Explicit Versions of the
Briançon–Skoda Theorem with Variations

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1. Introduction

Let ϕ and $f_1, \ldots, f_m$ be holomorphic functions in a neighborhood of the origin in $\mathbb{C}^n$. The Briançon–Skoda theorem [9] states that $\phi^{\min(n,m)}$ belongs to the ideal (f) generated by f_j if $|\phi| \leq C|f|$. This condition is equivalent to ϕ belonging to the integral closure of the ideal (f). The original proof is based on Skoda's L^2-estimates in [20] (see Remark 1 in this section) and actually gives the stronger statement that $\phi \in (f)$ if $|\phi| \leq C|f|^{\min(n,m)}$. An explicit proof based on Berndtsson's division formula [8] and multivariable residue calculus appeared in [5]; see also [14] for a special case. There are purely algebraic versions in more arbitrary rings due to Lipman and Teissier [17].

In general this result cannot be improved, but for certain tuples f_j a much weaker size condition on ϕ is enough to guarantee that ϕ belongs to (f). For instance, the ideal $(f)^2$ is generated by the $m(m+1)/2$ functions $g_{jk} = f_j f_k$; we have $|f|^2 \sim |g|$, so applying the previous result yields $\phi \in (f)^2$ if $|\phi| \leq C|f|^{\min(2n, m(m+1))}$. However, in this case actually the power $\min(n, m) + 1$ is enough. In general we have the following statement.

THEOREM 1.1 (Briançon–Skoda). *If $f = (f_1, \ldots, f_m)$ and ϕ are holomorphic at 0 in $\mathbb{C}^n$ and if $|\phi| \leq C|f|^{\min(m,n)+r-1}$, then $\phi \in (f)^r$.*

This more general formulation follows in a manner similar to the case $r = 1$ by L^2-methods as well as by (a small modification of) the argument in [5]. In [1] we gave a somewhat different proof of the case $r = 1$ by means of residue calculus, and in this paper we extend that method to achieve various related results for product ideals as well as the general case of Theorem 1.1. We consider several possibly different tuples in the first result, as follows.

THEOREM 1.2. *Let f_j ($j = 1, \ldots, r$) be m_j-tuples of holomorphic functions at $0 \in \mathbb{C}^n$, and assume that*

$$|\phi| \leq C|f_1|^{s_1} \cdots |f_r|^{s_r}$$

for all s such that $s_1 + \cdots + s_r \leq n + r - 1$ and $1 \leq s_j \leq m_j$. Then $\phi \in (f_1) \cdots (f_r)$.

Notice that this immediately implies Theorem 1.1 in the case $m \geq n$ by simply choosing all $f_j = f$. In certain cases Theorem 1.2 can be improved, as one can see by taking $f_j = f$ and $m < n$ and then comparing with Theorem 1.1. Another case is when all the functions in the various tuples f_j together form a regular sequence.

THEOREM 1.3.   *Let f_j ($j = 1, \ldots, m$) be m_j-tuples of holomorphic functions at $0 \in \mathbb{C}^n$, and assume that the codimension of $\{f_1 = \cdots = f_r = 0\}$ is $m_1 + \cdots + m_r$. If*

$$|\phi| \leq C \min(|f_1|^{m_1}, \ldots, |f_r|^{m_r}),$$

then $\phi \in (f_1) \cdots (f_r)$.

We have not seen these latter two results in the literature, although they might belong to the folklore. In the algebraic setting there are several results related to the Briançon–Skoda (and Lipman–Teissier) theorem (see e.g. [16; 22] and the references therein).

REMARK 1.   The Briançon–Skoda theorem follows by a direct application of Skoda's L^2-estimate [20; 21] if $m \leq n$. In fact, if ψ is any plurisubharmonic function then the L^2-estimate guarantees a holomorphic solution to $f \cdot u = \phi$ such that

$$\int_{X \setminus Z} \frac{|u|^2}{|f|^{2(\min(m,n+1)-1+\varepsilon)}} e^{-\psi} \, dV < \infty,$$

provided that

$$\int_{X \setminus Z} \frac{|\phi|^2}{|f|^{2(\min(m,n+1)+\varepsilon)}} e^{-\psi} \, dV < \infty.$$

If $|\phi| \leq C|f|^m$ then the second integral is finite (taking $\psi = 0$) provided ε is small enough, and thus Skoda's theorem provides the desired solution. The case when $r > 1$ is obtained by iteration. If $m > n$ then a direct use of the L^2-estimate will not give the desired result. However, in this case one can find an n-tuple $\tilde{f}$ such that $(\tilde{f}) \subset (f)$ and $|\tilde{f}| \sim |f|$ (see [11]), and the theorem then follows by applying the L^2-estimate to the tuple $\tilde{f}$.

   In the same way, Theorem 1.2 can easily be proved from the L^2-estimate if $m_1 + \cdots + m_r \leq n + r - 1$. To see this, assume for simplicity that $r = 2$ and that $|\phi| \leq C|f_1|^{m_1}|f_2|^{m_2}$. Choosing $\psi = 2(m_1 + \varepsilon) \log|f_1|$, Skoda's theorem gives a solution to $f_2 \cdot u = \phi$ such that

$$\int_{X \setminus Z} \frac{|u|^2}{|f_1|^{2(m_1+\varepsilon)}} \, dV < \infty.$$

Another application then gives v_j such that $f_1 \cdot v_j = u_j$. This means that ϕ belongs to $(f_1)(f_2)$. However, we do not know whether one can derive Theorem 1.3 from the L^2-estimate when $m_1 + \cdots + m_r > n + r - 1$.

Now consider an $r \times m$ matrix f_j^k of holomorphic functions, $r \leq m$, with rows $f_1, \ldots, f_r$. We let F be the $m!/(m-r)! \, r!$ tuple of functions $\det(f_j^{I_k})$ for increasing multi-indices I of length r. We will refer to F as the *determinant* of f. If f_j

are the rows of the matrix considered as sections of the trivial bundle E^*, then F is just the section $f_r \wedge \cdots \wedge f_1$ of the bundle $\Lambda^r E^*$. Our next result is a Briançon–Skoda-type result for the tuple F. It turns out that it is enough to use a power much less than $m!/(m-r)!\,r!$. Let Z be the zero set of F and observe that codim $Z \leq m - r + 1$; this is easily seen by Gauss elimination.

THEOREM 1.4. *Let F be the determinant of the holomorphic matrix f as before. If*

$$|\phi| \leq C|F|^{\min(n, m-r+1)},$$

then $\phi \in (F)$.

REMARK 2. This result is closely related to the following statement, which was proved in [3]. *Suppose that ϕ is an r-tuple of holomorphic functions and let $\|\phi\|$ be the pointwise norm induced by f; that is, $\|\phi\| = \det(ff^*)\langle (ff^*)^{-1}\phi, \phi \rangle$. If*

$$\|\phi\| \lesssim |F|^{\min(n, m-r+1)},$$

then $f\psi = \phi$ has a local holomorphic solution.

REMARK 3. Another related situation is when f is a section of a bundle E^*, ϕ takes values in $\Lambda^l E$, and we ask for a holomorphic section ψ of $\Lambda^{l+1}E$ such that $\delta_f \psi = \phi$ (provided that the necessary compatibility condition $\delta_f \phi = 0$ is fulfilled). Let $p = \mathrm{codim}\{f = 0\}$. Then a sufficient condition is that

$$|\phi| \leq C|f|^{\min(n, m-l)}$$

if $l \leq m - p$, whereas there is no condition at all if $l > m - p$; see Theorems 1.2 and 1.4 and Corollary 1.5 in [1].

Theorem 1.4 is proved by constructing a certain residue current R with support on the analytic set Z such that $R\phi = 0$ implies that ϕ belongs to the ideal (F) locally. The size conditions of ϕ then imply that $R\phi = 0$ by brute force (see Theorem 2.3 in the next section). There may be more subtle reasons for annihilation. For instance, in the generic case (i.e., when codim $Z = m - r + 1$) even the converse statement holds; if ϕ is in the ideal (F) then actually $R\phi = 0$ (see Theorem 2.3(iv)). The analogous statement also holds for the equation $f\psi = \phi$ in Remark 2 (see [3]). These results are thus extensions of the well-known duality theorem of Dickenstein–Sessa [12] and Passare [18] stating that, if f is a tuple that defines a complete intersection (i.e., if codim$\{f = 0\} = m$), then $\phi \in (f)$ if and only if ϕ annihilates the Coleff–Herrera current defined by f. For the analysis of the residue current R we use the basic tools developed in [5; 6; 7; 19]—that is, resolution of singularites by Hironaka's theorem followed by a toric resolution. In Section 3 we obtain Theorems 1.2 and 1.3 (as well as Theorem 1.1) along the same lines by analysis of special choices of the matrix f. It might happen that there are some similarities with the methods used here and the algebraic methods introduced in [13].

By means of the new construction in [4] of division formulas we obtain, for a given holomorphic function ϕ, a holomorphic decomposition

$$\phi = T\phi + S\phi \tag{1.1}$$

such that $T\phi$ belongs to the determinant ideal (F) and $S\phi$ vanishes as soon as ϕ annihilates the residue current R. In particular, this gives an explicit proof of Theorem 1.4 and also leads to explicit proofs of Theorems 1.1–1.3.

2. The Ideal Generated by the Determinant Section

Although in this paper we are mainly interested in local results, it is convenient to adopt an invariant perspective. Therefore, assume we have hermitian vector bundles E and Q of ranks m and $r \leq m$ (respectively) over a complex n-dimensional manifold X as well as a holomorphic morphism $f: E \to Q$. We also assume that f is generically surjective (i.e., that the analytic set Z where f is not surjective has codimension ≥ 1). If ε_j is a local holomorphic frame for Q then $f = f_1 \otimes \varepsilon_1 + \cdots + f_r \otimes \varepsilon_r$, where f_j are sections of the dual bundle E^*. Moreover, $F = f_r \wedge \cdots \wedge f_1 \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_r$ is an invariantly defined section of $\Lambda^r E^* \otimes \det Q^*$ that we will call the *determinant section* associated with f. Notice that if e_j is a local frame for E with dual frame e_j^* for E^*, then $f_j = \sum_1^m f_j^k e_k^*$ and

$$F = {\sum_{|I|=r}}' F_I e_{I_1}^* \wedge \cdots \wedge e_{I_r}^*,$$

where the sum runs over increasing multi-indices I and where $F_I = \det(F_j^{I_k})$. Let $S^l Q^*$ be the subbundle of $(Q^*)^{\otimes l}$ consisting of symmetric tensors. We consider the so-called Eagon–Northcott complex

$$\cdots \xrightarrow{\delta_f} \Lambda^{r+k-1} E \otimes S^{k-1} Q^* \otimes \det Q^* \xrightarrow{\delta_f} \cdots$$

$$\xrightarrow{\delta_f} \Lambda^{r+1} E \otimes Q^* \otimes \det Q^* \xrightarrow{\delta_f} \Lambda^r E \otimes \det Q^* \xrightarrow{\delta_F} \mathbb{C} \to 0. \tag{2.1}$$

Here

$$\delta_f = \sum_j \delta_{f_j} \otimes \delta_{\varepsilon_j},$$

with δ_{f_j} and δ_{ε_j} denoting interior multiplication on ΛE and from the left on $SQ^* \otimes \det Q^*$ (respectively), and

$$\delta_F = \delta_f^r / r! = \delta_{f_r} \cdots \delta_{f_1} \otimes \delta_{\varepsilon_1} \cdots \delta_{\varepsilon_r}.$$

It is readily checked that (2.1) actually is a complex. Observe that if $r = 1$ then (2.1) is the usual Koszul complex.

In $X \setminus Z$ we let σ_j be the sections of E with minimal norms such that $f_k \sigma_j = \delta_{jk}$. Then $\sigma = \sigma_1 \otimes \varepsilon_1^* + \cdots + \sigma_r \otimes \varepsilon_r^*$ is the section of $\mathrm{Hom}(Q, E)$ such that, for each section ϕ of Q, $v = \sigma\phi$ is the solution to $fv = \phi$ with pointwise minimal norm. We also have the invariantly defined section

$$\boldsymbol{\sigma} = \sigma_1 \wedge \cdots \wedge \sigma_r \otimes \varepsilon_r^* \wedge \cdots \wedge \varepsilon_1^*$$

of $\Lambda E \otimes = \det Q^*$, which in fact is the section with minimal norm such that $F\boldsymbol{\sigma} = 1$ (see e.g. [3]).

EXAMPLE 1. Assume that E and Q are trivial. Let ε_j be an ON-frame for Q and let e_j be an ON-frame for E, with dual frame e_j^*. If $F = \sum_{|I|=r}' F_I e_{I_1}^* \wedge \cdots \wedge e_{I_r}^*$ as before, then

$$\boldsymbol{\sigma} = \sum_{|I|=r}' \frac{\bar{F}_I}{|F|^2} e_{I_1} \wedge \cdots \wedge e_{I_r}.$$

We will consider $(0, q)$-forms with values in $\Lambda^{r+k-1} E \otimes S^{k-1} Q^* \otimes \det Q^*$; it is convenient to consider them as sections of $\Lambda^{r+k+q-1}(E \oplus T_{0,1}^*(X)) \otimes S^{k-1} Q^* \otimes \det Q^*$ so that δ_f anticommutes with $\bar{\partial}$ and $\delta_F \bar{\partial} = (-1)^r \delta_F \bar{\partial}$. In what follows we let $\otimes$ denote the usual tensor product of all Q^*-factors and the wedge product of $\Lambda(E \oplus T_{0,1}^*(X))$-factors. Thus, for instance,

$$\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} = \left(\sum_1^r \sigma_j \otimes \varepsilon_j^* \right) \otimes (\sigma_1 \wedge \cdots \wedge \sigma_r \otimes \varepsilon_1^* \wedge \cdots \wedge \varepsilon_r^*) = 0.$$

Moreover, $(\bar{\partial}\sigma)^{\otimes(k-1)}$ is a symmetric tensor for each $k \geq 1$; more precisely,

$$(\bar{\partial}\sigma)^{\otimes(k-1)} = \sum_{|\alpha|=k-1} (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \cdots \wedge (\bar{\partial}\sigma_r)^{\alpha_r} \otimes \varepsilon_\alpha^*. \tag{2.2}$$

Here $\varepsilon_\alpha^* = (\varepsilon_1^*)^{\alpha_1} \dot{\otimes} \cdots \dot{\otimes} (\varepsilon_r^*)^{\alpha_r} / \alpha_1! \cdots \alpha_r!$, with $\dot{\otimes}$ denoting symmetric tensor product. For each $k \geq 1$ we define in $X \setminus Z$ the $(0, k-1)$-forms

$$u_k = (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \boldsymbol{\sigma} = \sigma_1 \wedge \cdots \wedge \sigma_r \wedge (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \varepsilon^* \tag{2.3}$$

(where $\varepsilon^* = \varepsilon_r^* \wedge \cdots \wedge \varepsilon_1^*$) with values in $\Lambda^{r+k-1} E \otimes S^{k-1} Q^* \otimes \det Q^*$.

PROPOSITION 2.1. *In $X \setminus Z$ we have*

$$\delta_F u_1 = 1, \qquad \delta_f u_{k+1} = \bar{\partial} u_k, \quad k \geq 1. \tag{2.4}$$

Proof. Since the δ_{ε_j} act from the left and since $\delta_{f_j} \bar{\partial}\sigma_l = 0$ for all l, it follows that

$$\delta_f u_{k+1} = \delta_f [\sigma_1 \wedge \cdots \wedge \sigma_r \wedge (\bar{\partial}\sigma)^{\otimes k} \otimes \varepsilon^*]$$

$$= \delta_f [\sigma_1 \wedge \cdots \wedge \sigma_r \wedge \bar{\partial}\sigma] \otimes (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \varepsilon^*$$

$$= \sum_{j=1}^r \delta_{f_j} (\sigma_1 \wedge \cdots \wedge \sigma_r) \wedge \bar{\partial}\sigma_j \otimes (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \varepsilon^*$$

$$= \bar{\partial}(\sigma_1 \wedge \cdots \wedge \sigma_r) \wedge (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \varepsilon^* = \bar{\partial} u_k.$$

Since $\delta_F u_1 = F\boldsymbol{\sigma} = 1$, the proposition is proved. $\qquad\square$

Let $u = u_1 + u_2 + \cdots$ and let δ denote either δ_f or δ_F; then (2.4) can be written as $(\delta - \bar{\partial})u = 1$. We shall use the following lemma to analyze the singularities of u at Z.

LEMMA 2.2 [3, Lemma 4.1]. *If $F = F_0 F'$ for some holomorphic function F_0 and nonvanishing holomorphic section F', then*

$$s' = F_0\sigma \quad and \quad S' = F_0\boldsymbol{\sigma}$$

are smooth across Z.

Notice that $|F|^{2\lambda}u$ and $\bar{\partial}|F|^{2\lambda} \wedge u$ are well-defined forms in X for $\operatorname{Re}\lambda \gg 0$.

THEOREM 2.3.

(i) *The forms $|F|^{2\lambda}u$ and $\bar{\partial}|F|^{2\lambda} \wedge u$ have analytic continuations as currents in X to $\operatorname{Re}\lambda > -\varepsilon$. If $U = |F|^{2\lambda}u|_{\lambda=0}$ and $R = \bar{\partial}|F|^{2\lambda} \wedge u|_{\lambda=0}$, then*

$$(\delta - \bar{\partial})U = 1 - R.$$

(ii) *The current R has support on Z and $R = R_p + \cdots + R_\mu$, where $p = \operatorname{codim} Z$ and $\mu = \min(n, m - r + 1)$.*

(iii) *If ϕ is a holomorphic function and $R\phi = 0$, then locally $F\Psi = \phi$ has holomorphic solutions.*

(iv) *If $\operatorname{codim} Z = m - r + 1$ and $F\Psi = \phi$ has a holomorphic solution, then $R\phi = R_{m-r+1}\phi = 0$.*

(v) *If $|\phi| \le C|F|^\mu$, then $R\phi = 0$.*

Here, of course, $R_k = \bar{\partial}|F|^{2\lambda} \wedge u_k|_{\lambda=0}$ is the component of R that is a $(0, k)$-current with values in $\Lambda^{r+k-1}E \otimes S^{k-1}Q^* \otimes \det Q^*$.

Proof of Theorem 2.3. For $r = 1$ this theorem is contained in [1, Thms. 1.1–1.4], and most parts of the proof are completely analogous; we therefore merely point out the necessary modifications. By Hironaka's theorem and a further toric resolution (following the technique developed in [6; 19]), we may assume that locally $F = F_0 F'$ as in Lemma 2.2. Since moreover $\sigma \otimes \sigma = 0$ it follows that, locally in the resolution,

$$u_k = \frac{(\bar{\partial}s')^{\otimes(k-1)} \otimes S'}{F_0^k}.$$

It is then easy to see that the proposed analytic extensions exist; hence we have

$$U_k = \left[\frac{1}{F_0^k}\right](\bar{\partial}s')^{\otimes(k-1)} \otimes S' \tag{2.5}$$

and

$$R_k = \bar{\partial}\left[\frac{1}{F_0^k}\right] \wedge (\bar{\partial}s')^{\otimes(k-1)} \otimes S', \tag{2.6}$$

where $[1/F_0^k]$ is the usual principal value current. If $R\phi = 0$ then $(\delta - \bar{\partial})U\phi = \phi$, so by successively solving the $\bar{\partial}$-equations $\bar{\partial}w_k = U_k\phi + \delta w_{k+1}$ we finally obtain the holomorphic solution $\Psi = U_1\phi + \delta w_2$. All parts but (iv) now follow in a similar way as in [1]. Notice in particular that $k \le \min(n, m - r + 1)$ in (2.6) for degree reasons and so $R\phi = 0$ if the hypothesis in (v) is satisfied. As for (iv), assume we have a holomorphic section Ψ of $\Lambda^r E \otimes \det Q^*$ such that $F\Psi = \phi$. If $\Psi = \psi \otimes \varepsilon^*$, then $F\Psi = \delta_{f_r} \cdots \delta_{f_1}\psi$. Since u_{m-r+1} has full degree in e_j, it follows that

$$u_{m-r+1}\phi = \phi\sigma_1 \wedge \cdots \wedge \sigma_r \wedge (\bar{\partial}\sigma)^{\otimes(m-r)} \otimes \varepsilon^*$$
$$= (\delta_{f_r} \cdots \delta_{f_1}\psi)\sigma_1 \wedge \cdots \wedge \sigma_r \wedge (\bar{\partial}\sigma)^{\otimes(m-r)} \otimes \varepsilon^*$$
$$= \psi \wedge (\bar{\partial}\sigma)^{\otimes(m-r)} \otimes \varepsilon^*|_{\lambda=0}$$
$$= (\bar{\partial}\sigma)^{\otimes(m-r)} \otimes \Psi = \bar{\partial}(\sigma \otimes (\bar{\partial}\sigma)^{\otimes(m-r+1)}) \otimes \Psi = \bar{\partial}u'_{m-r} \otimes \Psi.$$

Since codim $Z = m - r + 1$, by part (ii) we have $R = R_{m-r+1}$; hence

$$R\phi = R_{m-r+1}\phi = \bar{\partial}|F|^{2\lambda} \wedge u_{m-r+1}\phi|_{\lambda=0} = -\bar{\partial}(\bar{\partial}|F|^{2\lambda} \wedge u'_{m-r} \otimes \Psi|_{\lambda=0}).$$

However,

$$\bar{\partial}|F|^{2\lambda} \wedge u'_{m-r} \otimes \Psi|_{\lambda=0}$$

vanishes for degree reasons, precisely in the same way as R_k vanishes for $k \leq m - r$. □

Proof of Theorem 1.4. If we consider the matrix f as a morphism $E \to Q$ for trivial bundles E and Q, then the theorem follows immediately from parts (v) and (iii) of Theorem 2.3. □

REMARK 4. As we have seen, the reason for the power $m - r + 1$ in Theorem 1.4 (and in part (v) of Theorem 2.3) when n is large is that the complex (2.1) terminates at $k = m - r + 1$. In attempting to analyze the section F by means of the usual Koszul complex with respect to the basis $(e_I)'_{|I|=r}$, one could hope that the corresponding forms u_k would miraculously vanish when $k > m - r + 1$—although one has $m!/(m-r)!\,r!$ dimensions (basis elements). However, this is not the case in general. Take for instance the simplest nontrivial case, $m = 3$ and $r = 2$, and choose $f_1 = (1, 0, \xi_1)$ and $f_2 = (0, 1, \xi_2)$ as well as the trivial metric. Then $F_{12} = 1$, $F_{13} = \xi_2$, $F_{23} = \xi_1$, and $\sigma = \bar{F}/|F|^2$, so that

$$\sigma_{12} = \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2}, \quad \sigma_{13} = \frac{\bar{\xi}_2}{1 + |\xi_1|^2 + |\xi_2|^2}, \quad \sigma_{23} = \frac{\bar{\xi}_1}{1 + |\xi_1|^2 + |\xi_2|^2}.$$

Now $m - r + 1 = 2$, but if we form the usual Koszul complex with (say) the basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$, so that

$$\sigma = \sigma_{12}\varepsilon_1 + \sigma_{13}\varepsilon_2 + \sigma_{23}\varepsilon_3 = \frac{1}{|F|^2}(\varepsilon_1 + \bar{\xi}_2\varepsilon_2 + \bar{\xi}_1\varepsilon_3),$$

then

$$\sigma \wedge (\bar{\partial}\sigma)^2 = \frac{2}{|F|^6}d\bar{\xi}_1 \wedge d\bar{\xi}_2 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3,$$

and this form is not zero. For an example where Z is nonempty, multiply f by a function f_0.

3. Products of Ideals

For $j = 1, \ldots, r$, let $E_j \to X$ be a hermitian vector bundle of rank m_j and let f_j be a section of E_j^*. Let $E = \bigoplus_1^r E_j$, and let $Q \simeq \mathbb{C}^r$ with ON-basis $\varepsilon_1, \ldots, \varepsilon_r$. If we consider f_j as sections of E, then $f = \sum_1^r f_j \otimes \varepsilon_j$ is a morphism $E \to Q$.

Furthermore, $F\Psi = \phi$ (with $\Psi = \psi \otimes \varepsilon^*$ as before) means that $\delta_{f_r} \cdots \delta_{f_1} \psi = \phi$ and hence that ϕ belongs to the product ideal $(f_1) \cdots (f_r)$. To obtain such a solution Ψ we proceed as in the previous section. Notice that now σ_j can be identified with the section of E_j having minimal norm such that $f_j \sigma_j = 1$. Moreover, $|F| = |f_1| \cdots |f_r|$. In this case we thus have

$$R_k = \bar{\partial}|F|^{2\lambda} \wedge u_k = \bar{\partial}(|f_1|^{2\lambda} \cdots |f_r|^{2\lambda}) \wedge \sigma_1 \wedge \cdots \wedge \sigma_r$$

$$\wedge \sum_{|\alpha|=k-1} (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \cdots \wedge (\bar{\partial}\sigma_r)^{\alpha_r} \otimes \varepsilon^*_\alpha \otimes \varepsilon^*|_{\lambda=0}.$$

For degree reasons, R_k will vanish unless

$$0 \le \alpha_j \le m_j - 1 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_r \le n - 1. \tag{3.1}$$

Proof of Theorem 1.2. Consider the tuples f_j as sections of E_j. For each j, let e_{ji} ($i = 1, \ldots, m_j$) be a local frame for E_j so that $f_j = \sum_{i=1}^{m_j} f^i_j e^*_{ji}$. After a suitable resolution as before we may assume that $f_1 = f_1^0 f_1'$, where f_1^0 is holomorphic and f_1' is a nonvanishing section of E_1^*. After a further resolution we may also assume that $f_2 = f_2^0 f_2'$, and so forth. Hence we may finally assume for each j that $f_j = f_j^0 f_j'$, where f_j^0 is holomorphic and f_j' is a nonvanishing section of E_j^*. Therefore, R_k is a sum of terms like

$$\bar{\partial}(|f_1^0|^{2\lambda} \cdots |f_r^0|^{2\lambda} v^\lambda) \wedge \frac{\beta}{(f_1^0)^{\alpha_1+1} \cdots (f_r^0)^{\alpha_r+1}}\bigg|_{\lambda=0},$$

where v is smooth and nonvanishing. By the same argument as before, this current is annihilated by ϕ if $|\phi| \le C|f_1|^{\alpha_1+1} \cdots |f_r|^{\alpha_r+1}$; in view of (3.1) and the theorem's hypothesis, taking $s_j = \alpha_j + 1$ thus yields that ϕ annihilates R. It now follows from Theorem 2.3(iii) that $F\Psi = \phi$ has a holomorphic solution and so $\psi \in (f_1) \cdots (f_r)$. $\square$

We can also easily obtain the Briançon–Skoda theorem.

Proof of Theorem 1.1. Assume that the tuple $f = (f^1, \ldots, f^m)$ is given. Choose disjoint isomorphic bundles $E_j \simeq \mathbb{C}^m$ with isomorphic bases e_{ji}, and let $f_j = \sum_{i=1}^m f^i e^*_{ji}$. Outside $Z = \{f = 0\}$ we have $\sigma_j = \sum_1^m \sigma^i e_{ji}$. Now the $\bar{\partial}\sigma^i$ are linearly dependent, since $\sum_1^m f^i \bar{\partial}\sigma^i = \bar{\partial} \sum_1^m f^i \sigma^i = \bar{\partial}1 = 0$. Hence the form u_k must vanish if $k - 1 > m - 1$ and so R_k vanishes unless $k \le \min(n, m)$. Since $|f_j| = |f|$ locally in the resolution, we have

$$R_k = \bar{\partial}|f|^{2r\lambda} \wedge \frac{\beta}{(f^0)^{k+r-1}}\bigg|_{\lambda=0};$$

therefore, R_k is annihilated by ϕ if $|\phi| \le C|f|^{\min(m,n)+r-1}$. $\square$

It remains to consider the case when the f_j together define a complete intersection. The proof is very much inspired by similar proofs in [24].

Proof of Theorem 1.3. We now assume that codim$\{f_1 = \cdots = f_r = 0\} = m_1 + \cdots + m_r$; in particular, $m_1 + \cdots + m_r \leq n$. Let ξ be a test form times ϕ. If the support is small enough then, after a resolution of singularities and further localization, the action of R on ξ becomes a sum of terms the worst of which are like

$$\int \bar{\partial}(|f_1^0|^{2\lambda} \cdots |f_r^0|^{2\lambda}) \wedge \frac{s_1' \wedge \cdots \wedge s_r' \wedge (\bar{\partial}s_1')^{m_1-1} \wedge \cdots \wedge (\bar{\partial}s_r')^{m_r-1} \wedge \tilde{\xi}\rho}{(f_1^0)^{m_1} \cdots (f_r^0)^{m_r}}\bigg|_{\lambda=0},$$

where $\tilde{\xi}$ is the pull-back of ξ and ρ is a cutoff function in the resolution. We may assume that each f_j^0 is a monomial times a nonvanishing factor in a local coordinate system τ_k. Let τ be one of the coordinate factors in, say, f_1 (with order l), and consider the integral that appears when $\bar{\partial}$ falls on $|\tau^l|^{2\lambda}$. If τ does not occur in any other f_j^0, then the assumption $|\phi| \leq C|f_1|^{m_1}$ implies that $\tilde{\phi}$ is divisible by τ^{lm_1}. Hence $\tilde{\phi}$ and thus also $\tilde{\xi}$ annihilates the singularity as before, so the integral vanishes. We now claim that if, on the other hand, τ occurs in some of the other factors then the integral vanishes because of the complete intersection assumption. Hence let us assume that τ occurs in $f_2^0, \ldots, f_k^0$ but not in $f_{k+1}^0, \ldots, f_r^0$. The forms $s_j = |f_j|^2 \sigma_j$ are smooth; moreover,

$$\tilde{\gamma} = \frac{s_{k+1}' \wedge \cdots \wedge s_r' \wedge (\bar{\partial}s_{k+1}')^{m_{k+1}-1} \wedge \cdots \wedge (\bar{\partial}s_r')^{m_r-1} \wedge \tilde{\xi}}{(f_{k+1}^0)^{m_{k+1}} \cdots (f_r^0)^{m_r}}$$

is the pull-back of

$$\gamma = \frac{s_{k+1} \wedge \cdots \wedge s_r \wedge (\bar{\partial}s_{k+1})^{m_{k+1}-1} \wedge \cdots \wedge (\bar{\partial}s_r)^{m_r-1} \wedge \xi}{|f_{k+1}|^{2m_{k+1}} \cdots |f_r^0|^{2m_r}}.$$

Since in $d\bar{z}$ the form γ has codegree $1 + (m_1 - 1) + \cdots + (m_k - 1)$, which is strictly less than $m_1 + \cdots + m_k = \text{codim}\{f_1 = \cdots = f_k = 0\}$, it follows that the antiholomorphic factor of the denominator vanishes on $\{f_1 = \cdots = f_k = 0\}$. Therefore, each term of its pull-back vanishes where $\tau = 0$, so it must contain either a factor $\bar{\tau}$ or $d\bar{\tau}$. However, by assumption the (pull-back of the) denominator contains no factor $\bar{\tau}$, so each term of $\tilde{\gamma}$ will contain $\bar{\tau}$ or $d\bar{\tau}$. Hence the integral that appears when $\bar{\partial}$ falls on $|\tau|^{2\lambda l}$ will vanish when $\lambda = 0$. □

4. Explicit Integral Representation

We shall now supply explicit proofs of Theorems 1.1–1.4. Since all of them are local, we may assume (using the notation from Section 2) that both $f : E \to Q$ and the function ϕ are holomorphic in a convex neighborhood X of the closure of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. We also fix global holomorphic frames e_j and ε_k for E and Q, respectively, and use the trivial metric with respect to these frames.

To give a hint of how the formulas are built up, first suppose that f is a function in the unit disk with no zeros on the unit circle—that is, $n = r = m = 1$. The construction of representation (1.1) is a generalization of the simple one-variable formula

$$\phi(z) = f(z) \int_{|\zeta|=1} \frac{1}{f(\zeta)} \frac{d\zeta}{\zeta - z} \phi(\zeta) + \frac{1}{2\pi i} \int_{|\zeta|<1} \bar{\partial} \frac{1}{f} \wedge h(\zeta, z) \phi(\zeta); \quad (4.1)$$

here $h = (f(\zeta) - f(z)) d\zeta / (\zeta - z) 2\pi i$, which follows directly from Cauchy's integral formula. Notice that the second term vanishes as soon as ϕ annihilates the residue $R = \bar{\partial}(1/f)$. Moreover, for an arbitrary holomorphic function ϕ, this term interpolates ϕ at each zero of f up to the order of the zero. If the order is 1 then this follows immediately from the simple observation that $\bar{\partial}(1/f) \wedge df/2\pi i$ is the point mass at the zero.

We now turn our attention to the general case. One can verify that, if in $R = \bar{\partial}|f|^{2\lambda} \wedge u|_{\lambda=0}$ we contract each $\sigma = \sum \sigma_j \otimes \varepsilon_j$ with $\sum df_j \otimes \varepsilon_j^*$ and contract $\boldsymbol{\sigma}$ with dF, then we obtain a d-closed $(*, *)$-current of order 0 that in some sense generalizes the Lelong current over Z; see [2] for the case when $r = 1$. The recipe for obtaining a division interpolation formula like (1.1) (and (4.1)) is to replace the differentials by Hefer forms and then multiply by a Cauchy-type form. This idea is developed in a quite general setting in [4], so we only sketch our special situation here.

For fixed $z \in X$, let δ_η denote interior multiplication with the vector field $2\pi i \sum (\zeta_j - z_j)(\partial/\partial \zeta_j)$ and let $\nabla_\eta = \delta_\eta - \bar{\partial}$. Moreover, let χ be a cutoff function in X that is identically 1 in a neighborhood of $\bar{\mathbb{B}}$ and let

$$s(\zeta, z) = \frac{1}{2\pi i} \frac{\partial |\zeta|^2}{|\zeta|^2 - \bar{\zeta} \cdot z}.$$

Then, for each $z \in \mathbb{B}$, it follows (see [4]) that

$$g = \chi - \bar{\partial} \chi \wedge \frac{s}{\nabla_\eta s} = \chi - \bar{\partial} \chi \wedge \sum_{k=1}^{n} \frac{1}{(2\pi i)^k} \frac{\partial |\zeta|^2 \wedge (\bar{\partial} \partial |\zeta|^2)^{k-1}}{(|\zeta|^2 - \bar{\zeta} \cdot z)^k} \quad (4.2)$$

is a compactly supported ∇_η-closed form such that $g_{0,0}(z) = 1$, where lower indices denote bidegree. Furthermore, g depends holomorphically on z.

We then choose holomorphic $(1,0)$-forms h_j in X (Hefer forms) such that $\delta_\eta h_j = f_j(\zeta) - f_j(z)$. Let $h = \sum_1^m h_j \otimes \varepsilon_j^*$; we may also assume that h_j (and hence h) depend holomorphically on the parameter z. Now $\delta_h \colon E_{k+1} \to E_k$ for $k \geq 1$, so $(\delta_h)_k \colon E_{k+1} \to E_1$ for $k \geq 0$ if $(\delta_h)_l = \delta_h^l / l!$. It is easily seen that

$$\delta_\eta (\delta_h)_k = (\delta_h)_{k-1} \delta_f - \delta_{f(z)} (\delta_h)_{k-1}. \quad (4.3)$$

So far, δ_F has acted only on $(0,0)$-forms with values in $\Lambda^r E$. We now extend it to general (p, q)-forms, with the convention that a negative sign is inserted when $p + q$ is odd. Thus we let

$$\delta_F \alpha = (-1)^{(r+1)(\deg \alpha + 1)} \delta_{f_r} \cdots \delta_{f_1} \otimes \delta_{\varepsilon_1} \cdots \delta_{\varepsilon_r},$$

where $\deg \alpha$ is the degree of α in $\Lambda(E \otimes T^*(X))$. With this convention, both δ_F and δ_f will anticommute with $\bar{\partial}$ and δ_η. It is possible (see [4]) to find explicit holomorphic $(1,0)$-form-valued mappings $H_k^0 \colon E_k \to \mathbb{C}$ that depend holomorphically on the parameter z and such that

$$\delta_\eta H_1^0 = \delta_{F(\zeta)} - \delta_{F(z)} \quad \text{and}$$
$$\delta_\eta H_k^0 = H_{k-1}^0 \delta_{f(\zeta)} - \delta_{F(z)} (\delta_h)_{k-1}, \quad k \geq 2. \quad (4.4)$$

If we define

$$H^1 U = \sum_{k=1}^{\min(n+1, m-r+1)} (\delta_h)_{k-1} U_k$$

and

$$H^0 R = \sum_{k=1}^{\min(n, m-r+1)} H_k^0 R_k,$$

then it follows from (4.3) and (4.4) that $g' = (\delta_{F(z)} H^1 U + H^0 R) \wedge g$ is ∇_η-closed and that $g'_{0,0}(z) = 1$. This yields (see [4]) the following result.

THEOREM 4.1. *If ϕ is holomorphic in X and if g is the Cauchy form* (4.2), *then we have the holomorphic decomposition*

$$\phi(z) = \delta_{F(z)} \int H^1 U \wedge g\phi + \int H^0 R \wedge g\phi, \quad z \in \mathbb{B}. \tag{4.5}$$

In particular, $\Psi(z) = \int H^1 U \wedge g\phi$ is an explicit solution to $\delta_{F(z)}\Psi = \phi$ if $R\phi = 0$. We now consider this solution in more detail. In view of (2.2) and (2.3), outside Z we have that

$(\delta_h)_{k-1} u_k$

$$= \sum_{|\alpha|=k-1} (\delta_{h_1})_{\alpha_1} \cdots (\delta_{h_r})_{\alpha_r} [\sigma_1 \wedge \cdots \wedge \sigma_r \wedge (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \cdots \wedge (\bar{\partial}\sigma_r)^{\alpha_r}] \otimes \varepsilon^*.$$

Moreover, since we have used a trivial metric, it follows that the

$$\sigma_j = \sum_{i=1}^{m} \sigma_{ij} e_j, \quad j = 1, \ldots, r,$$

are just the columns in the matrix $f^*(ff^*)^{-1}$. Suppressing the nonvanishing section ε^* results in our first corollary.

COROLLARY 4.2. *Let f be a generically surjective holomorphic $r \times m$ matrix in X with rows f_j considered as sections of the trivial bundle E^*, and assume that the hypothesis of Theorem 1.4 is fulfilled. Then*

$$\psi(z) = \int H^1 U \wedge g\phi$$

is an explicit solution to $\delta_{F(z)}\psi(z) = \delta_{f_1(z)} \cdots \delta_{f_r(z)}\psi(z) = \phi(z)$ in $\mathbb{B}$, where $H^1 U \phi$ is the value at $\lambda = 0$ of (*the analytic continuation of*)

$$|f|^{2\lambda} \sum_{k=1}^{\min(n+1, m-r+1)} \sum_{|\alpha|=k-1} (\delta_{h_1})_{\alpha_1} \cdots (\delta_{h_r})_{\alpha_r} [\sigma_1 \wedge \cdots \wedge \sigma_r$$
$$\wedge (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \cdots \wedge (\bar{\partial}\sigma_r)^{\alpha_r}]\phi. \tag{4.6}$$

If $m - r + 1 \leq n$, then $H^1 U \phi$ is locally integrable and the value at $\lambda = 0$ exists in the ordinary sense.

Proof. It remains to verify the claim about local integrability. In fact, after a resolution of singularities (cf. (2.5)) it follows that $U_k\phi$ is locally integrable if

$|\phi| \lesssim |F|^k$. If $m - r + 1 \le n$, then the sum terminates at $k = m - r + 1$ and so the current is locally integrable; otherwise, the worst term is like $U_{n+1}\phi$ and will not be locally integrable in general.   □

If all the f_j take values in different bundles E_j^* and if $E = \bigoplus_j E_j$, then we can further simplify the expression for $H^1 U$. In this case (cf. Section 3)

$$\sigma_j = \sum_{i=1}^{m_j} \frac{\bar{f}_j^i}{|f_j|^2} e_{ij}, \quad j = 1, \ldots, r.$$

With natural choices of Hefer forms h_j, the δ_{h_j} will vanish on forms with values in E_k for $k \neq j$; this yields our final result as follows.

COROLLARY 4.3.   *Let f_j be m_j-tuples of functions considered as sections of the trivial bundles E_j^* over X. If the conditions of Theorem 1.2 or 1.3 are fulfilled, or if all f_j are equal to some fixed m-tuple f and the condition in Theorem 1.1 is fulfilled, then*

$$\psi(z) = \int H^1 U\phi \wedge g$$

is an explicit solution to $\delta_{f_1(z)} \cdots \delta_{f_r(z)} \psi(z) = \phi(z)$ in $\mathbb{B}$. Here $H^1 U\phi$ is the value at $\lambda = 0$ of (*the analytic continuation of*)

$$|f|^{2\lambda} \sum_{k=1}^{n+1} \sum_{|\alpha|=k-1} (\delta_{h_1})_{\alpha_1} [\sigma_1 \wedge (\bar{\partial}\sigma_1)^{\alpha_1}] \wedge \cdots \wedge (\delta_{h_r})_{\alpha_r} [\sigma_r \wedge (\bar{\partial}\sigma_r)^{\alpha_r}]\phi, \quad (4.7)$$

and $N = \min(n + 1, m - r + 1)$.

In the case of Theorems 1.2 and 1.3, the only terms that actually occur are those such that $\alpha_j \le m_j$. In the case of Theorem 1.1 we have only terms such that $k \le m$.

The division formulas discussed here are different from Berndtsson's classical formulas [8]. As already mentioned, an explicit proof of Theorem 1.1 in the case $r = 1$, based on Berndtsson's division formula, has already appeared in [5] (see the proof of Theorem 3.25 there); the case with general r follows in essentially the same way (see [15]). However, we see no way to prove any of the variations discussed in this paper by classical Berndtsson-type formulas.

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