Michigan Math. J. 54 (2006)

On Geometric Properties of Smooth Maps That Preserve $H^2(\mathbb{B}_n)$

Warren R. Wogen

Introduction

Suppose that Ω is a domain in $\mathbb{C}^n$ and that $\phi\colon \Omega \to \Omega$ is analytic on Ω. If X is a Banach space of analytic functions on Ω, let $C_\phi f = f \circ \phi$ for $f \in X$; here C_ϕ is the composition operator on X with symbol ϕ. A great deal of research has been done (see e.g. [CoM; S] and their extensive references) on composition operators for many choices of Ω and X. In particular, when Ω is the unit disk in $\mathbb{C}$ and X is the Hardy space H^p $(p \geq 1)$, it is classical that every C_ϕ is bounded on H^p.

The situation is different for $\Omega \subset \mathbb{C}^n$ with $n > 1$. We restrict our attention to $\Omega = \mathbb{B}_n = \mathbb{B}$, the open unit ball in $\mathbb{C}^n$. Write $\mathbb{S}_n = \mathbb{S}$ for the unit sphere in $\mathbb{C}^n$. Several authors [CSW; CW; M1] have constructed examples of analytic self-maps ϕ of $\mathbb{B}$ such that C_ϕ is unbounded on $H^p(\mathbb{B})$. Versions of most of these examples appear in [CoM, Chap. 6]. In particular, one can even take ϕ to be a univalent polynomial map (see [CW] and [CoM, Chap. 6.3]).

In [W1] the author proved a necessary and sufficient condition for boundedness of C_ϕ on $H^2(\mathbb{B})$ for the case when ϕ is a C^3 map on $\mathbb{B} \cup \mathbb{S} = \bar{\mathbb{B}}$; this paper is a sequel to [W1]. We first describe the main result of [W1] in Theorem 1. Then we establish an analytic consequence (Theorem 2) and a geometric consequence (Theorem 3). We also produce some new examples of symbols that induce unbounded composition operators.

Results

We begin by setting some notation. Suppose that $\psi : \mathbb{B} \to \mathbb{C}$ is a C^1-map and that $\xi \in S$. Then $D_\xi(z)$ denotes the (complex) directional derivative of ψ at z in the ξ direction.

Suppose that $\phi\colon \bar{\mathbb{B}} \to \mathbb{C}^n$ is analytic on $\mathbb{B}$ and is C^1 on $\bar{\mathbb{B}}$. For $z \in \bar{\mathbb{B}}$, $D\phi(z)$ is the (complex) Jacobian matrix. Also, if $\eta \in S$ then $\phi_\eta(z) = \langle \phi(z), \eta\rangle$ will denote the coordinate of ϕ in the η direction.

We state the result of [W1]. See also [CoM, Chap. 6.2] for a discussion of this theorem.

Theorem 1. *Suppose that $\phi\colon \bar{\mathbb{B}} \to \bar{\mathbb{B}}$ is analytic on $\mathbb{B}$ and is C^3 on $\bar{\mathbb{B}}$. Then the following statements are equivalent.*

Received January 3, 2005. Revision received March 24, 2005.
The author wishes to thank J. A. Cima and P. R. Mercer for helpful discussions.

(i) *C_ϕ is unbounded on $H^2(\mathbb{B})$.*
(ii) *There exist points ξ_1, ξ_2, and η in S such that ξ_1 and ξ_2 are orthogonal, $\phi(\xi_1) = \eta$, and*

$$D_{\xi_1}\phi_\eta(\xi_1) = |D_{\xi_2\xi_2}\phi_\eta(\xi_1)|. \tag{1}$$

Thus, to test C_ϕ for boundedness, find all $\xi_1 \in S$ such that $\phi(\xi_1) \in S$. Then, letting $\eta = \phi(\xi_1)$, compare $D_{\xi_1}\phi_\eta(\xi_1)$ with $|D_{\xi_2\xi_2}\phi_\eta(\xi_1)|$.

The proof of Theorem 1 uses Carleson measures. Equality (1) yields a collapse of the surface measure on S near ξ_1 under the mapping ϕ that violates the Carleson measure condition [M2] for boundedness of C_ϕ.

REMARK 1. Let $\{e_k\}_{k=1}^n$ be the standard basis for $\mathbb{C}^n$. By pre- and post-composing ϕ by unitary maps of $\mathbb{C}^n$, we may assume the following normalization: $\xi_1 = \eta = e_1$ and $\xi_2 = e_2$. Then write $D_{e_k} = D_k$ and $\phi_k = \phi_{e_k}$, $1 \le k \le n$. Replacing e_2 by λe_2 for an appropriate λ ($|\lambda| = 1$), we can also assume that $D_{22}\phi_1(e_1) \ge 0$ (cf. [CoM, pp. 231, 232]). Given this normalization, equality (1) becomes

$$D_1\phi_1(e_1) = D_{22}\phi_1(e_1). \tag{2}$$

Finally we note that one always has $D_k\phi_1(e_1) = 0$ and $|D_{kk}\phi_1(e_1)| \le D_1\phi_1(e_1)$ for $k \ge 2$; see [CoM, Lemma 6.6].

THEOREM 2. *Suppose ϕ satisfies the hypotheses of Theorem 1 and that C_ϕ is unbounded on $H^2(\mathbb{B})$. If* (1) *holds at $\xi_1 \in S$, then $D\phi(\xi_1)$ is singular.*

Proof. We assume the normalization as in Remark 1. We will analyze the second-order Taylor expansions about e_1 for the coordinate functions of ϕ.

Let $A_j = D_j\phi_1(e_1)$ and $A_{ij} = D_{ij}\phi_1(e_1)$, $1 \le i, j \le n$. Also, let $g(t) = (\cos t)e_1 + (\sin t)e_2 = (\cos t, \sin t, 0, \dots, 0)$. Here g parameterizes a smooth unit speed complex tangential curve in S, with $g(0) = e_1$. Suppose that $\phi(g(t)) = h(t) = (h_1(t), \dots, h_n(t))$. Recall that $A_2 = D_2\phi_1(e_1) = 0$ and that ϕ is a C^3-map. From the Taylor expansion of ϕ_1 about e_1, we have

$$\begin{aligned} h_1(t) &= 1 + A_1(\cos t - 1) \\ &\quad + \tfrac{1}{2}[A_{11}(\cos t - 1)^2 + 2A_{12}(\cos t - 1)\sin t + A_{22}\sin^2 t] + O(t^3). \end{aligned} \tag{3}$$

Substitute the Maclaurin series for $\sin t$ and $\cos t$ into (3). Then

$$h_1(t) = 1 + \left(-\tfrac{1}{2}A_1 + \tfrac{1}{2}A_{22}\right)t^2 + O(t^3) = 1 + O(t^3), \tag{4}$$

since (2) gives $A_1 = A_{22}$.

Next let $B_k = D_2\phi_k(e_1)$ for $k \ge 2$. Using the second Taylor polynomial for ϕ_k about e_1, we see that

$$h_k(t) = \phi_k(g(t)) = B_k \sin t + O(t^2) = B_k t + O(t^2). \tag{5}$$

From (4) and (5) it follows that

$$\|h(t)\|^2 \ge |h_1(t)|^2 + |h_k(t)|^2 = 1 + |B_k|^2 t^2 + O(t^3),$$

so if $B_k \neq 0$ then $\|h(t)\|^2 > 1$ for small t, a contradiction. We have shown that all entries in the second column of $D\phi(e_1)$ are zero, so that $D\phi(e_1)$ is singular. □

REMARK 2. Condition (2) is key for the Carleson measure estimates that prove (ii) implies (i) in Theorem 1.

Theorem 2 may significantly simplify the use of Theorem 1 in testing a specific C_ϕ for boundedness. Namely, given a smooth ϕ, one need only check condition (1) at those $\xi \in S$ such that $\phi(\xi) \in S$ *and* such that $D\phi(\xi)$ is singular. This may be most useful in case ϕ is univalent on $\mathbb{B}$ or at least locally univalent. Then $D\phi(z)$ will be invertible for all $z \in \mathbb{B}$, and invertibility of $D\phi(z)$ may well persist at points $z \in S$.

Next we analyze the geometry of the smooth mapping ϕ for C_ϕ unbounded. We continue to assume the normalization of Remark 1. Thus, in the terminology of the proof of Theorem 2, we have $A_1 = A_{22}$.

Fix λ with $|\lambda| = 1$ and $\lambda \neq \pm 1$, and let $g_\lambda(t) = (\cos t)e_1 + \lambda(\sin t)e_2$. Let $h_\lambda(t) = \phi(g_\lambda(t))$. The same computation that led to (4) gives us

$$(h_\lambda)_1(t) = 1 + \tfrac{1}{2}(-A_1 + \lambda^2 A_{22})t^2 + O(t^3). \tag{6}$$

Let $T_1 = \frac{1}{2}(-A_1 + \lambda^2 A_{22})$, and note that $T_1 \neq 0$. For $k \geq 2$, we saw in Theorem 2 that $B_k = D_2\phi_k(e_1) = 0$ and so $(h_\lambda)_k(t) = T_k t^2 + O(t^3)$, where $T_k = \frac{1}{2}(-D_1\phi_k(e_2) + \lambda^2 D_{22}\phi_k(e_1))$. Thus we have shown that

$$h_\lambda(t) - h_\lambda(0) = Tt^2 + O(t^3), \tag{7}$$

where $T = (T_1, \ldots, T_n)$. This means that h_λ, the image of g_λ under ϕ, has a cusp at $\phi(e_1) = e_1$. The "tangent" vector of this cusp is T. Note that $T_1 \neq 0$ (in fact $\operatorname{Re} T_1 \neq 0$), so T is transverse to the tangent plane to S at e_1.

Now let's suppose in addition that ϕ is univalent on $\bar{\mathbb{B}}$. We have seen that the smooth curves g_λ ($\lambda^2 \neq 1$) are "pinched" by ϕ into cusps h_λ. This pinching can be quantified in another way. For $t > 0$ and t small, $\|g_\lambda(t) - g_\lambda(-t)\| \approx 2t$. By (7) we have

$$\|\phi(g_\lambda(t)) - \phi(g_\lambda(t))\| = \|h_\lambda(t) - h_\lambda(-t)\| = O(t^3).$$

It follows that, if $\psi = \phi^{-1}$, then $\psi\colon \phi(\bar{\mathbb{B}}) \to \bar{\mathbb{B}}$ cannot be in the class $\operatorname{Lip}\alpha$ for any $\alpha > \frac{1}{3}$.

Now we apply the work of Mercer (see [Me, Prop. 2.6] and also [FSte, Prop. 12.2]) to deduce that $\phi(\mathbb{B})$ cannot be convex. In fact, the results of [Me] show that, for any proper map ψ of a convex domain Ω in $\mathbb{C}^n$ onto $\mathbb{B}_n$, we must have that ψ is in $\operatorname{Lip}\frac{1}{2}$. As Mercer has pointed out to the author, the proof requires taking $\alpha = 1$ and $m = 2$ in the proof of Proposition 2.6 of [Me]. We omit further details.

The preceding discussion proves the following conjecture of J. A. Cima.

THEOREM 3. *If ϕ is a biholomorphic map of $\mathbb{B}$ into $\mathbb{B}$ that extends to be C^3 on $\bar{\mathbb{B}}$ and if $\phi(\mathbb{B})$ is convex, then C_ϕ is bounded on $H^2(B)$.*

Examples

We begin by constructing a simple example that we then use to construct new unbounded operators C_ϕ with univalent symbols.

For $n \geq 2$, define $f\colon \bar{\mathbb{B}}_n \to \mathbb{C}$ by $f(z) = z_1 + \frac{1}{2}z_2^2$. For $z \in \mathbb{B}_n$, let $r = |z_1|$ and note that $|f(z)| \leq r + \frac{1}{2}(1 - r^2)$. Elementary arguments show that $|f(z)| < 1$ unless $r = 1$. So we see that $f(\mathbb{B}_n) \subset \mathbb{B}_1$ and $|f(\lambda e_1)| = 1$ if $|\lambda| = 1$. Also, $D_1 f(e_1) = 1 = D_{22} f(e_1)$. In fact, if $\xi_1 = \lambda e_1$ and $\xi_2 = \mu e_2$ where $|\lambda| = |\mu| = 1$, then $D_{\xi_1} f(\xi_1) = |D_{\xi_2\xi_2} f(\xi_1)|$.

EXAMPLE 1. Let $\phi = (f, 0, \ldots, 0)\colon \mathbb{B}_n \to \mathbb{B}_n$. Then C_ϕ is unbounded on $H^2(\mathbb{B}_n)$. This is immediate from Theorem 1 and our preceding discussion of f. This ϕ qualifies as the simplest possible example such that C_ϕ is unbounded.

Next we construct a new univalent example that illustrates Theorem 2.

EXAMPLE 2. For $n \geq 2$, define ϕ on $\bar{\mathbb{B}}_n$ by

$$\phi(z) = \tfrac{1}{2}(1 + f(z), z_2(1 - f(z)), \ldots, z_n(1 - f(z))).$$

If $|z| < 1$, then

$$\begin{aligned} |\phi(z)|^2 &= \frac{1}{4}\left\{|1 + f(z)|^2 + |1 - f(z)|^2 \sum_2^n |z_k|^2\right\} \\ &\leq \tfrac{1}{4}\{|1 + f(z)|^2 + |1 - f(z)|^2\} = \tfrac{1}{2}(1 + |f(z)|^2) < 1. \end{aligned}$$

Thus $\phi(\mathbb{B}_n) \subset \mathbb{B}_n$. Next we show that ϕ is univalent on $\bar{\mathbb{B}}_n$. Suppose that $z, w \in \bar{\mathbb{B}}_n$ and that $\phi(z) = \phi(w)$. Then $\phi_1(z) = \phi_1(w)$, so $f(z) = f(w)$. Also $\phi_k(z) = \phi_k(w)$ for $2 \leq k \leq n$ implies that $f(z) = f(w) = 1$ or $z_k = w_k$. But $f(z) = f(w) = 1$ yields $z = w = e_1$. If $z_k = w_k$ for $2 \leq k \leq n$, then from $f(z) = f(w)$ we see that $z_1 = w_1$ also.

Finally, observe that $\phi(e_1) = e_1$ and that $D_1\phi_1(e_1) = D_{22}\phi_1(e_1) = \frac{1}{2}$. Thus ϕ is a univalent polynomial map such that C_ϕ is unbounded. More complicated such maps may be found in [CW] and in [CoM, Chap. 6.3]. One can check that the singular matrix $D\phi(e_1)$ is $\frac{1}{2}P$, where P is the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}e_1$. Thus rank $D\phi(e_1) = 1$. We show in the next example how to modify this ϕ so that if $2 \leq k \leq n - 1$ then rank $D\phi(e_1) = k$.

EXAMPLE 3. We outline an example for the case $k = n - 1$. Let

$$\phi(z) = \tfrac{1}{2}(1 + f(z), z_2(1 - f(z)), z_3, \ldots, z_n)$$

for $z \in \bar{\mathbb{B}}_n$. By modifying the arguments of Example 2, we see that ϕ is univalent on $\bar{\mathbb{B}}_n$, $\phi(e_1) = e_1$, and $D_1\phi(e_1) = D_{22}\phi(e_1) = \frac{1}{2}$. We show that $\phi(\mathbb{B}_n) \subset \mathbb{B}_n$. For $z \in \mathbb{B}_n$ write $z = (z', z'')$, where $z' = (z_1, z_2)$. Then

$$\begin{aligned} |\phi_1(z)|^2 + |\phi_2(z)|^2 &\leq \tfrac{1}{4}\{|1 + f(z)|^2 + |1 - f(z)|^2\} \\ &= \tfrac{1}{2}(1 + |f(z)|^2) \leq \tfrac{1}{2}(1 + |z'|^2). \end{aligned}$$

Thus $|\phi(z)|^2 \leq \frac{1}{2}(1 + |z'|^2) + \frac{1}{4}|z''|^2 < 1$.

It follows that C_ϕ is unbounded on $H^2(\mathbb{B}_n)$. Note that $D\phi(e) = \frac{1}{2}Q$, where Q is the orthogonal projection of $\mathbb{C}^n$ onto $\{e_2\}^\perp$. Thus $D\phi(e)$ has rank $n-1$. It should be clear that we can interpolate Examples 2 and 3 to achieve rank $D\phi(e_1) = k$ for any k with $1 \le k \le n-1$.

We also point out that the previous constructions can be modified to produce families of symbols ϕ that induce unbounded C_ϕ. To illustrate, refer to Example 2. Given $0 < r < 1$, if we define ϕ by

$$\phi(z) = \bigl(r + (1-r)f(z), \sqrt{r(1-r)}z_2(1-f(z)), \dots, \sqrt{r(1-r)}z_n(1-f(z))\bigr)$$

then one can show that ϕ satisfies the same conditions as the mapping of Example 2. See also [CoM, Chap. 6.3].

Example 4. In the discussion preceding Theorem 3 we saw that, with the normalization of Remark 1, if C_ϕ is unbounded then ϕ maps the family of complex tangential curves g_λ ($\lambda \ne \pm 1$) into cusps. This might be expected since the tangent vectors to g_λ at e_1 lie in the kernel of $D\phi(e_1)$ and since the affine approximation $L(z) = \phi(e_1) + D\phi(e_1)(z - e_1)$ carries S into a complex plane of dimension $< n$. But the following example shows that the special curve g_1 need not be mapped to a cusp.

Define $\phi\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $\phi(z) = \frac{1}{2}\bigl(1 + f(z), \frac{1}{2}z_2^3\bigr)$. We first verify that $\phi(\mathbb{B}_2) \subset \mathbb{B}_2$. If $|z| < 1$ and $|z_1| = r$, then

$$\begin{aligned} |\phi(z)|^2 &\le \tfrac{1}{4}\bigl\{(1 + |f(z)|)^2 + \tfrac{1}{4}(1-r^2)^3\bigr\} \\ &\le \tfrac{1}{4}\bigl\{\bigl(1 + r + \tfrac{1}{2}(1-r^2)\bigr)^2 + \tfrac{1}{4}(1-r^2)^2\bigr\} \\ &= \tfrac{1}{4}\bigl\{(1+r)^2 + (1+r)(1-r^2) + \tfrac{1}{2}(1-r^2)^2\bigr\} = g(r). \end{aligned}$$

Check that g increases on $[0,1]$ and that $g(1) = 1$, so $\phi(\mathbb{B}_2) \subset \mathbb{B}_2$. Theorem 1 then shows that C_ϕ is unbounded, since $D_1\phi_1(e_1) = D_{22}\phi_1(e_1)$.

Let $h_1(t) = \phi(g_1(t)) = \frac{1}{2}\bigl(1 + \cos t + \frac{1}{2}\sin^2 t, \frac{1}{2}\sin^3 t\bigr)$. Using Maclaurin expansions for sin and cos, we obtain $h_1(t) - h_1(0) = \bigl(\frac{7}{24}t^4, \frac{1}{2}t^3\bigr) + O(t^5)$, so that

$$\frac{h_1(t) - h_1(0)}{t^3} \to \frac{1}{2}e_2 \quad \text{as } t \to 0.$$

Thus the approach of h_1 to e_1 as $t \to 0$ is complex tangential to S_2.

The map ϕ is not univalent on $\mathbb{B}_2$ (it is 3-to-1). It is an open question whether the phenomenon just described is possible for a univalent map.

References

[CSW] J. A. Cima, C. S. Stanton, and W. R. Wogen, *On boundedness of composition operators on $H^2(\mathbb{B}_2)$*, Proc. Amer. Math. Soc. 91 (1984), 217–222.

[CW] J. A. Cima and W. R. Wogen, *Unbounded composition operators on $H^2(\mathbb{B}_2)$*, Proc. Amer. Math. Soc. 99 (1987), 477–483.

[CoM] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions,* Stud. Adv. Math., CRC Press, Boca Raton, FL, 1995.

[FSte] J. E. Fornæss and B. Stensones, *Lectures on counterexamples in several complex variables,* Math. Notes, 33, Princeton Univ. Press, Princeton, NJ, 1987.

[M1] B. D. MacCluer, *Spectra of compact composition operators on* $H^2(B_N)$, Analysis 4 (1984), 87–103.

[M2] ———, *Compact composition operators on* $H^2(B_n)$, Michigan Math. J. 32 (1985), 237–248.

[Me] P. R. Mercer, *A general Hopf lemma and proper holomorphic mappings between convex domains in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 119 (1993), 573–578.

[R] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehern Math. Wiss., 241, Springer-Verlag, New York, 1980.

[S] J. H. Shapiro, *Composition operators and classical function theory,* Springer-Verlag, New York, 1993.

[W1] W. R. Wogen, *The smooth mappings which preserve the Hardy space* $H^2(\mathbb{B}_n)$, Oper. Theory Adv. Appl., 35, pp. 249–267, Birkhäuser, Basel, 1988.

[W2] ———, *Composition operators acting on spaces of holomorphic functions on domains in* $\mathbb{C}^n$, Proc. Sympos. Pure Math., 51 (part 2), pp. 361–366, Amer. Math. Sci., Providence, RI, 1991.

Department of Mathematics
University of North Carolina
Chapel Hill, NC 27599-3250

wrw@email.unc.edu