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The Multipole Lempert Function Is Monotone under Inclusion of Pole Sets

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Let D be a domain in $\mathbb{C}^n$ and let $A = (a_j)_{j=1}^l$, $1 \le l \le \infty$, be a countable (i.e. $l = \infty$) or nonempty finite (i.e. $l \in \mathbb{N}$) subset of D. Moreover, fix a function $\boldsymbol{p}: D \to \mathbb{R}_+$ with

$$|\boldsymbol{p}| := \{a \in D : \boldsymbol{p}(a) > 0\} = A;$$

$\boldsymbol{p}$ is called a *pole function for* A on D and $|\boldsymbol{p}|$ its *pole set*. When $B \subset A$ is a nonempty subset we put $\boldsymbol{p}_B := \boldsymbol{p}$ on B and $\boldsymbol{p}_B := 0$ on $D \setminus B$. Then $\boldsymbol{p}_B$ is a pole function for B.

For $z \in D$ we set

$$l_D(\boldsymbol{p}, z) = \inf\left\{\prod_{j=1}^{l} |\lambda_j|^{\boldsymbol{p}(a_j)}\right\},$$

where the infimum is taken over all subsets $(\lambda_j)_{j=1}^l$ of $\mathbb{D}$ (in this paper, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$) for which there is an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(\lambda_j) = a_j$ for all j. Here we call $l_D(\boldsymbol{p}, \cdot)$ the *Lempert function with* $\boldsymbol{p}$*-weighted poles at* A ([8; 9]; see also [5], where this function is called the *Coman function for* $\boldsymbol{p}$).

Wikström [8] has proved that, if A and B are finite subsets of a convex domain $D \subset \mathbb{C}^n$ with $\emptyset \ne B \subset A$ and if $\boldsymbol{p}$ is a pole function for A, then $l_D(\boldsymbol{p}, \cdot) \le l_D(\boldsymbol{p}_B, \cdot)$ on D.

On the other hand, in [9] Wikström gave an example of a complex space for which this inequality fails to hold, and he asked whether it remains true for an arbitrary domain in $\mathbb{C}^n$. The main purpose of this note is to present a positive answer to that question, even for countable pole sets. (In particular, it follows that the infimum in the definition of the Lempert function is always taken over a nonempty set.)

Theorem 1. *For any domain $D \subset \mathbb{C}^n$, any countable or nonempty finite subset A of D, and any pole function $\boldsymbol{p}$ for A, we have*

$$l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : \emptyset \ne B \text{ a finite subset of } A\}.$$

Therefore,

$$l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : \emptyset \ne B \subset A\}.$$

The proof of this result will be based on the following theorem.

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THEOREM (Arakelian's theorem [2]). *Let $E \subset \Omega \subset \mathbb{C}$ be a relatively closed subset of the domain Ω. Assume that $\Omega^* \setminus E$ is connected and locally connected. (Here Ω^* denotes the one-point compactification of Ω.)*

If f is a complex-valued continuous function on E that is holomorphic in the interior of E and if $\varepsilon > 0$, then there is a $g \in \mathcal{O}(\Omega)$ with $|g(z) - f(z)| < \varepsilon$ for any $z \in E$.

Proof. Fix a point $z \in D$. First, we shall verify the inequality

$$l_D(\boldsymbol{p}, z) \le \inf\{l_D(\boldsymbol{p}_B, z) : \emptyset \ne B \text{ a finite subset of } A\}. \tag{1}$$

Take a nonempty proper finite subset B of A. We may assume without loss of generality that $B = A_m := (a_j)_{j=1}^m$ for a certain $m \in \mathbb{N}$, where $A = (a_j)_{j=1}^l$ for $m < l \le \infty$.

Now, let $\varphi\colon \mathbb{D} \to D$ be an analytic disc with $\varphi(\lambda_j) = a_j$ $(0 \le j \le m)$, where $\lambda_0 := 0$ and $a_0 := z$. Fix $t \in [\max_{0\le j\le m}|\lambda_j|, 1)$ and put $\lambda_j := 1 - (1-t)/j^2$, $j \in A(m)$, where $A(m) := \{m+1, \dots, l\}$ if $l < \infty$ or $A(m) := \{j \in \mathbb{N} : j > m\}$ if $l = \infty$. Consider a continuous curve $\varphi_1\colon [t, 1) \to D$ such that $\varphi_1(t) = \varphi(t)$ and $\varphi_1(\lambda_j) = a_j$ for $j \in A(m)$. Define

$$f = \begin{cases} \varphi|_{t\overline{\mathbb{D}}} \\ \varphi_1|_{[t,1)} \end{cases}$$

on the set $F_t := t\overline{\mathbb{D}} \cup [t, 1) \subset \mathbb{D}$. Observe that F_t satisfies the geometric condition in Arakelian's theorem.

Since $(\lambda_j)_{j=0}^l$ satisfy the Blaschke condition, for any k we may find a Blaschke product B_k with zero set $(\lambda_j)_{j=0, j\ne k}^l$. Moreover, we denote by d the function $\operatorname{dist}(\partial D, f)$ on F_t, where the distance arises from the l^∞-norm. Let η_1, η_2 be continuous real-valued functions on F_t with

$$\eta_1, \eta_2 \le \log\frac{d}{9}, \qquad \eta_1, \eta_2 = \min_{t\overline{\mathbb{D}}} \log\frac{d}{9} \ \text{ on } (t\overline{\mathbb{D}}),$$

$$\eta_1(\lambda_j) = \eta_2(\lambda_j) + \log(2^{-j-1}|B_j(\lambda_j)|), \quad j \in A(m).$$

Applying Arakelian's theorem three times, we can find functions $\zeta_1, \zeta_2 \in \mathcal{O}(\mathbb{D})$ and a holomorphic mapping h on $\mathbb{D}$ such that

$$|\zeta_1 - \eta_1| \le 1, \qquad |\zeta_2 - \eta_2| \le 1,$$

$$|h - f| \le \varepsilon|e^{\zeta_1 - 1}| \le \varepsilon e^{\eta_1} \ \text{ on } F_t,$$

where $\varepsilon := \min\{|B_j(\lambda_j)|/2^{j+1} : j = 0, \dots, m\} < 1$ (in the last case apply Arakelian's theorem componentwise to the mapping $e^{1-\zeta_1} f$).

In particular, we have

$$|h - f| \le \frac{d}{9} \ \text{ on } F_t,$$

$$|\gamma_j| \le e^{\eta_1(\lambda_j)} 2^{-j-1}|B_j(\lambda_j)| = e^{\eta_2(\lambda_j)} 2^{-j-1}|B_j(\lambda_j)|, \quad j = 0, \dots, m,$$

$$|\gamma_j| \le e^{\eta_1(\lambda_j)} \le e^{\eta_2(\lambda_j)} 2^{-j-1}|B_j(\lambda_j)|, \quad j \in A(m);$$

here $\gamma_j := h(\lambda_j) - f(\lambda_j)$ with $j \in \mathbb{Z}_+$ if $l = \infty$ or with $0 \le j \le l$ if $l \in \mathbb{N}$. Then, by virtue of $e^{\eta_2(\lambda_j)} \le e^{1+\operatorname{Re}\zeta_2(\lambda_j)}$, the function

$$g := e^{\zeta_2} \sum_{j=0}^{l} \frac{B_j}{e^{\zeta_2(\lambda_j)} B_j(\lambda_j)} \gamma_j$$

is holomorphic on $\mathbb{D}$ with $g(\lambda_j) = \gamma_j$, and

$$|g| \le e^{\operatorname{Re}\zeta_2 + 1} \le e^{\eta_2 + 2} \le \frac{e^2}{9} d \ \text{ on } F_t;$$

for $q_t := h - g$ it follows that $q_t(\lambda_j) = f(\lambda_j)$ and

$$|q_t - f| \le \frac{e^2 + 1}{9} d < d \ \text{ on } F_t.$$

Thus we have found a holomorphic mapping q_t on $\mathbb{D}$ with $q_t(\lambda_j) = a_j$ and $q_t(F_t) \subset D$. Hence there is a simply connected domain E_t such that $F_t \subset E_t \subset \mathbb{D}$ and $q_t(E_t) \subset D$.

Let $\rho_t : \mathbb{D} \to E_t$ be the Riemann mapping with $\rho_t(0) = 0$, $\rho_t'(0) > 0$, and $\rho_t(\lambda_j^t) = \lambda_j$. Considering the analytic disc $q_t \circ \rho_t : \mathbb{D} \to D$, we deduce that

$$l_D(\boldsymbol{p}, z) \le \prod_{j=1}^{l} |\lambda_j^t|^{\boldsymbol{p}(a_j)} \le \prod_{j=1}^{m} |\lambda_j^t|^{\boldsymbol{p}(a_j)}.$$

Note that, by the Carathéodory kernel theorem, ρ_t tends (locally uniformly) to the identity map of $\mathbb{D}$ as $t \to 1$. This shows that the last product converges to $\prod_{j=1}^{m} |\lambda_j|^{\boldsymbol{p}(a_j)}$. Since φ was an arbitrary competitor for $l_D(\boldsymbol{p}_{A_m}, z)$, inequality (1) follows.

On the other hand, the existence of an analytic disc whose graph contains A and z implies that

$$l_D(\boldsymbol{p}, z) \ge \limsup_{m \to \infty} l_D(\boldsymbol{p}_{A_m}, z),$$

which completes the proof. □

REMARK 2. In the foregoing proof we have demonstrated an approximation and simultaneous interpolation result—that is, the constructed function q_t approximates and interpolates the given function f.

We first mention that the proof of Theorem 1 could be simplified by using a nontrivial result of Carleson on interpolation sequences (see e.g. [1, Chap. 7, Thm. 3.1]). Moreover, it is possible to prove the following general result, which extends an approximation and simultaneous interpolation result of Gauthier and Hengartner [4] and Nersesyan [6].

Let $D \subset \mathbb{C}$ be a domain, $E \subset D$ a relatively closed subset satisfying the condition in Arakelian's theorem, and $\Lambda \subset E$ such that Λ has no accumulation point in D and $\Lambda \cap \operatorname{int} E$ is a finite set. Then, for given functions $f, h \in \mathcal{C}(E) \cap \mathcal{O}(\operatorname{int} E)$, there exists a $g \in \mathcal{O}(D)$ such that

$$|g - f| < e^{\operatorname{Re} h} \text{ on } E \quad \text{and} \quad f(\lambda) = g(\lambda),\ \lambda \in \Lambda.$$

It is even possible to prescribe a finite number of the derivatives for g at all the points in Λ.

As a by-product, we have the following result.

COROLLARY 3. *Let* $D \subset \mathbb{C}^n$ *be a domain and let* $\boldsymbol{p}, \boldsymbol{q} \colon D \to \mathbb{R}_+$ *be two pole functions on* D, *where* $\boldsymbol{p} \le \boldsymbol{q}$ *with* $|\boldsymbol{q}|$ *at most countable. Then* $l_D(\boldsymbol{q}, z) \le l_D(\boldsymbol{p}, z)$ *for* $z \in D$.

Hence, the Lempert function is monotone with respect to pole functions having a support that is at most countable.

REMARK 4. In general, the Lempert function is not strictly monotone under inclusion of pole sets. For example, take $D := \mathbb{D} \times \mathbb{D}$ and $A := \{a_1, a_2\} \subset \mathbb{D}$ with $a_1 \ne a_2$, and observe that $l_D(\boldsymbol{p}, (0,0)) = |a_1|$ where $\boldsymbol{p} := \chi|_{A \times \{a_1\}}$ (use the product property from [3; 5; 7]).

REMARK 5. Let $D \subset \mathbb{C}^n$ be a domain, let $z \in D$, and let now $\boldsymbol{p} \colon D \to \mathbb{R}_+$ be a "general" pole function (i.e., $|\boldsymbol{p}|$ is uncountable). Then there are two cases as follows.

1. There exists a $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(\lambda_{\varphi,a}) = a$, $\lambda_{\varphi,a} \in \mathbb{D}$, for all $a \in |\boldsymbol{p}|$ and $\varphi(0) = z$. If we define

$$l_D(\boldsymbol{p}, z) := \inf\Big\{ \prod |\lambda_{\psi,a}|^{\boldsymbol{p}(a)} : \psi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \psi(\lambda_{\psi,a}) = a \text{ for all } a \in |\boldsymbol{p}|,\ \psi(0) = z \Big\},$$

then $l_D(\boldsymbol{p}, z) = 0$. Observe that $l_D(\boldsymbol{p}, z) = \inf\{l_D(\boldsymbol{p}_B, z) : \emptyset \ne B \text{ a finite subset of } A\}$.

2. There is no analytic disc as in case 1; hence, in this case we may define

$$l_D(\boldsymbol{p}, z) := \inf\{l_D(\boldsymbol{p}_B, z) : \emptyset \ne B \text{ a finite subset of } A\}$$

(cf. the definition of the Coman function in [5]).

Example 7 may show that the definition in case 2 is more sensitive than the one used in [5].

Before giving the example, we use our preceding definition of $l_D(\boldsymbol{p}, \cdot)$ for an arbitrary pole function $\boldsymbol{p}$ to conclude as follows.

COROLLARY 6. *Let* $D \subset \mathbb{C}^n$ *be a domain and let* $\boldsymbol{p}, \boldsymbol{q} \colon D \to \mathbb{R}_+$ *be arbitrary pole functions with* $\boldsymbol{p} \le \boldsymbol{q}$. *Then* $l_D(\boldsymbol{q}, \cdot) \le l_D(\boldsymbol{p}, \cdot)$.

EXAMPLE 7. Put $D := \mathbb{D} \times \mathbb{D}$ and let $A \subset \mathbb{D}$ be uncountable—for example, $A = \mathbb{D}$. Then there is no $\varphi \in \mathcal{O}(\mathbb{D}, D)$ passing through $A \times \{0\}$ and $(0, w)$ with $w \in \mathbb{D}_*$. Put $\boldsymbol{p} := \chi|_{A \times \{0\}}$ on D as a pole function.

Let $B \subset A$ be a nonempty finite subset. Then, applying the product property (see [3; 5; 7]), we obtain

$$\log l_D(\boldsymbol{p}_{B\times\{0\}}, (0,w)) = g_D(B \times \{0\}, (0,w)) = \max\{g_{\mathbb{D}}(B,0), g_{\mathbb{D}}(0,w)\},$$

where $g_{\mathbb{D}}(A,\cdot)$ denotes the Green function in $\mathbb{D}$ with respect to the pole set A.

Therefore, $l_D(\boldsymbol{p},(0,w)) = |w|$.

To conclude this note we mention that the Lempert function is not holomorphically contractible, even if the holomorphic map is a proper covering.

Example 8. Let $\pi : \mathbb{D}_* \to \mathbb{D}_*$ with $\pi(z) := z^2$. Obviously, π is proper and a covering. Fix two different points $a_1, a_2 \in \mathbb{D}_*$ with $a_1^2 = a_2^2 =: c$. Let $z \in \mathbb{D}_*$ with $z \neq a_j$, $j = 1,2$. Recalling how to calculate the classical Lempert function via the covering map π, we have

$$\begin{aligned} l_{\mathbb{D}_*}(\chi|_{\{c\}}, z^2) &= \min\{l_{\mathbb{D}_*}(\chi|_{\{a_1\}}, z), l_{\mathbb{D}_*}(\chi|_{\{a_2\}}, z)\} \\ &> l_{\mathbb{D}_*}(\chi|_{\{a_1\}}, z) l_{\mathbb{D}_*}(\chi|_{\{a_2\}}, z), \end{aligned}$$

where $\chi|_B$ is the characteristic function for the set $B \subset \mathbb{D}_*$. On the other hand, we can see (using e.g. [7, Prop. 2]) that the last product is equal to $l_{\mathbb{D}_*}(\chi|_{\{a_1,a_2\}}, z)$. Hence, $l_{\mathbb{D}_*}(\boldsymbol{p}, \pi(z)) > l_{\mathbb{D}_*}(\boldsymbol{p} \circ \pi, z)$ for $\boldsymbol{p} := \chi|_{\{c\}}$.

Therefore, the Lempert function with a multipole behaves worse than the Green function with a multipole.

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Added in Proof. After we submitted this paper for publication, the preprint "Holomorphic discs with dense images" by F. Forstnerič and J. Winkelmann appeared (arXiv:math.CV/0410390; see also Math. Res. Lett. 12 (2005), 265–268). There—using an idea similar to that in the proof of our Theorem 1 and replacing Arakelian's theorem by an approximation theorem due to Forstnerič—the following result is proved.

Let M be a connected complex manifold endowed with a complete Riemannian metric, d the induced distance, A a countable subset of M, $f \in \mathcal{O}(\mathbb{D}, X)$, and $r \in (0,1)$. Then there exists a $g \in \mathcal{O}(\mathbb{D}, X)$ such that $A \subset g(\mathbb{D})$ and $d(f(z), g(z)) < 1 - r$ for any $z \in r\mathbb{D}$.

A slight modification to the proof of this result may show that, if (in addition to A, f, r) a finite subset Λ of $\mathbb{D}$ is given, then one can provide a function g as before and a sequence $(\mu_\lambda)_{\lambda\in\Lambda} \subset \mathbb{D}$, $r|\mu_\lambda| < |\lambda|$, such that $f(\lambda) = g(\mu_\lambda)$ for $\lambda \in \Lambda$. Letting $r \to 1$, this implies that Theorem 1 still holds for connected complex manifolds.

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