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Tight Closure Test Exponents for Certain Parameter Ideals

Rodney Y. Sharp

0. Introduction

Throughout the paper, R will denote a commutative Noetherian ring of prime characteristic p. We shall always denote by $f: R \to R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. Let $\mathfrak{a}$ be an ideal of R. The nth *Frobenius power* $\mathfrak{a}^{[p^n]}$ of $\mathfrak{a}$ is the ideal of R generated by all p^nth powers of elements of $\mathfrak{a}$.

We use R° to denote the complement in R of the union of the minimal prime ideals of R. An element $r \in R$ belongs to the *tight closure* $\mathfrak{a}^*$ *of* $\mathfrak{a}$ if and only if there exists a $c \in R^\circ$ such that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \gg 0$. We say that $\mathfrak{a}$ is *tightly closed* precisely when $\mathfrak{a}^* = \mathfrak{a}$. The theory of tight closure was invented by M. Hochster and C. Huneke [4], and many applications have been found for the theory (see [7]). For the definition of the tight closure N^*_M of a submodule N in an ambient R-module M (and explanation of the notations $N_M^{[p^n]}$ and m^{p^n} for $m \in M$ and a nonnegative integer n), the reader is referred to [4, (8.1)–(8.3)].

A p^{w_0}*-weak test element* for R (where w_0 is a nonnegative integer) is an element $c' \in R^\circ$ such that, for every finitely generated R-module M and every submodule N of M and for $m \in M$, we have $m \in N^*_M$ if and only if $c'm^{p^n} \in N_M^{[p^n]}$ for all $n \geq w_0$. A p^0-weak test element is called a *test element*. A *locally stable* p^{w_0}*-weak test element* (respectively, *completely stable* p^{w_0}*-weak test element*) for R is an element $c' \in R$ such that, for every prime ideal $\mathfrak{p}$ of R, the natural image $c'/1$ of c' in the localization $R_\mathfrak{p}$ is a p^{w_0}-weak test element for $R_\mathfrak{p}$ (respectively, for the completion $\widehat{R_\mathfrak{p}}$ of $R_\mathfrak{p}$). When $w_0 = 0$, we omit the adjective "p^{w_0}-weak". A locally stable p^{w_0}-weak test element for R is a p^{w_0}-weak test element for R, and a completely stable p^{w_0}-weak test element for R is a locally stable p^{w_0}-weak test element for R; see [4, Prop. (8.13)].

It is a result of Hochster and Huneke [5, Thm. (6.1)(b)] that an algebra of finite type over an excellent local ring of characteristic p has a completely stable p^{w_0}-weak test element for some w_0; furthermore, such an algebra that is also reduced actually has a completely stable test element.

This paper is concerned with the concept of a *test exponent* in tight closure theory introduced by Hochster and Huneke in [6, Def. 2.2]. Let c be a test element

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for a reduced commutative Noetherian ring R of characteristic p, and let $\mathfrak{a}$ be an ideal of R. A test exponent for $c, \mathfrak{a}$ is a power $q = p^{e_0}$ (where $e_0 \in \mathbb{N}_0$, the set of nonnegative integers) such that, if for an $r \in R$ we have $cr^{p^e} \in \mathfrak{a}^{[p^e]}$ for *one single* $e \geq e_0$, then $r \in \mathfrak{a}^*$ (so that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \in \mathbb{N}_0$). In [6] it is shown that this concept has strong connections with the major open problem of whether tight closure commutes with localization; indeed, to quote Hochster and Huneke, "roughly speaking, test exponents exist if and only if tight closure commutes with localization". Although the question of whether tight closure commutes with localization is open in general, it is known that it does commute in many particular cases (see [1]); consequently, the results of Hochster and Huneke in [6] imply (via a rather circuitous route) that test exponents must exist rather often.

In [6, Disc. 5.1], Hochster and Huneke state that "it would be of considerable interest to solve the problem of determining test exponents effectively even for parameter ideals". The main purpose of this paper is to provide a short direct proof that, for a test element c for a reduced excellent equidimensional local ring $(R, \mathfrak{m})$, there exists $e_0 \in \mathbb{N}_0$ such that p^{e_0} is a test exponent for $c, \mathfrak{a}$ for every parameter ideal $\mathfrak{a}$ of R. (For such $(R, \mathfrak{m})$, a *parameter ideal* of R is simply one that can be generated by a subset of a system of parameters for R.) The fact that p^{e_0} is a test exponent for $c, \mathfrak{a}$ for *every* parameter ideal $\mathfrak{a}$ of R is relevant to [6, Disc. 5.3], where Hochster and Huneke raise the question of whether there might conceivably exist (when R (not necessarily local) and c satisfy certain conditions) a "uniform test exponent" for c, that is, a power of p that is a test exponent for $c, \mathfrak{b}$ for *all* ideals $\mathfrak{b}$ of R simultaneously.

1. Left Modules over the Skew Polynomial Ring $R[x, f]$

1.1. NOTATION. We shall work with the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x over R. Recall that $R[x, f]$ is, as a left R-module, freely generated by $(x^i)_{i \in \mathbb{N}_0}$ and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule

$$xr = f(r)x = r^p x \quad \text{for all } r \in R.$$

Note that the decomposition $R[x, f] = \bigoplus_{n \in \mathbb{N}_0} Rx^n$ provides R with a structure as a positively graded ring.

We use $\mathbb{N}$ to denote the set of positive integers.

Our first lemma enables one to see quickly that, in certain circumstances, an R-module M has a structure as a left $R[x, f]$-module extending its R-module structure.

1.2. LEMMA (see e.g. [8, Lemma 1.3]). *Let G be an R-module and let $\xi : G \to G$ be a $\mathbb{Z}$-endomorphism of G such that $\xi(rg) = r^p \xi(g)$ for all $r \in R$ and $g \in G$. Then the R-module structure on G can be extended to a structure of a left $R[x, f]$-module in such a way that $xg = \xi(g)$ for all $g \in G$.*

1.3. Definitions. Let H be a left $R[x, f]$-module. The $R[x, f]$-submodule

$$\Gamma_x(H) := \{h \in H : x^j h = 0 \text{ for some } j \in \mathbb{N}\}$$

of H is called the *x-torsion submodule* of H. We say that H is *x-torsion* precisely when $H = \Gamma_x(H)$ and that H is *x-torsion-free* precisely when $\Gamma_x(H) = 0$.

It is easy to check that, in general, the left $R[x, f]$-module $H/\Gamma_x(H)$ is x-torsion-free.

1.4. Definitions. Let H be a left $R[x, f]$-module. The *annihilator of H* will be denoted by $\operatorname{ann}_{R[x,f]} H$ or $\operatorname{ann}_{R[x,f]}(H)$. Thus $\operatorname{ann}_{R[x,f]}(H) = \{\theta \in R[x, f] : \theta h = 0 \text{ for all } h \in H\}$, and this is a (two-sided) ideal of $R[x, f]$.

For a two-sided ideal $\mathfrak{B}$ of $R[x, f]$, we shall use $\operatorname{ann}_H \mathfrak{B}$ or $\operatorname{ann}_H(\mathfrak{B})$ to denote the *annihilator of* $\mathfrak{B}$ *in* H. Thus $\operatorname{ann}_H \mathfrak{B} = \operatorname{ann}_H(\mathfrak{B}) = \{h \in H : \theta h = 0$ for all $\theta \in \mathfrak{B}\}$, and this is an $R[x, f]$-submodule of H.

1.5. Remark. It is easy to see that a subset $\mathfrak{B}$ of $R[x, f]$ is a graded left ideal if and only if there is a family $(\mathfrak{b}_n)_{n\in\mathbb{N}_0}$ of ideals of R such that $\mathfrak{b}_n \subseteq f^{-1}(\mathfrak{b}_{n+1})$ for all $n \in \mathbb{N}_0$ and $\mathfrak{B} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{b}_n x^n$. Similarly, a subset $\mathfrak{C}$ of $R[x, f]$ is a graded two-sided ideal if and only if there is a family $(\mathfrak{c}_n)_{n\in\mathbb{N}_0}$ of ideals of R such that $\mathfrak{c}_n \subseteq \mathfrak{c}_{n+1}$ for all $n \in \mathbb{N}_0$ (so that the sequence $(\mathfrak{c}_n)_{n\in\mathbb{N}_0}$ is eventually stationary) and $\mathfrak{C} = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{c}_n x^n$.

1.6. Lemma. *Let G be an x-torsion-free left $R[x, f]$-module. Suppose that, for some $w_0 \in \mathbb{N}_0$ and some ideal $\mathfrak{c}$ of R, the graded two-sided ideal $\bigoplus_{n\geq w_0} \mathfrak{c}x^n$ of $R[x, f]$ annihilates G. Then G is annihilated by $\bigoplus_{n\geq 0}(\sqrt{\mathfrak{c}})x^n$.*

Proof. Let $a \in R$ be such that $a^t \in \mathfrak{c}$ for some $t \in \mathbb{N}$. We show that, for $g \in G$, necessarily $ax^n g = 0$ for each $n \in \mathbb{N}_0$. Choose $m \in \mathbb{N}$ such that $p^m \geq t$ and $m \geq w_0$; then $x^m ax^n g = a^{p^m} x^{m+n} g = 0$ because $a^{p^m} \in \mathfrak{c}$ and $m + n \geq w_0$. Since G is x-torsion-free, it follows that $ax^n g = 0$. □

In this paper, substantial use will be made of the following extension, due to G. Lyubeznik, of a result of R. Hartshorne and R. Speiser. It shows that, when R is local, an x-torsion left $R[x, f]$-module that is Artinian (or "cofinite", in the terminology of Hartshorne and Speiser) as an R-module exhibits a certain uniformity of behavior.

1.7. Theorem ([9, Prop. 4.4]; compare [3, Prop. 1.11]). *Suppose that $(R, \mathfrak{m})$ is local, and let H be a left $R[x, f]$-module that is Artinian as an R-module. Then there exists an $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(H) = 0$.*

Hartshorne and Speiser first proved this result in the case where R is local and contains its residue field, and that residue field is perfect. Lyubeznik applied his theory of F-modules to obtain the result without restriction on the local ring R of characteristic p.

The following corollary extends the Hartshorne–Speiser–Lyubeznik theorem to nonlocal situations.

1.8. COROLLARY. *Let H be a left $R[x, f]$-module that is Artinian as an R-module. Then there exists an $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(H) = 0$.*

Proof. Suppose that $H \neq 0$. For each ideal $\mathfrak{a}$ of R, let

$$\Gamma_{\mathfrak{a}}(H) := \bigcup_{n\in\mathbb{N}}(0 :_H \mathfrak{a}^n).$$

The ideas of [11, Exers. 8.48 and 8.49] can be used to show that there are only finitely many maximal ideals $\mathfrak{m}$ of R such that $\Gamma_{\mathfrak{m}}(H) \neq 0$ and that, if we denote the distinct such maximal ideals by $\mathfrak{m}_1, \dots, \mathfrak{m}_t$, then H decomposes as a direct sum of R-submodules

$$H = \Gamma_{\mathfrak{m}_1}(H) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_t}(H).$$

Let $i \in \{1, \dots, t\}$. In fact, $\Gamma_{\mathfrak{m}_i}(H)$ is an $R[x, f]$-submodule of H, since if $h \in \Gamma_{\mathfrak{m}_i}(H)$ and $\mathfrak{m}_i^k h = 0$ for a $k \in \mathbb{N}$ then $(\mathfrak{m}_i^k)^{[p]}xh = 0$. In addition, for $s \in R \setminus \mathfrak{m}_i$ we have $\mathfrak{m}_i^k + Rs = R$, so there exists an $s' \in R$ such that $s'sh = h$. It follows that multiplication by s provides an R-automorphism of $\Gamma_{\mathfrak{m}_i}(H)$, so that the latter left $R[x, f]$-module has a natural structure as an $R_{\mathfrak{m}_i}$-module in which $(r/s)h$, for $r \in R$ and s as before, is equal to the unique element $h' \in H$ for which $sh' = rh$. It can easily be checked that this structure is such that $x(r/s)h = (r^p/s^p)xh$, so it follows from Lemma 1.2 that this $R_{\mathfrak{m}_i}$-module structure on $\Gamma_{\mathfrak{m}_i}(H)$ can be extended to a structure as a left $R_{\mathfrak{m}_i}[x, f]$-module that is compatible with its structure as a left $R[x, f]$-module. We can now use Theorem 1.7 to deduce that there exists an $e_i \in \mathbb{N}$ such that $x^{e_i}\Gamma_x(\Gamma_{\mathfrak{m}_i}(H)) = 0$.

Since

$$\Gamma_x(H) = \Gamma_x(\Gamma_{\mathfrak{m}_1}(H)) \oplus \cdots \oplus \Gamma_x(\Gamma_{\mathfrak{m}_t}(H)),$$

the integer $e := \max\{e_1, \dots, e_t\}$ has the property that $x^e\Gamma_x(H) = 0$. □

1.9. DEFINITION. Let H be a left $R[x, f]$-module that is Artinian as an R-module. By Corollary 1.8, there exists an $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(H) = 0$; we call the smallest such e the *Hartshorne–Speiser–Lyubeznik number* (or *HSL-number,* for short) of H.

1.10. LEMMA. *Let H be a left $R[x, f]$-module that is Artinian as an R-module, and let m_0 be the HSL-number of H. Let $\mathfrak{c}$ be an ideal of R and let $t_0 \in \mathbb{N}_0$.*

Then $\operatorname{ann}_H\big(\bigoplus_{n\geq m_0+t_0} \mathfrak{c}^{[p^{m_0}]}x^n\big)$ *is an $R[x, f]$-submodule of H that contains $\Gamma_x(H)$; furthermore,*

$$\begin{aligned}\big(\operatorname{ann}_H\big(\textstyle\bigoplus_{n\geq m_0+t_0} \mathfrak{c}^{[p^{m_0}]}x^n\big)\big)/\Gamma_x(H) &= \operatorname{ann}_{H/\Gamma_x(H)}\big(\textstyle\bigoplus_{n\geq t_0} \mathfrak{c}x^n\big)\\ &= \operatorname{ann}_{H/\Gamma_x(H)}\big(\textstyle\bigoplus_{n\geq 0}(\sqrt{\mathfrak{c}})x^n\big)\\ &= \big(\operatorname{ann}_H\big(\textstyle\bigoplus_{n\geq m_0}(\sqrt{\mathfrak{c}})^{[p^{m_0}]}x^n\big)\big)/\Gamma_x(H).\end{aligned}$$

Proof. Since $\bigoplus_{n\geq m_0+t_0}\mathfrak{c}^{[p^{m_0}]}x^n \subseteq \bigoplus_{n\geq m_0} Rx^n$, it is immediate that

$$\Gamma_x(H) = \operatorname{ann}_H\big(\bigoplus_{n\geq m_0} Rx^n\big) \subseteq \operatorname{ann}_H\big(\bigoplus_{n\geq m_0+t_0}\mathfrak{c}^{[p^{m_0}]}x^n\big).$$

Now let $h \in H$, $n \in \mathbb{N}_0$, and $c \in \mathfrak{c}$. Then $cx^n(h+\Gamma_x(H)) = 0$ in $H/\Gamma_x(H)$ if and only if $cx^nh \in \Gamma_x(H)$; by definition of the HSL-number of H, this is the case if and only if $x^{m_0}cx^nh = 0$ in H, that is, if and only if $c^{p^{m_0}}x^{m_0+n}h = 0$. It is now easy to prove all the claims by means of this observation and Lemma 1.6. □

The next theorem is the key result of this paper.

1.11. Theorem. *Let G be an x-torsion-free left $R[x, f]$-module that is Artinian as an R-module, and let $\mathfrak{c}$ be an ideal of R. Let N be the R-submodule $(0 :_G \mathfrak{c})$ of G; for each $i \in \mathbb{N}_0$, set*

$$N_i := \{g \in G : x^i g \in N\} = \{g \in G : cx^i g = 0 \text{ for all } c \in \mathfrak{c}\}$$
$$= \{g \in G : \mathfrak{c}x^i g = 0\}.$$

Then the following statements hold:

(i) *each N_i $(i \in \mathbb{N}_0)$ is an R-submodule of G*;
(ii) *$N_i \supseteq N_{i+1}$ for all $i \in \mathbb{N}_0$*;
(iii) *if $N_i = N_{i+1}$ for some $i \in \mathbb{N}_0$, then $N_{i+1} = N_{i+2}$.*

Since G is Artinian as an R-module, it follows from (i), (ii), *and* (iii) *that there exists a* (*uniquely determined*) *$v_0 \in \mathbb{N}_0$ such that*

$$N = N_0 \supset N_1 \supset \cdots \supset N_{v_0} = N_{v_0+1} = \cdots = N_{v_0+j} = \cdots$$

(*where $\supset$ denotes strict containment*). *Then N_{v_0} is the largest $R[x, f]$-submodule of G that is contained in N.*

Henceforth, we shall refer to the integer v_0 as the $\mathfrak{c}$-stability index of G. *Note that it has the following property*: *for $g \in G$, if $\mathfrak{c}x^{n_1}g = 0$ for one single integer $n_1 \geq v_0$ then $\mathfrak{c}x^n g = 0$ for all $n \in \mathbb{N}_0$.*

Proof. (i) It is clear that N_i is an Abelian subgroup of G. Let $g \in N_i$ and let $r \in R$. Thus $cx^ig = 0$ for all $c \in \mathfrak{c}$. Hence $cx^i(rg) = cr^{p^i}x^ig = r^{p^i}cx^ig = 0$ for all $c \in \mathfrak{c}$, so that $rg \in N_i$.

(ii) Let $g \in N_{i+1}$ and let $c \in \mathfrak{c}$; hence $cx^{i+1}g = 0$. Therefore $xcx^ig = c^px^{i+1}g = 0$, and since G is x-torsion-free it follows that $cx^ig = 0$. As a result, $g \in N_i$.

(iii) Assume that $N_i = N_{i+1}$. By part (ii) we have $N_{i+1} \supseteq N_{i+2}$. Let $g \in N_{i+1}$ and let $c \in \mathfrak{c}$; hence $cx^{i+1}g = 0$. Therefore $cx^i(xg) = 0$, so that (as this is true for all $c \in \mathfrak{c}$) we must have $xg \in N_i = N_{i+1}$. Thus $cx^{i+1}(xg) = 0$; that is, $cx^{i+2}g = 0$ for all $c \in \mathfrak{c}$. Hence $g \in N_{i+2}$.

The only remaining claim that still requires proof is the one that N_{v_0} is the largest $R[x, f]$-submodule of G that is contained in N. To see this, first note that if $g \in N_{v_0}$ then $g \in N_{v_0+1}$, so that $cx^{v_0}(xg) = cx^{v_0+1}g = 0$ for all $c \in \mathfrak{c}$ and $xg \in N_{v_0}$. This shows that N_{v_0} is an $R[x, f]$-submodule of G; it is contained in $N = N_0$ by part (ii). On the other hand, if L is any $R[x, f]$-submodule of G that is contained in N and if $g \in L$, then we must have $x^{v_0} g \in L \subseteq N$ and so $g \in N_{v_0}$; hence $L \subseteq N_{v_0}$. □

Next, we use Corollary 1.8 to produce a consequence of Theorem 1.11 that applies to a left $R[x, f]$-module that is Artinian as an R-module but that is not necessarily x-torsion-free.

1.12. COROLLARY. *Let H be a left $R[x, f]$-module that is Artinian as an R-module, and let m_0 be its HSL-number. Let $\mathfrak{c}$ be an ideal of R, and let v_0 be the $\mathfrak{c}$-stability index of the x-torsion-free left $R[x, f]$-module $G := H/\Gamma_x(H)$. Let $h \in H$. Then the following statements are equivalent*:

(i) *there exists one single integer $n_1 \geq m_0 + v_0$ such that $\mathfrak{c}^{[p^{m_0}]} x^{n_1} h = 0$;*
(ii) $\mathfrak{c}^{[p^{m_0}]} x^n h = 0$ *for all $n \geq m_0$.*

Proof. Suppose that $n_1 \in \mathbb{N}_0$ is such that $n_1 \geq m_0 + v_0$ and $\mathfrak{c}^{[p^{m_0}]} x^{n_1} h = 0$. Then $x^{m_0} \mathfrak{c} x^{n_1 - m_0} h = 0$, so $\mathfrak{c} x^{n_1 - m_0} h \subseteq \Gamma_x(H)$ and $\mathfrak{c} x^{n_1 - m_0}(h + \Gamma_x(H)) = 0$ in G. Since $n_1 - m_0 \geq v_0$, the $\mathfrak{c}$-stability index of G, it follows from Theorem 1.11 that $\mathfrak{c} x^n (h + \Gamma_x(H)) = 0$ for all $n \in \mathbb{N}_0$. Therefore, by Lemma 1.10,

$$h + \Gamma_x(H) \in \operatorname{ann}_{H/\Gamma_x(H)}\Big(\bigoplus_{n \geq 0} \mathfrak{c} x^n\Big) = \Big(\operatorname{ann}_H\Big(\bigoplus_{n \geq m_0} (\sqrt{\mathfrak{c}})^{[p^{m_0}]} x^n\Big)\Big) \Big/ \Gamma_x(H).$$

Hence h is annihilated by $\bigoplus_{n \geq m_0} \mathfrak{c}^{[p^{m_0}]} x^n$. □

2. Applications to Test Exponents for Tight Closure

The main strategy employed in this paper involves application of the key result, and its corollary, of Section 1 to the top local cohomology module of $(R, \mathfrak{m})$ in the case where the latter is an equidimensional excellent local ring (of characteristic p). We therefore review the $R[x, f]$-module structure carried by this local cohomology module.

2.1. REMINDER. Suppose that $(R, \mathfrak{m})$ is a local ring of dimension $d > 0$. In this reminder, we shall sometimes use R' to denote R regarded as an R-module by means of f.

(i) With this notation, $f \colon R \to R'$ becomes a homomorphism of R-modules and so induces an R-homomorphism $H^d_{\mathfrak{m}}(f) \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R')$. The independence theorem for local cohomology (see [2, 4.2.1]) applied to the ring homomorphism $f \colon R \to R$ yields an R-isomorphism $\nu^d_R \colon H^d_{\mathfrak{m}}(R') \xrightarrow{\cong} H^d_{\mathfrak{m}^{[p]}}(R)$, where $H^d_{\mathfrak{m}^{[p]}}(R)$ is regarded as an R-module via f. Since $\mathfrak{m}$ and $\mathfrak{m}^{[p]}$ have the same radical, $H^d_{\mathfrak{m}}$ and $H^d_{\mathfrak{m}^{[p]}}$ are the same functor. Composition yields a $\mathbb{Z}$-endomorphism $\xi := \nu^d_R \circ H^d_{\mathfrak{m}}(f) \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ which is such that $\xi(r\gamma) = r^p \xi(\gamma)$ for all $\gamma \in H^d_{\mathfrak{m}}(R)$ and $r \in R$. Hence it follows from Lemma 1.2 that $H^d_{\mathfrak{m}}(R)$ has a natural structure as a left $R[x, f]$-module in which $x\gamma = \xi(\gamma)$ for all $\gamma \in H^d_{\mathfrak{m}}(R)$.

(ii) It is important to note that this $R[x, f]$-module structure on $H^d_{\mathfrak{m}}(R)$ does not depend on any choice of system of parameters for R. The reader might like to consult [8, 2.1] for amplification of this point.

(iii) Let $a_1, \dots, a_d$ be a system of parameters for R, and represent $H^d_{\mathfrak{m}}(R)$ as the dth cohomology module of the Čech complex of R with respect to $a_1, \dots, a_d$, that is, as the residue class module of $R_{a_1 \dots a_d}$ modulo the image, under the Čech complex "differentiation" map, of $\bigoplus_{i=1}^d R_{a_1 \dots a_{i-1} a_{i+1} \dots a_d}$; see [2, Sec. 5.1]. We use $[\cdot]$ to denote natural images of elements of $R_{a_1 \dots a_d}$ in this residue class module. It is worth remarking that, for $r \in R$ and $n \in \mathbb{N}_0$,

$$\left[\frac{r}{(a_1 \dots a_d)^n}\right] = 0 \text{ in } H^d_{\mathfrak{m}}(R)$$

if and only if there exists a $k \in \mathbb{N}_0$ such that $(a_1 \dots a_d)^k r \in (a_1^{n+k}, \dots, a_d^{n+k})R$.

(iv) The left $R[x, f]$-module structure on $H^d_{\mathfrak{m}}(R)$ is such that

$$x\left[\frac{r}{(a_1 \dots a_d)^n}\right] = \left[\frac{r^p}{(a_1 \dots a_d)^{np}}\right] \quad \text{for all } r \in R \text{ and } n \in \mathbb{N}_0.$$

The reader may wish to consult [8, 2.3] for more details.

In [6, Def. 2.2], Hochster and Huneke defined the concept of test exponent in a *reduced* commutative Noetherian ring R of prime characteristic. They also defined the concept for c, N, M, where (c is a test element for R and) N is a submodule of the finitely generated R-module M. However, the definition for modules is not pursued in this paper; we are concerned with test exponents for $c, \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of R (and where R is understood to be the ambient module for $\mathfrak{a}$). On the other hand, there are advantages to extending the concept to weak test elements in nonreduced rings.

2.2. Definition. Let c be a p^{w_0}-weak test element, where $w_0 \in \mathbb{N}_0$, for the (not necessarily reduced) ring R, and let $\mathfrak{a}$ be an ideal of R. We say that p^{e_0}, where $e_0 \in \mathbb{N}_0$, is a *test exponent for* $c, \mathfrak{a}$ if, whenever $r \in R$ is such that $cr^{p^e} \in \mathfrak{a}^{[p^e]}$ for *one single* $e \geq e_0$, then $r \in \mathfrak{a}^*$ (so that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \geq w_0$).

Recall that a *parameter ideal* in a commutative Noetherian ring is a proper ideal of height h that can be generated by h elements for some $h \in \mathbb{N}_0$. In an equidimensional catenary local ring, an ideal is a parameter ideal if and only if it can be generated by a subset of a system of parameters.

2.3. Theorem. *Let $(R, \mathfrak{m})$ as in Notation 1.1 be an equidimensional excellent local ring of dimension $d > 0$.*

By Reminder 2.1, the Artinian R-module $H := H^d_{\mathfrak{m}}(R)$ has a natural structure as a left $R[x, f]$-module; let m_0 be its HSL-number. Let $c \in R^\circ$ and let v_0 be the Rc-stability index of the x-torsion-free left $R[x, f]$-module $G := H/\Gamma_x(H)$.

Then, for each parameter ideal $\mathfrak{a}$ of R, the following statement is true: if $r \in R$ is such that $cr^{p^{n_1}} \in \mathfrak{a}^{[p^{n_1}]}$ for one single $n_1 \geq m_0 + v_0$, then $r \in \mathfrak{a}^*$.

Proof. We shall first prove the claim when $\mathfrak{a}$ is an ideal $\mathfrak{q}$ of R generated by a full system of parameters $a_1, \dots, a_d$ for R. Use the representation of H as the dth cohomology module of the Čech complex of R with respect to $a_1, \dots, a_d$ recalled in 2.1(iii), and write a for the product $a_1 \dots a_d$.

Set $\zeta := [r/a] \in H$. The assumption that $cr^{p^{n_1}} \in \mathfrak{q}^{[p^{n_1}]}$ for an $r \in R$ and an $n_1 \geq m_0 + v_0$ implies (by 2.1(iv)) that $cx^{n_1}\zeta = [cr^{p^{n_1}}/a^{p^{n_1}}] = 0$. Therefore, $Rc^{p^{m_0}}x^{n_1}\zeta = 0$ and so it follows from Corollary 1.12 that $Rc^{p^{m_0}}x^{n}\zeta = 0$ for all $n \geq m_0$. Hence

$$\left[\frac{c^{p^{m_0}}r^{p^n}}{a^{p^n}}\right] = c^{p^{m_0}}x^n\zeta = 0 \quad \text{for all } n \geq m_0.$$

It now follows from 2.1(iii) that, for all $n \geq m_0$, there exists a $k(n) \in \mathbb{N}_0$ such that

$$c^{p^{m_0}}r^{p^n}(a_1 \dots a_d)^{k(n)} \in (a_1^{p^n+k(n)}, \dots, a_d^{p^n+k(n)})R.$$

The next part of the argument is due to K. E. Smith; see the proof of [12, Prop. 3.3(i)]. By repeated use of the colon-capturing properties of tight closure described in [12, Thm. 2.9], it follows that

$$c^{p^{m_0}}r^{p^n} \in (\mathfrak{q}^{[p^n]})^* \quad \text{for all } n \geq m_0.$$

Since R is an excellent local ring, it has a p^{w_0}-weak test element c' for some $w_0 \in \mathbb{N}_0$ (by [5, Thm. 6.1(b)]). Consequently,

$$c'(c^{p^{m_0}}r^{p^n})^{p^{w_0}} \in (\mathfrak{q}^{[p^n]})^{[p^{w_0}]} \quad \text{for all } n \geq m_0;$$

that is, $c'c^{p^{m_0+w_0}}r^{p^{n+w_0}} \in \mathfrak{q}^{[p^{n+w_0}]}$ for all $n \geq m_0$. Since $c'c^{p^{m_0+w_0}} \in R^\circ$, we see that $r \in \mathfrak{q}^*$.

It remains to extend the result to an arbitrary parameter ideal $\mathfrak{a}$ of R. Since R is equidimensional and excellent, there exist a full system of parameters $u_1, \dots, u_d$ for R and $i \in \{0, 1, \dots, d\}$ such that $\mathfrak{a} = (u_1, \dots, u_i)R$. Suppose that $r \in R$ is such that $cr^{p^{n_1}} \in \mathfrak{a}^{[p^{n_1}]}$ for an $n_1 \geq m_0 + v_0$. Then, for all $t \in \mathbb{N}$, we have $cr^{p^{n_1}} \in (u_1, \dots, u_i, u_{i+1}^t, \dots, u_d^t)^{[p^{n_1}]}$; hence $r \in (u_1, \dots, u_i, u_{i+1}^t, \dots, u_d^t)^*$ by the first part of this proof. We can now use the p^{w_0}-weak test element c' to deduce that

$$c'r^{p^n} \in (u_1^{p^n}, \dots, u_i^{p^n}, u_{i+1}^{p^n t}, \dots, u_d^{p^n t})R \quad \text{for all } n \geq w_0 \text{ and } t \in \mathbb{N}.$$

Therefore, by Krull's intersection theorem,

$$c'r^{p^n} \in (u_1^{p^n}, \dots, u_i^{p^n})R \quad \text{for all } n \geq w_0,$$

so that $r \in \mathfrak{a}^*$. □

If, in Theorem 2.3, we take the element c to be a weak test element for R, then we can immediately deduce the existence of a test exponent for $c, \mathfrak{a}$ for each parameter ideal of R; it should be noted that this test exponent is "partially uniform" in the sense that the one test exponent for c works for *every* parameter ideal of R. These results are recorded in part (i) of Corollary 2.4. I am very grateful to the referee for suggesting part (ii) of the corollary, which shows (loosely speaking) that a slightly higher power of p is not only a test exponent for $c, \mathfrak{a}$ for all parameter ideals $\mathfrak{a}$ of R simultaneously but also that, for this test exponent, one need only check that the "ideal membership test" is satisfied "up to tight closure".

2.4. COROLLARY. *Let* $(R, \mathfrak{m})$ *as in Notation 1.1 be an equidimensional excellent local ring of dimension* $d > 0$. *Let* c *be a* p^{w_0}*-weak test element* (*where* $w_0 \in \mathbb{N}_0$) *for* R.

As in Theorem 2.3, let m_0 be the HSL-number of $H := H^d_{\mathfrak{m}}(R)$, and let v_0 be the Rc-stability index of $G := H/\Gamma_x(H)$.

(i) *Then $p^{m_0+v_0}$ is a test exponent for $c, \mathfrak{a}$ for all parameter ideals $\mathfrak{a}$ of R simultaneously.*

(ii) *The power $p^{m_0+v_0+1}$ has the following property: for each parameter ideal $\mathfrak{a}$ of R, we have $r \in \mathfrak{a}^*$ whenever $r \in R$ is such that $cr^{p^{n_1}} \in (\mathfrak{a}^{[p^{n_1}]})^*$ for one single $n_1 \geq m_0 + v_0 + 1$.*

Proof. Part (i) is immediate from Theorem 2.3, and so we prove (ii).

We first show that the $Rc^{p^{w_0}+1}$-stability index v_1 of G satisfies $v_1 \leq v_0+w_0+1$. By Theorem 1.11, to prove this inequality it suffices to show that $c^{p^{w_0}+1}x^{v_0+w_0+1}g = 0$ (for $g \in G$) implies that $c^{p^{w_0}+1}x^n g = 0$ for all $n \in \mathbb{N}_0$. But $c^{p^{w_0}+1}x^{v_0+w_0+1}g = 0$ implies that $c^{p^{w_0+1}}x^{v_0+w_0+1}g = 0$, so

$$x^{w_0+1}cx^{v_0}g = c^{p^{w_0+1}}x^{v_0+w_0+1}g = 0;$$

therefore $cx^{v_0}g = 0$ because G is x-torsion-free. Since v_0 is the Rc-stability index of G, this implies that $cx^n g = 0$ for all $n \in \mathbb{N}_0$, so $c^{p^{w_0}+1}x^n g = 0$ for all $n \in \mathbb{N}_0$. Therefore $v_1 \leq v_0 + w_0 + 1$.

Now suppose $\mathfrak{a}$ is a parameter ideal of R and $r \in R$ is such that $cr^{p^{n_1}} \in (\mathfrak{a}^{[p^{n_1}]})^*$ for one single $n_1 \geq m_0 + v_0 + 1$. Since c is a p^{w_0}-weak test element for R, we have $c(cr^{p^{n_1}})^{p^{w_0}} \in (\mathfrak{a}^{[p^{n_1}]})^{[p^{w_0}]}$; that is, $c^{p^{w_0}+1}r^{p^{n_1+w_0}} \in \mathfrak{a}^{[p^{n_1+w_0}]}$. Now $n_1 + w_0 \geq m_0 + v_0 + w_0 + 1 \geq m_0 + v_1$, so it follows from Theorem 2.3 that $r \in \mathfrak{a}^*$. □

In the final theorem of this paper we deduce a nonlocal result from Corollary 2.4. I am again very grateful to the referee for suggestions that have led to improvements in Theorem 2.5. Note that R is said to be *locally equidimensional* precisely when the localization $R_{\mathfrak{p}}$ is equidimensional for every prime ideal $\mathfrak{p}$ of R. The reader is referred to [1, p. 87] for an explanation of what it means to say that R is "of acceptable type".

2.5. THEOREM. *Let R as in Notation 1.1 be a locally equidimensional ring of acceptable type. Suppose that there exists a completely stable p^{w_0}-weak test element c for R, where $w_0 \in \mathbb{N}_0$. (By* [1, Prop. (5.4)] *and* [5, Thm. (6.1)(b)], *these conditions would all be satisfied if R were an integral domain and an algebra of finite type over an excellent local ring of characteristic p.)*

Let $\mathfrak{a}$ be a parameter ideal of R of positive height. Let $\mathcal{P} := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be a finite set of prime ideals of R of positive height such that

$$\bigcup_{\mathfrak{p} \in \operatorname{ass} \mathfrak{a}^*} \mathfrak{p} \subseteq \bigcup_{i=1}^{t} \mathfrak{p}_i;$$

for example, $\mathcal{P}$ could be ass $\mathfrak{a}^*$.

For each $i = 1, \ldots, t$, let m_i denote the HSL-number of the top local cohomology module $H_i := H^{\operatorname{ht} \mathfrak{p}_i}_{\mathfrak{p}_i R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i})$ of the local ring $R_{\mathfrak{p}_i}$, and let v_i denote the $R_{\mathfrak{p}_i}(c/1)$-stability index of $H_i/\Gamma_x(H_i)$. Then

$$u_0 := \max\{m_1 + v_1, \dots, m_t + v_t\}$$

has the following property: if $r \in R$ *is such that* $cr^{p^{n_1}} \in (\mathfrak{a}^{[p^{n_1}]})^*$ *for one single* $n_1 \geq u_0 + 1$, *then* $r \in \mathfrak{a}^*$.

Consequently, p^{u_0+1} *is a test exponent for* $c, \mathfrak{a}$.

Proof. Temporarily, let A be a local commutative Noetherian ring of characteristic p and positive dimension d, and suppose that H is a left $A[x, f]$-module that is Artinian as an A-module. Then H has a natural structure as a module over the completion $\hat{A}$ of A (see [2, 8.2.4]), and it is easy to use Lemma 1.2 to show that this $\hat{A}$-module structure on H can be extended to a structure as a left $\hat{A}[x, f]$-module that is compatible with its structure as a left $A[x, f]$-module. Thus $\Gamma_x(H)$ is the same whether calculated over A or $\hat{A}$, and a similar comment applies to the HSL-number of H. Note also that, for $c' \in A$, the Ac'-stability index of $H/\Gamma_x(H)$ as left $A[x, f]$-module is the same as its $\hat{A}c'$-stability index as left $\hat{A}[x, f]$-module.

Again by [2, 8.2.4], there is an isomorphism of $\hat{A}$-modules $H \cong H \otimes_A \hat{A}$. It follows from 2.1(iv) and our comments in the preceding paragraph that, for each $i \in \{1, \dots, t\}$, the HSL-number of the top local cohomology module H_i of $R_{\mathfrak{p}_i}$ is equal to the corresponding number for $\widehat{R_{\mathfrak{p}_i}}$ and that the $R_{\mathfrak{p}_i}(c/1)$-stability index for $H_i/\Gamma_x(H_i)$ is equal to the corresponding index (for $\widehat{R_{\mathfrak{p}_i}}(c/1)$).

Since R is of acceptable type, it and all its localizations are universally catenary; therefore, by Ratliff's theorem (see [10, Thm. 31.7]), all the $R_{\mathfrak{p}_i}$ $(i = 1, \dots, t)$ are formally catenary (see [10, p. 252]). Since R is locally equidimensional, it follows that all the $\widehat{R_{\mathfrak{p}_i}}$ $(i = 1, \dots, t)$ are equidimensional; they are also excellent, because every complete Noetherian local ring is excellent. Thus Corollary 2.4(ii) can be applied over each $\widehat{R_{\mathfrak{p}_i}}$, $i = 1, \dots, t$.

For each $i = 1, \dots, t$, let the indices e_i and c_i stand for extension and contraction with respect to the natural ring homomorphism $R \to R_{\mathfrak{p}_i}$. Let $r \in R$ be such that $cr^{p^{n_1}} \in (\mathfrak{a}^{[p^{n_1}]})^*$ for one single $n_1 \geq u_0 + 1$. Choose $i \in \{1, \dots, t\}$. Then, in the local ring $R_{\mathfrak{p}_i}$ and its completion, we have

$$\begin{aligned}\frac{c}{1}\left(\frac{r}{1}\right)^{p^{n_1}} &= \frac{cr^{p^{n_1}}}{1} \in ((\mathfrak{a}^{[p^{n_1}]})^*)^{e_i} \subseteq ((\mathfrak{a}^{[p^{n_1}]})^{e_i})^* \\ &= ((\mathfrak{a}^{e_i})^{[p^{n_1}]})^* \subseteq ((\mathfrak{a}^{e_i}\widehat{R_{\mathfrak{p}_i}})^{[p^{n_1}]})^*.\end{aligned}$$

Now $\mathfrak{a}^{e_i}\widehat{R_{\mathfrak{p}_i}}$, if proper, is a parameter ideal of $\widehat{R_{\mathfrak{p}_i}}$. Since $n_1 \geq m_i + v_i + 1$, it thus follows from Corollary 2.4(ii) that $r/1 \in (\mathfrak{a}^{e_i}\widehat{R_{\mathfrak{p}_i}})^*$.

Since c is a completely stable p^{w_0}-weak test element for R, it follows that

$$\frac{c}{1}\left(\frac{r}{1}\right)^{p^n} \in ((\mathfrak{a}^{e_i})^{[p^n]}\widehat{R_{\mathfrak{p}_i}}) \cap R_{\mathfrak{p}_i} = (\mathfrak{a}^{e_i})^{[p^n]} \quad \text{for all } n \geq w_0.$$

Therefore, $r/1 \in (\mathfrak{a}^{e_i})^*$. Since R is locally equidimensional and of acceptable type, we can use [1, Thm. (8.3)(a)] to see that localization commutes with tight closure for the pair $\mathfrak{a} \subseteq R$; hence $r/1 \in (\mathfrak{a}^*)^{e_i}$ and $r \in (\mathfrak{a}^*)^{e_i c_i}$. This is true for all $i = 1, \dots, t$.

However, it follows from elementary facts about primary decomposition that the hypotheses about $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ ensure that $\mathfrak{a}^* = \bigcap_{i=1}^{t} (\mathfrak{a}^*)^{e_i c_i}$, so it follows that $r \in \mathfrak{a}^*$ as required. It is then immediate that p^{u_0+1} is a test exponent for $c, \mathfrak{a}$. $\square$

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Department of Pure Mathematics
University of Sheffield
Sheffield S3 7RH
United Kingdom

R.Y.Sharp@sheffield.ac.uk