundary point of Ω then $W(\Omega) = \infty$.

PROPOSITION 3.3. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$. The circular width $W(\Omega)$ is infinite if and only if there is a sequence of annuli $A_n = \{w \in \mathbb{C} : r_n < |w| < R_n\}$ with $A_n \subset \Omega$ such that $R_n/r_n \to \infty$.*

As a preparation, we first show the following.

LEMMA 3.4. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ and set*

$$M(\Omega) = \sup_{w\in\Omega} \inf_{b\in\mathbb{C}\setminus\Omega} \left|\log\left|\frac{w}{b}\right|\right|.$$

Then the inequalities

$$\frac{4}{\pi} M(\Omega) \le W(\Omega) \le 2M(\Omega) + C$$

hold, where $C = \Gamma(1/4)^4/(2\pi^2) \approx 8.7538$.

Proof. For the proof we shall need the following estimate (see [26, Thm. 1.5]):

$$\frac{1}{2m(w) + C} \le |w|\rho_\Omega(w) \le \frac{\pi}{4m(w)}, \quad w \in \Omega,$$

where

$$m(w) = \inf_{b\in\mathbb{C}\setminus\Omega} \left|\log\left|\frac{w}{b}\right|\right|.$$

By the definitions of $W(\Omega)$ and $M(\Omega)$, the required inequalities now follow. □

Proof of Proposition 3.3. By Lemma 3.4, $W(\Omega) = \infty$ if and only if $M(\Omega) = \infty$. Suppose that $A = \{r < |w| < R\} \subset \Omega$. Then $m(w) \ge (1/2)\log(R/r)$ for $|w| = \sqrt{rR}$. Therefore, we have $M(\Omega) \ge (1/2)\,\overline{\lim}\log(R_n/r_n) = \infty$ if $A_n = \{r_n < |w| < R_n\} \subset \Omega$ satisfies $R_n/r_n \to \infty$. Conversely, assume that $W(\Omega) = \infty$; equivalently, $M(\Omega) = \infty$. Then there exists a sequence w_n such that $m_n = m(w_n) \to \infty$ as $n \to \infty$. Hence the annulus $A_n = \{e^{-m_n}|w_n| < |w| < e^{m_n}|w_n|\}$ does not meet $\mathbb{C} \setminus \Omega$ by the definition of the function m and so $A_n \subset \Omega$. It is evident that the sequence A_n is what we wanted. □

The circular width may not behave continuously in Ω. For instance, consider the sequence of domains $\Omega_n = \{w \in \mathbb{C}^* : |w - 1| < 1 + 1/n\}$. Then Ω_n converges to $\Omega_\infty = \{w \in \mathbb{C}^* : |w - 1| < 1\}$ in the Hausdorff topology. Since the origin is an isolated boundary point of Ω_n, by Proposition 3.3 it follows that $W(\Omega_n) = \infty$. On the other hand, $W(\Omega_\infty) = 2$ (see Example 5.4). Therefore, $W(\Omega_n)$ does not converge to $W(\Omega)$ in this case. We can, however, show a continuity property of circular width in the following form.

PROPOSITION 3.5. *Let Ω_n be a sequence of domains with $\Omega_n \subset \Omega_{n+1}$ such that the union $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ is a proper subdomain of $\mathbb{C}^*$. Then $W(\Omega_n) \to W(\Omega)$ as $n \to \infty$.*

Proof. By the monotonicity of circular width (Theorem 3.2(ii)),

$$W(\Omega_1) \le W(\Omega_2) \le \cdots \le W(\Omega);$$

therefore, $\lim_{n\to\infty} W(\Omega_n) \le W(\Omega)$. On the other hand, for any number $m < W(\Omega)$ we can find a point $w_0 \in \Omega$ such that $|w_0|\rho_\Omega(w_0) < 1/m$. Since $\rho_{\Omega_n}(w_0) \to \rho_\Omega(w_0)$ (see e.g. [12, Thm. 1]), we obtain

$$\frac{1}{m} > \lim_{n\to\infty} |w_0|\rho_{\Omega_n}(w_0) \ge \lim_{n\to\infty} W(\Omega_n)^{-1}.$$

Because m was arbitrary as far as $m < W(\Omega)$, we now obtain

$$W(\Omega)^{-1} \ge \lim_{n\to\infty} W(\Omega_n)^{-1},$$

that is, $\lim_{n\to\infty} W(\Omega_n) \ge W(\Omega)$. The proof is now complete. □

The circular width $W(\Omega)$ dominates the quantity $V_D(\varphi)$ for holomorphic maps $\varphi\colon D \to \Omega$.

THEOREM 3.6. *Let Ω be a proper subdomain of $\mathbb{C}^*$ and let $\varphi\colon D \to \Omega$ be holomorphic. Then $V_D(\varphi) \le W(\Omega)$.*

Proof. By Theorem 2.2(c), $V_D(\varphi) = V_D(\mathrm{id}_\Omega \circ \varphi) \le V_\Omega(\mathrm{id}_\Omega) = W(\Omega)$. □

Combining this with Proposition 2.1 yields the following statement, which is a slight generalization of a result of Zheng [32].

COROLLARY 3.7. *Under the same hypotheses as in Theorem 3.6,*

$$\exp\{-W(\Omega)d_D(z_0, z_1)\} \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\{W(\Omega)d_D(z_0, z_1)\}, \quad z_0, z_1 \in D.$$

We remark that a similar result can be obtained by applying the Harnack inequality to the harmonic function $\log|\varphi|$ on D. The latter idea is even efficient for quasiregular mappings in higher-dimensional Euclidean space (see [27, Sec. 13]).

4. Connection with Uniform Perfectness

In general, we can define the circular width $W_a(D)$ of a hyperbolic domain D about a point $a \in \mathbb{C} \setminus D$ by

$$W_a(D) = \frac{1}{\inf_{z\in D}|z-a|\rho_D(z)}.$$

Note that this can also be written as $W_a(D) = V_D(\tau_a)$, where $\tau_a(z) = z - a$. It is known that the domain constant

$$C(D) = \sup_{a \in \partial D} W_a(D) = \sup_{z \in D} \frac{1}{\delta_D(z)\rho_D(z)}$$

is finite if and only if the set ∂D is uniformly perfect (see e.g. [22] or [25]). Here we recall that $\delta_D(z) = \operatorname{dist}(z, \partial D)$. In this context, the constant $W_a(D)$ appeared essentially in a paper by Zheng [32]. We remark that ∂D may be replaced by the complement of D in the equation without any essential change. The constant $C(D)$ or, equivalently, the constant $c(D) = 1/C(D)$ has been studied by many authors (see e.g. [11; 16; 25; 30]). Observe that $C(D) \geq 2$ holds for an arbitrary hyperbolic domain D, with equality if and only if D is convex [11, Thm. 4].

Let us introduce a variant of the quantity $V_D(\varphi)$. For a nonvanishing analytic function φ on D, set

$$\hat{V}_D(\varphi) = \sup_{z \in D} \delta_D(z) \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$

Since $\rho_D(z)\delta_D(z) \leq 1$ for $z \in D$, we have $\hat{V}_D(\varphi) \leq V_D(\varphi)$. Let $N(D)$ be the least number such that

$$V_D(\varphi) \leq N(D)\hat{V}_D(\varphi)$$

holds for every holomorphic map $\varphi \colon D \to \mathbb{C}^*$. If there is no such number then we set $N(D) = +\infty$. It is interesting to observe that the quantities $C(D)$ and $N(D)$ are equal.

PROPOSITION 4.1. *Let D be a hyperbolic plane domain. Then $C(D) = N(D)$ holds. In particular, ∂D is uniformly perfect if and only if $N(D) < \infty$.*

Proof. Since $\rho_D^{-1} \leq C(D)\delta_D$, the inequality $N(D) \leq C(D)$ is trivial. We now show $C(D) \leq N(D)$. First we note the simple fact that $\hat{V}_D(\tau_a) \leq 1$ holds for each $a \in \mathbb{C} \setminus D$. Then we apply the inequality $V_D(\varphi) \leq N(D)\hat{V}_D(\varphi)$ to the function $\varphi = \tau_a$ to obtain $W_a(D) = V_D(\tau_a) \leq N(D)$ for $a \in \partial D$. Taking the supremum over a, we obtain $C(D) \leq N(D)$. □

As a simple application of this proposition we give a proof of Osgood's theorem. In order to state it, we introduce the domain constant

$$U(D) = \sup_f V_D(f'),$$

where the supremum is taken over all univalent analytic functions f on D. Note that $V_D(f')$ is nothing but the hyperbolic sup-norm of the pre-Schwarzian derivative f''/f' of f. Osgood's theorem [21, Thm. 2] states that ∂D is uniformly perfect if and only if $U(D) < \infty$. In view of his proof, a quantitative form can be presented in the following way.

THEOREM 4.2 (Osgood). *Let D be a hyperbolic plane domain. Then*

$$2C(D) \leq U(D) \leq 4C(D).$$

Proof. First we show the inequality $2C(D) \le U(D)$. For $a \in \partial D$, consider the function $f_a(z) = 1/(a-z)$. It is clear that f_a is univalent analytic on D; in particular, $V_D(f_a') \le U(D)$. Since $f_a' = \tau_a^{-2}$, the relation (2.2) implies that $V_D(f_a') = 2V_D(\tau_a) = 2W_a(D)$. Thus $2W_a(D) \le U(D)$, from which the required inequality follows.

We now show the inequality $U(D) \le 4C(D)$. Let $f\colon D \to \mathbb{C}$ be univalent and analytic. Then the sharp inequality $\hat{V}_D(f') \le 4$ holds (see [21, Lemma 1]) and so $U(D) \le 4N(D)$. Now we employ Proposition 4.1 to obtain $U(D) \le 4C(D)$. □

5. Computations of Circular Widths

In this section we give exact values of circular width for several concrete examples. These will be useful to give upper bounds of circular width for various domains. In view of Theorem 3.2(iii), we see that circularly symmetric domains are particularly important.

EXAMPLE 5.1 (Sectors). For $S(\beta) = \{w : |\arg w| < \pi\beta/2\}$, $0 < \beta \le 2$, we have $W(S(\beta)) = 2\beta$.

Indeed, by Theorem 2.4 we have $W(\mathbb{H}) = 2$. Since $S(\beta) = \varphi_\beta(\mathbb{H})$ for $\varphi_\beta(z) = z^\beta$, it follows from (2.2) that

$$W(S(\beta)) = V_{\mathbb{H}}(\varphi_\beta) = |\beta| V_{\mathbb{H}}(\mathrm{id}) = \beta \cdot W(\mathbb{H}) = 2\beta.$$

We remark that this computation remains valid even when β is a complex number. It is easy to see that φ_β is univalent in $\mathbb{H}$ if $|\beta - 1| \le 1$ and $\beta \ne 0$; hence we have also $W(\varphi_\beta(\mathbb{H})) = 2|\beta|$ for such a β. Note that $\varphi_\beta(\mathbb{H})$ is a Jordan domain bounded by two logarithmic spirals ending at 0 and ∞ when $|\beta - 1| < 1$.

EXAMPLE 5.2 (Half-sectors). Let $S(\beta, r) = \{w : |\arg w| < \pi\beta/2, |w| < r\}$ and let $S'(\beta, r) = \{w : |\arg w| < \pi\beta/2, |w| > 1/r\}$ for $0 < \beta \le 2$ and $0 < r < \infty$. Then $W(S(\beta, r)) = W(S'(\beta, r)) = 2\beta$.

Because circular width is invariant under dilations, we can see that $W(S(\beta, r)) = W(S(\beta, 1))$. On the other hand, by Proposition 3.5 we have $\lim_{r\to\infty} W(S(\beta, r)) = W(S(\beta)) = 2\beta$. Thus we obtain $W(S(\beta, r)) = W(S(\beta, 1)) = 2\beta$. For the other case, the same computation works.

It is interesting to see that an arbitrary domain Ω with $S(\beta, r) \subset \Omega \subset S(\beta)$ for some $r > 0$ has circular width 2β because $2\beta = W(S(\beta, r)) \le W(\Omega) \le W(S(\beta)) = 2\beta$ by monotonicity.

EXAMPLE 5.3 (Annuli). For the annulus $A(r, R) = \{w : r < |w| < R\}$ with $0 < r < R < \infty$, we have $W(A(r, R)) = (2/\pi)\log(R/r)$.

For the proof we may assume that $R = e^m$ and $r = e^{-m}$ for some $m > 0$. Then the mapping $\varphi(z) = \exp\{(2mi/\pi)\log z\} = z^{2mi/\pi}$ gives an analytic universal covering projection of the right half-plane $\mathbb{H}$ onto $A(r, R)$. Thus the same computation as in Example 5.1 gives $W(A(r, R)) = V_{\mathbb{H}}(\varphi) = 2|2mi/\pi| = 4m/\pi = (2/\pi)\log(R/r)$.

EXAMPLE 5.4 (Disks). Let $\mathbb{D}(a, r) = \{w : |w - a| < r\}$ for $0 < r \le a$. Then

$$W(\mathbb{D}(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

Let $\varphi(z) = a + rz$. Then

$$W(\mathbb{D}(a, r)) = V_{\mathbb{D}}(\varphi) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{r}{|a + rz|} = \sup_{0 \le x < 1} \frac{r(1 - x^2)}{a - rx}.$$

Since $r(1 - x^2)/(a - rx)$ takes its maximum at $x = \left(a - \sqrt{a^2 - r^2}\right)/r$, we obtain the required expression of $W(\mathbb{D}(a, r))$.

Observe that $W(\mathbb{D}(a, a)) = 2$ for $a > 0$. Since circular width is invariant under the inversion $z \mapsto 1/z$ (see Theorem 3.2(i)), we also obtain $W(H) = 2$ for the half-plane $H = \{w : \operatorname{Re} w > b\}$ when $b = 1/(2a) > 0$.

EXAMPLE 5.5 (Parallel Strips). Let $P(a, b) = \{w : a < \operatorname{Re} w < b\}$ for $0 \le a < b < \infty$. Then

$$W(P(a, b)) = \max_{0 \le \theta \le \pi/2} \frac{2t\cos\theta}{1 - t\theta},$$

where t is a number with $0 < t \le 2/\pi$ determined by

$$\frac{\pi t}{2} = \frac{b - a}{b + a}.$$

Note that the function $\varphi(z) = 1 + it\log z$ maps the right half-plane $\mathbb{H}$ onto the parallel strip $P(1 - \pi t/2, 1 + \pi t/2)$. Therefore, if we choose t as before then this strip is similar to $P(a, b)$ and thus they have the same circular width. Writing $z = re^{i\theta}$, we may compute

$$\begin{aligned} W(P(a, b)) = V_{\mathbb{H}}(\varphi) &= \sup_{z \in \mathbb{H}} 2\operatorname{Re} z \frac{t/|z|}{|1 + it\log z|} \\ &= \sup_{0 < r < \infty,\, -\pi/2 < \theta < \pi/2} \frac{2t\cos\theta}{|1 - t\theta + it\log r|} \\ &= \sup_{-\pi/2 < \theta < \pi/2} \frac{2t\cos\theta}{1 - t\theta}. \end{aligned}$$

Clearly we can discard the case $\theta < 0$, and thus we have the required form.

We remark that these parallel strips are *not* circularly symmetric.

EXAMPLE 5.6 (Truncated Wedges). Let $S(\beta, r, R) = \{w : |\arg w| < \pi\beta/2, r < |w| < R\}$ for $0 < \beta \le 2$ and $0 < r < R < \infty$. Then

$$W(\Omega) = \frac{\log(R/r)}{(1 + t)\mathbf{K}(t)},$$

where

$$\mathbf{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the complete elliptic integral of the first kind and where $0 < t < 1$ is a number such that

$$\frac{\mathbf{K}\bigl(\sqrt{1-t^2}\bigr)}{\mathbf{K}(t)} = \frac{2\pi\beta}{\log(R/r)}.$$

Observe that the quantity $\mu(t) = (\pi/2)\mathbf{K}\bigl(\sqrt{1-t^2}\bigr)/\mathbf{K}(t)$ is the modulus of the Grötzsch ring $\mathbb{D} \setminus [0,t]$ for $0 < t < 1$ and is decreasing from $+\infty$ to 0 (see e.g. [1]). Therefore, we can always take such a t satisfying the displayed equality.

We set $K = \mathbf{K}(t)$ and $K' = \mathbf{K}\bigl(\sqrt{1-t^2}\bigr)$. Since the rectangles $Q_1 = (-K, K) \times (0, K')$ and $Q_2 = (\log r, \log R) \times (-\pi\beta/2, \pi\beta/2)$ are similar by the choice of t, there is a linear function $L(z) = az + b$ with $a > 0$ such that $L(Q_1) = Q_2$. Note that

$$a = \frac{\log(R/r)}{2K} = \frac{\pi\beta}{K'}. \tag{5.1}$$

It is well known that the function

$$F(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-t^2\zeta^2)}}$$

maps the upper half-plane H onto the rectangle Q_1. Therefore, the composed function $\varphi(z) = \exp\{aF(z) + b\}$ is a conformal map of H onto $S(\beta, r, R)$. We now have

$$W(S(\beta, r, R)) = V_H(\varphi) = \sup_{z\in H} 2\,\mathrm{Im}\, z \cdot a|F'(z)| = \sup_{z\in H} \frac{2a\,\mathrm{Im}\, z}{|(1-z^2)(1-t^2z^2)|^{1/2}}.$$

We write $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$. Then

$$\begin{aligned}
&|(1-z^2)(1-t^2z^2)|^2 - (1+t)^4y^4 \\
&\quad = (1 - x^2 - t^2x^2 + t^2x^4 - t^2y^4)^2 + 2(1+t^2)y^2(tx^2 + ty^2 - 1)^2 \\
&\qquad + 2x^2y^2[2t(1-t)^2 + (t^2x^2 + t^2y^2 - 1)^2 + t^4(x^2 + y^2 - 1)^2]
\end{aligned}$$

and thus

$$|(1-z^2)(1-t^2z^2)|^{1/2} \ge (1+t)y,$$

where equality holds when $x = 0$ and $ty^2 = 1$. Hence, in view of (5.1) we obtain

$$W(S(\beta, r, R)) = \frac{2a}{1+t} = \frac{\log(R/r)}{(1+t)K} = \frac{2\pi\beta}{(1+t)K'}.$$

Observe that the limiting case $S(\beta, 0, \infty) = S(\beta)$ corresponds to $t = 1^-$. Since $K' \to \pi/2$ as $t \to 1^-$, we reproduce the relation $W(S(\beta)) = 2\beta$. We also see that the other limiting case $S(\infty, r, R) = A(r, R)$ corresponds to $t = 0^+$. (We must regard $S(\beta, r, R)$ as an overlapped domain when $\beta > 2$.) Since $K \to \pi/2$ as $t \to 0^+$, we reproduce the relation $W(A(r, R)) = (2/\pi)\log(R/r)$.

We remark that essentially the same observations were made by Avhadiev and Aksent'ev [2] (see also Corollary 6.9 to follow), though they did not make systematic use of circular width.

We end the present section with a criterion for a subdomain of the right half-plane $\mathbb{H}$ to have circular width 2.

PROPOSITION 5.1. *Let Ω be a subdomain of $\mathbb{H}$. Suppose that, for each number $\beta \in (0,1)$, there is a number $\delta > 0$ such that $S(\beta,\delta) \subset \Omega$. Then $W(\Omega) = 2$.*

Proof. Since $\Omega \subset \mathbb{H}$, it follows that $W(\Omega) \le W(\mathbb{H}) = 2$. On the other hand, by assumption we have $W(\Omega) \ge W(S(\beta,\delta)) = 2\beta$ for each $\beta < 1$ (see Example 5.2). Thus we conclude that $W(\Omega) = 2$. $\square$

Obviously, we may replace $S(\beta,\delta)$ by $S'(\beta,\delta)$ in the assertion of this proposition.

As an example, if Ω contains a disk whose boundary contains the origin then $W(\Omega) \ge 2$ (see also Example 5.4).

6. Applications

In this section we give a few applications of circular width. More concrete applications can be found in [15] and [23].

Let us introduce some notation. For a locally univalent function f on $\mathbb{D}$, the quantity $T_f = f''/f'$ is called the *pre-Schwarzian derivative* of f and is measured by the norm

$$\|T_f\|_{\mathbb{D}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)|.$$

Note that this can be described by $\|T_f\|_{\mathbb{D}} = V_{\mathbb{D}}(f')$. Let $\mathcal{A}$ denote the class of holomorphic functions f on $\mathbb{D}$ normalized by $f(0)=0$, $f'(0)=1$.

THEOREM 6.1. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ with $W(\Omega) < 2$. If $f \in \mathcal{A}$ satisfies $f'(\mathbb{D}) \subset \Omega$, then $|f(z)| < M$ for $z \in \mathbb{D}$. Here M is a constant depending only on $W(\Omega)$.*

The assumption implies that $\|T_f\|_{\mathbb{D}} = V_{\mathbb{D}}(f') \le W(\Omega) < 2$ by Theorem 3.6. Though it is known that the condition $\|T_f\|_{\mathbb{D}} < 2$ implies boundedness of f (see [14]), we will give a proof for completeness.

Proof of Theorem 6.1. Set $\lambda = W(\Omega)/2$. By Corollary 3.7, we have $|f'(z)| \le \exp\{2\lambda d_{\mathbb{D}}(z,0)\}$. Since $d_{\mathbb{D}}(z,0) = \operatorname{arctanh}(|z|) = (1/2)\log((1+|z|)/(1-|z|))$, this inequality is equivalent to

$$|f'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{\lambda}.$$

Since $\lambda < 1$, the function $((1+x)/(1-x))^{\lambda}$ is integrable over $(0,1)$. Therefore,

$$|f(z)| \le \int_0^1 \left(\frac{1+x}{1-x}\right)^{\lambda} dx < 2^{\lambda}\int_0^1 (1-x)^{-\lambda}\,dx = \frac{2^{\lambda}}{1-\lambda} = \frac{2^{W(\Omega)/2}}{1-W(\Omega)/2}. \qquad \square$$

We remark that the preceding integral can be expressed by

$$\int_0^1 \left(\frac{1+x}{1-x}\right)^{\lambda} dx = \lambda\left[\psi\left(-\frac{\lambda}{2}\right) - \psi\left(\frac{1-\lambda}{2}\right)\right] - 1,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function (see [14]).

If $W(\Omega) \geq 2$ then there is no guarantee that f is bounded. For instance, consider the function $f(z) = -2\log(1-z) - z$. Though $f'(\mathbb{D}) \subset \mathbb{H}$ and $W(\mathbb{H}) = 2$, the function f is unbounded.

It may be interesting to find a characterization of such subdomains Ω of $\mathbb{H}$ whereby $f'(\mathbb{D}) \subset \Omega$ implies boundedness of $f \in \mathcal{A}$. Theorem 6.1 gives the sufficient condition $W(\Omega) < 2$ for such an implication. A similar problem was considered by MacGregor and Rønning [17], who tried to find conditions for subdomains Ω of $\mathbb{H}$ whereby $g'(z)/g(z) \in \Omega$ $(z \in \mathbb{D})$ implies boundedness of $\log|g(z)|$ for a nonvanishing locally univalent function g on $\mathbb{D}$. Letting $f = \log g$, we see that the latter conclusion is weaker than the former. In particular, the condition $W(\Omega) < 2$ is sufficient for MacGregor–Rønning's problem. Their conditions, however, are more refined because they cover also those cases where $W(\Omega) = 2$.

Note as well that the condition $f'(\mathbb{D}) \subset \mathbb{H}$ implies univalence of f (Noshiro–Warschawski theorem). Recently, Chuaqui and Gevirtz [6] gave a characterization of such subdomains Ω of $\mathbb{H}$ whereby $f'(\mathbb{D}) \subset \Omega$ implies quasiconformal extensibility of $f \in \mathcal{A}$.

We now consider sufficient conditions for proper subdomains Ω of $\mathbb{C}^*$ to satisfy $W(\Omega) < 2$, which implies a boundedness criterion by Theorem 6.1. Note that if $\Omega \subset \mathbb{H}$ then $W(\Omega) \leq 2$. Thus, the following result gives a sufficient condition for such Ω to have circular width less than 2. We also remind the reader that we have already given a sufficient condition for subdomains Ω of $\mathbb{H}$ to have circular width 2 (see Proposition 5.1).

Let $\tau_\Omega(r)$ denote half the length of the set $\{\theta \in [-\pi, \pi] : re^{i\theta} \in \Omega\}$. Then, by Theorem 3.2, $W(\Omega) \leq 2$ if $\tau_\Omega(r) \leq \pi/2$ for every $r > 0$. Furthermore, we have the following result.

THEOREM 6.2. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ with the property that $\tau_\Omega \leq \pi/2$ on $(0,\infty)$. If $\overline{\lim}_{r\to 0} \tau_\Omega(r) < \pi/2$ and if $\overline{\lim}_{r\to\infty} \tau_\Omega(r) < \pi/2$, then $W(\Omega) < 2$.*

Proof. Let Ω^* be the circular symmetrization of Ω. By Theorem 3.2(iii), we have $W(\Omega) \leq W(\Omega^*)$. Observe that Ω^* is contained in the right half-plane $\mathbb{H}$ by assumption.

For positive constants m and R, define the domains $\Omega_\infty(m,R)$ and $\Omega_0(m,R)$ by $\{w = u + iv : u > 0, |v| < mu + R\}$ and $\{1/w : w \in \Omega_\infty(m,R)\}$, respectively. By assumption, Ω^* is contained in the domain $\Omega' = \Omega_0(m,R) \cap \Omega_\infty(m,R)$ for sufficiently large m and R. Since $W(\Omega) \leq W(\Omega^*) \leq W(\Omega')$, it suffices to show $W(\Omega') < 2$.

Let $\psi : \mathbb{H} \to \Omega'$ be a conformal homeomorphism. Since Ω' is a Jordan domain, it follows by the Carathéodory extension theorem that ψ extends uniquely to a homeomorphism from $\overline{\mathbb{H}}$ onto $\overline{\Omega'}$. We may take ψ so that $\psi(0) = 0$ and $\psi(\infty) = \infty$. Consider the function

$$\Phi(z) = \operatorname{Re} z \left| \frac{\psi'(z)}{\psi(z)} \right|$$

in $\mathbb{H}$. Since $\psi((1+z)/(1-z))$ is Gelfer, Theorem 2.4 implies that $\Phi(z) < 2$ for every $z \in \mathbb{H}$. Therefore, in order to show $W(\Omega') < 2$, it is enough to show that $\overline{\lim}_{z\to\zeta} \Phi(z) < 2$ for each $\zeta \in \partial\Omega'$. Since ψ is symmetric, we may further assume that $\operatorname{Im} \zeta \geq 0$. Let ia and ib be the inverse images of i/R and iR (respectively) under the mapping ψ. We can see that the function $\psi'(z)/\psi(z)$ analytically extends to a holomorphic function across the boundary point iy for $y > 0$ except for $y = a, b$. Hence $\overline{\lim}_{z\to iy} \Phi(z) = 0$ for such y.

The case $y = b$ or $y = a$ requires more effort. First we note that the opening angle of Ω' at iR is $\pi\beta = \arctan m + \pi/2$. Therefore, $\varphi = (\psi - iR)^{1/\beta}$ extends to a conformal map around ib. In particular, $\varphi(z) = c(z - ib)(1 + o(1))$ and $\varphi'(z) = c(1 + o(1))$ as $z \to ib$, where $c = \varphi'(ib) \neq 0$. Since $\psi = iR + \varphi^\beta$, it follows that

$$\begin{aligned}
\Phi(z) &= \operatorname{Re}(z - ib) \frac{\beta|\varphi(z)|^{\beta-1}|\varphi'(z)|}{|iR + \varphi(z)^\beta|} \\
&\leq (1 + o(1))|z - ib| \frac{\beta|c|^\beta|z - ib|^{\beta-1}}{R} \\
&= (R^{-1} + o(1))\beta|c|^\beta|z - ib|^\beta = o(1)
\end{aligned}$$

as $z \to ib$. Considering $1/\psi$ instead of ψ, we can also see that $\Phi(z) = o(1)$ as $z \to ia$.

Finally, we examine the cases where $\zeta = 0$ and $\zeta = \infty$. We first claim that

$$\overline{\lim_{\mathbb{H}\ni z\to 0}} \Phi(z) \leq \frac{4}{\pi} \arctan m < 2.$$

In order to show this, we let $\alpha > (2/\pi)\arctan m$ and consider the function $h(z) = \psi(\delta z)$ in $D_0 = \{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ for $\delta > 0$. We can choose δ so small that $h(D_0)$ is contained in the sector $S = \{w : |\arg w| < \pi\alpha/2\}$. As we saw in Example 5.1, $W(S) = 2\alpha$. Therefore, by Theorem 3.6, $V_{D_0}(h) \leq W(S) = 2\alpha$. Note that the function $f(z) = \left(\frac{1+iz}{1-iz}\right)^2$ maps the right half-disk D_0 conformally onto the upper half-plane. A direct computation shows that the hyperbolic density $\rho_{D_0}(z) = |f'(z)|/2 \operatorname{Im} f(z)$ satisfies the equality $\rho_{D_0}(z)^{-1} = 2 \operatorname{Re} z + O(|z|^2)$ as $z \to 0$ in D_0. Hence,

$$\overline{\lim_{z\to 0}} \Phi(z) = \overline{\lim_{z\to 0}} \operatorname{Re} \delta z \left| \frac{\psi'(\delta z)}{\psi(\delta z)} \right| = \overline{\lim_{z\to 0}} \frac{1}{2} \rho_{D_0}(z)^{-1} \left| \frac{h'(z)}{h(z)} \right| \leq \frac{1}{2} V_{D_0}(h) \leq \alpha.$$

Because α is arbitrary provided $\alpha > (2/\pi)\arctan m$ is satisfied, we have now proved our initial claim. Considering $1/\psi$ yields the same inequality when $z \to \infty$ in $\mathbb{H}$. $\square$

We next apply Theorem 3.6 to the problem of quasiconformal extensibility. Our result is based on the following theorem due to Becker (for sharpness see Becker and Pommerenke [5]).

THEOREM 6.3 (Becker [4]). *Let $f \in \mathcal{A}$ be locally univalent. If $\|T_f\|_{\mathbb{D}} \le 1$ then f is univalent. Furthermore, if $\|T_f\| \le k$ for $k \in [0,1)$ then f has a K-quasiconformal extension to the whole plane, where $K = (1+k)/(1-k)$.*

We are now in a position to show the following result.

THEOREM 6.4. *Suppose that a proper subdomain Ω of the punctured plane $\mathbb{C}^*$ satisfies $W(\Omega) \le k$ for some $k \le 1$. If $f'(\mathbb{D}) \subset \Omega$ for $f \in \mathcal{A}$, then f is univalent and, moreover, f has a K-quasiconformal extension to the whole plane when $K = (1+k)/(1-k) < \infty$.*

See [23] for a counterpart to this theorem for meromorphic functions.

Proof of Theorem 6.4. As already noted, the condition $f'(\mathbb{D}) \subset \Omega$ implies that $\|T_f\|_{\mathbb{D}} \le W(\Omega) \le k$. We now apply Theorem 6.3 to deduce the assertions. □

Combining Theorem 6.4 with examples presented in Section 5, we obtain a series of corollaries. (Remember that circular width is invariant under rotations.) Note that, since most domains are contained in half-planes, the univalence assertion is implied by the Noshiro–Warschawski theorem in those cases.

The first corollary was noted by Avhadiev and Aksent'ev [3, pp. 33–34] for the case $\gamma = 0$.

COROLLARY 6.5. *Let $0 < k \le 1$ and $f \in \mathcal{A}$. If $|\arg f'(z) - \gamma| < \pi k/4$ in $|z| < 1$ for some real constant γ, then f is univalent and, moreover, f extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Observe that the condition $|\arg f'(z)| < M$, $|z| < 1$, implies quasiconformal extensibility of f when $M < \pi/2$ (see [6]).

COROLLARY 6.6. *Let k, r, R be positive numbers with $0 < \log(R/r) \le \pi k/2$ for $k \le 1$ and let $f \in \mathcal{A}$. If $r < |f'(z)| < M$ for $|z| < 1$, then f is univalent and, moreover, f extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

This sort of univalence criterion was first given by John [13]. The greatest number $\gamma > 1$ such that $1 < |f'(z)| < \gamma$ for $|z| < 1$ implies univalence of f is called the *John constant*. He proved that $\log \gamma \ge \pi/2$; Gevirtz [10] showed that $\log \gamma < 0.6279\pi$.

COROLLARY 6.7. *Let $k \in (0,1)$, $a \in \mathbb{C}$, and $r > 0$ with $r \le |a|$ and $2r \le k\left(|a| + \sqrt{|a|^2 - r^2}\right)$. If $f \in \mathcal{A}$ satisfies $|f'(z) - a| < r$ in $|z| < 1$, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Note that the inequality $|a - 1| < r$ must be satisfied under the assumptions of Corollary 6.7 because $f'(0) = 1$.

Corollary 6.8. *Let a, b, k be positive numbers with $0 \leq k \leq 1$ such that $2t\cos\theta \leq k(1-t\theta)$ for all $0 \leq \theta \leq \pi/2$, where $t = (2/\pi)(b-a)/(b+a)$. If $f \in \mathcal{A}$ admits the inequality $a < \mathrm{Re}(e^{i\gamma} f'(z)) < b$ in $|z| < 1$ for some real constant γ, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Corollary 6.9. *Let k, r, R, β be positive numbers with*

$$0 < \frac{\log(R/r)}{(1+t)\mathbf{K}(t)} \leq k \leq 1,$$

where t is as in Example 5.6, and let $f \in \mathcal{A}$. If $|\arg f'(z) - \gamma| < \pi\beta/2$ and $r < |f'(z)| < R$ in $|z| < 1$ for some real constant γ, then f is univalent and extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.

We remark that this last result was first shown by Avhadiev and Aksent'ev [2] (see also [3, Thm. 34]) for the case $\gamma = 0$. Related results are also given by Minda and Wright [20].

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