Michigan Math. J. 53 (2005)

The Rank-2 Lattice-Type Vertex Operator Algebras V_L^+ and Their Automorphism Groups

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1. Introduction

This article continues a program to study automorphism groups of vertex operator algebras (VOAs). See references in the survey [G2] and the more recent articles [G3], [DG1], [DG2], [DGR], and [DN1].

Here we investigate the fixed point subVOA of a lattice-type VOA with respect to a group of order 2 lifting the -1 map on a positive definite lattice. We can obtain a definitive answer for the automorphism group of this subVOA in two extreme cases. The first is where the lattice has no vectors of norm 2 or 4, and the second is where the lattice has rank 2.

We use the standard notation V_L for a lattice VOA based on the positive definite even integral lattice L. For a subgroup G of $\mathrm{Aut}(L)$, V_L^G denotes the subVOA of points fixed by G. When G is a group of order 2 lifting -1_L, it is customary to write V_L^+ for the fixed points (even though, strictly speaking, G is defined only up to conjugacy; see the discussion in [DGH] or [GH]).

The rank-2 case is a natural extension of work on the rank-1 case, where $\mathrm{Aut}(V_L^G)$ was determined for all rank-1 lattices L and all choices of finite group $G \leq \mathrm{Aut}(V_L)$. The styles of proofs are different. In the rank-1 case, there was heavy analysis of the representation theory of the principal Virasoro subVOA on the ambient VOA. In the rank-2 case, there is a lot of work on idempotents and solving nonlinear equations as well as work with several subVOAs associated to Virasoro elements. For rank 2, the case of nontrivial degree-1 part is harder to settle than in rank 1.

Our strategy follows this model. Let V be one of our V_L^+. We get information about $G := \mathrm{Aut}(V)$ by its action on the finite-dimensional algebra $A := (V_2, 1^{st})$. We take a subset S of A that is G-invariant and understand S well enough to limit the possibilities for G (usually, there are no automorphisms besides the ones naturally inherited from V_L). A natural choice for S is the set of idempotents or conformal vectors. Usually, S spans A or at least generates A. In the main case of a rank-2 lattice, we prove that $\mathrm{Aut}(V)$ fixes a subalgebra of A that is the natural $M(1)_2^+$. The structure of V is controlled by $M(1)^+$, which is generated by $M(1)_2^+$ and its eigenspaces, so we eventually determine G.

Received September 21, 2004. Revision received December 22, 2004.
The first author is supported by NSF grants, a NSF grant of China, and a research grant from UC Santa Cruz. The second author is supported by NSA Grant no. USDOD-MDA904-03-1-0098.

For several results, we give more than one proof. For the case of a lattice L without roots, the automorphism group of V_L^+ was studied in [S]. We thank Harm Derksen for help with computer algebra.

2. Background Definitions and Notation

NOTATION 2.1. Let L be an even integral lattice. For an integer m, define $L_m := \{x \in L \mid (x,x) = 2m\}$. Let $H := \mathbb{C} \otimes L$, the ambient complex vector space. For a subset S of L, define $\mathrm{rank}(S)$ to be the rank of the sublattice spanned by S.

DEFINITION 2.2. For a lattice L, the group of automorphisms of the free abelian group L that preserves the bilinear form is called the *group of automorphisms,* the *isometry group,* the *group of units,* or the *orthogonal group of* L. This group is denoted $\mathrm{Aut}(L)$ or $O(L)$. We will use the notation $O(L)$ in this article as well as the associated $\mathrm{SO}(L)$ for the elements of determinant 1, $\mathrm{PO}(L)$ for $O(L)/\{\pm 1\}$, and $\mathrm{PSO}(L)$ for $\mathrm{SO}(L)/\mathrm{SO}(L) \cap \{\pm 1\}$.

DEFINITION 2.3. For an even integral lattice L, we let $\hat{L}$ be the 2-fold cover of L described in [DGH; FLMe; GH]. We may write bars for the map $\hat{L} \to L$. The *group of automorphisms,* the *isometry group,* the *group of units,* or the *orthogonal group* is the set of group automorphisms of $\hat{L}$ that preserve the bilinear form on the quotient of $\hat{L}$ by the normal subgroup of order 2; it is denoted $\mathrm{Aut}(\hat{L})$ or $O(\hat{L})$ and has shape $2^{\mathrm{rank}(L)}.O(L)$. A bar (cf. §3.2) denotes the natural map $O(\hat{L}) \to O(L)$.

We next list some notation that is used for work with lattice-type VOAs.

$\mathcal{D}(L)$: the discriminant group of the integral lattice L is $\mathcal{D}(L) := L^*/L$.

e^α: standard basis element for $\mathbb{C}[L]$.

FVOA: framed vertex operator algebra [DGH].

LVOA: lattice vertex operator algebra [FLMe].

LVOA type: the fixed points of a lattice vertex operator algebra under a finite group of automorphisms [DG1; DGR].

LVOA$^+$: V_L^+ for an even lattice L.

LVOAG(L): the subgroup of $\mathrm{Aut}(V_L)$, for an even integral lattice L, as described in [DN1]. It is denoted $\mathbb{N}(\hat{L})$ and is an extension of the form $T.\mathrm{Aut}(L)$ (possibly nonsplit), where T is a natural copy of the torus $\mathbb{C} \otimes L/L^*$ obtained by exponentiating the maps $2\pi x_0$ for $x \in V_1$; the quotient of this group by the normal subgroup T is naturally isomorphic to $\mathrm{Aut}(L)$. Also, $\mathbb{N}(\hat{L})$ is the product of subgroups TS, where $S \cong O(\hat{L})$ and $S \cap T = \{x \in T \mid x^2 = 1\} \cong \mathbb{Z}_2^{\mathrm{rank}(L)}$. We may take S to be the centralizer in LVOAG(L) of a lift of -1; it has the form $2^{\mathrm{rank}(L)}.\mathrm{Aut}(L)$, and in fact any such S has this form. Denote the groups S and T by $\mathbb{O}(\hat{L})$ and $\mathbb{T}(\hat{L})$, respectively.

LVOA group for L: this means LVOAG(L).

LVOAG: this means LVOAG(L) for some L.

LVOAG$^+$(L): this is the centralizer in LVOAG(L) of a lift of -1 modulo the group of order 2 generated by the lift; it has the form $2^{\mathrm{rank}(L)}.[\mathrm{Aut}(L)/\langle -1\rangle]$ and is the *inherited group.*

LVOAG$^+$: this means LVOAG$^+(L)$ for some L.
LVOA$^+$ group: same as LVOAG$^+$.
$M(1), M(1)^+$: see Section 3.
$\mathbb{N}(\hat{L})$: see LVOAG(L).
o: linear map from V to End(V).
$\mathbb{O}(\hat{L})$: see LVOAG(L).
$\mathbb{T}(\hat{L})$: see LVOAG(L).
v_α: $e^\alpha + e^{-\alpha}$.
$\mathbb{X}$ or $\mathbb{X}(L)$: given an even integral lattice L, this is a group of shape $2^{1+\mathrm{rank}(L)}$ for which commutation corresponds to inner products modulo 2; see an appendix of [GH].
$\mathbb{XO}$ or $\mathbb{XO}(\hat{L})$: an extension of $\mathbb{X}$ upward by $O(L)$.
$\mathbb{XPO}$ or $\mathbb{XPO}(\hat{L})$: a quotient of $\mathbb{XO}$ by a central involution that corresponds to -1_L under the natural epimorphism to $O(L)$.

REMARK 2.4. If $(L, L) \subset 2\mathbb{Z}$, then $\hat{L} \cong L \times \langle \pm 1 \rangle$. Thus $O(\hat{L})$ contains a copy of $O(L)$ that complements the normal subgroup of order $2^{\mathrm{rank}(L)}$ consisting of automorphisms that are trivial on the quotient group L of $\hat{L}$. This splitting passes to the groups $\mathrm{PO}(\hat{L})$ and $\mathbb{XPO}(L)$.

3. Automorphism Group of V_L^+ with $L_1 = L_2 = \emptyset$

In this section, we determine the automorphism group of V_L^+ with $L_1 = L_2 = \emptyset$ and assume only that $\mathrm{rank}(L) > 1$. The automorphism group of V_L^+ in case $\mathrm{rank}(L) = 1$ is determined in [DG1] without any restriction on L. The assumption that $L_1 = L_2 = \emptyset$ ensures that any automorphism of V_L^+ preserves the subspace $M(1)_2^+$, which can be identified with the Jordan algebra S^2H.

Since $M(1)^+$ is generated by $M(1)_2^+$ if $\dim H > 1$ and since V_L^+ is a direct sum of eigenspaces for $M(1)_2^+$ (cf. [AD]), the structure of $\mathrm{Aut}(V_L^+)$ can be determined easily. We shall use a classic result.

PROPOSITION 3.1. *The automorphism group of the Jordan algebra of symmetric $n \times n$ matrices is* $\mathrm{PO}(n, \mathbb{C})$, *acting by conjugation.*

Proof. See [J]. □

3.1. $\mathrm{Aut}(M(1)^+)$

We first recall the construction of $M(1)^+$. Let H be an n-dimensional complex vector space with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$, and let $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the corresponding affine Lie algebra. Consider the induced $\hat{H}$-module

$$M(1) = \mathcal{U}(\hat{H}) \otimes_{\mathcal{U}(H \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C} \simeq S(H \otimes t^{-1}\mathbb{C}[t^{-1}]) \quad \text{(linearly)},$$

where $H \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$ and where c acts as 1. For $\alpha \in H$ and $n \in \mathbb{Z}$ we set $\alpha(n) := \alpha \otimes t^n$. Let τ be the automorphism of $M(1)$ such that

$$\tau(\alpha_1(-n_1)\cdots\alpha_k(-n_k)) = (-1)^k\alpha_1(-n_1)\cdots\alpha_k(-n_k)$$

for $\alpha_i \in H$ and $n_1 \geq \cdots \geq n_k \geq 1$. Then $M(1)^+$ is the fixed point subspace of τ.

PROPOSITION 3.2. *The automorphism group of* $M(1)^+$ *is* $\mathrm{PO}(n,\mathbb{C})$.

Proof. We first deal with the case dim $H > 1$. Then $M(1)^+$ is generated by $M(1)_2^+$ (cf. [DN2]), which is a Jordan algebra under $u \cdot v = u_1 v$ for $u, v \in M(1)_2^+$. So any automorphism of $M(1)^+$ restricts to an automorphism of the Jordan algebra $M(1)_2^+$. On the other hand, the automorphism group of $M(1)$ is $O(n,\mathbb{C})$ [DM2], which preserves $M(1)^+$. Clearly, the kernel of the action of $O(n,\mathbb{C})$ on $M(1)^+$ is $\{\pm 1\}$ and so $\mathrm{PO}(n,\mathbb{C})$ is a subgroup of the automorphism group of $M(1)^+$. By Proposition 3.1, any automorphism of $M(1)_2^+$ extends to an automorphism of $M(1)^+$.

We now assume that dim $H = 1$. Then $M(1)^+$ is not generated by $M(1)_2^+$. By Lemma 2.6 and Theorem 2.7 of [DG1], for any nonnegative even integer n there is a unique lowest weight vector u^n (up to scalar multiple) of weight n^2, and $M(1)^+$ is generated by the Virasoro vector and u^n. Using the fusion rule given in Lemma 2.6 of [DG1], we immediately see that the automorphism group of $M(1)^+$ in this case is trivial. Clearly, $\mathrm{PO}(1,\mathbb{C}) = 1$. This finishes the proof. □

3.2. $\mathrm{Aut}(V_L^+)$

First we review from [B] and [FLMe] the construction of a lattice vertex operator algebra V_L for any positive definite even lattice L. Let $H = \mathbb{C}\otimes_{\mathbb{Z}} L$. Recall that $\hat{L}$ is the canonical central extension of L by the cyclic group $\langle\pm 1\rangle$ such that the commutator map is given by $c(\alpha,\beta) = (-1)^{(\alpha,\beta)}$. We fix a bimultiplicative 2-cocycle $\varepsilon\colon L\times L \to \langle\pm 1\rangle$ such that $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = c(\alpha,\beta)$ for $\alpha,\beta \in L$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}]\otimes_{\mathbb{C}[\langle\pm 1\rangle]}\mathbb{C} \simeq \mathbb{C}[L] \quad \text{(linearly)},$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and -1 acts on $\mathbb{C}$ as multiplication by -1. For $a \in \hat{L}$, write $\iota(a)$ for $a\otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by $a\cdot\iota(b) = \iota(ab)$ for $a,b\in\hat{L}$. If $(L,L)\subset 2\mathbb{Z}$ then $\mathbb{C}\{L\}$ and $\mathbb{C}[L]$ are isomorphic algebras. The lattice vertex operator algebra V_L is defined to be $M(1)\otimes\mathbb{C}\{L\}$, as a vector space.

It follows that $O(\hat{L})$ is a naturally defined subgroup of $\mathrm{Aut}(\hat{L})$, that $\mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z})$ may be identified with a subgroup of $O(\hat{L})$ (see [DN1; FLMe; GH]), and that there is an exact sequence

$$1 \to \mathrm{Hom}(L,\mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \xrightarrow{-} O(L) \to 1.$$

It is proved in [DN1] that $\mathrm{Aut}(V_L)$ has shape $N\cdot O(\hat{L})$, where N is the normal subgroup of $\mathrm{Aut}(V_L)$ generated by e^{u_0} for $u \in (V_L)_1$. Observe that $\mathrm{Hom}(L,\mathbb{Z}/2\mathbb{Z})$ can furthermore be identified with the intersection of N and $O(\hat{L})$; see the notation listed after Definition 2.3.

Let $e\colon L\to\hat{L}$ be a section associated to the 2-cocycle ε, written $\alpha\mapsto e_\alpha$, such that $e_0 = 1$. Let θ be the automorphism of $\hat{L}$ of order 2 such that $\theta e_\alpha = e_{-\alpha}$ for

$\alpha \in L$. Then θ extends to an automorphism of V_L, still denoted by θ, such that $\theta|_{M(1)}$ is identified with τ and $\theta\iota(a) = \iota(\theta a)$ for all $a \in \hat{L}$. Set $e^\alpha = \iota(e_\alpha)$. Then $\theta e^\alpha = e^{-\alpha}$.

Let V_L^+ be the fixed points of θ. In order to determine the automorphism group of V_L^+, it is important to understand which automorphism of V_L restricts to an automorphism of V_L^+. Clearly, the centralizer of θ in $\mathrm{Aut}(V_L)$ acts on V_L^+, so we get an action of $O(\hat{L})/\langle \pm 1\rangle$ on V_L^+. Let $h \in H$. Then $e^{2\pi i h(0)}$ preserves V_L^+ if and only if $(h,\alpha) \equiv (h,-\alpha)$ modulo $\mathbb{Z}$ for any $\alpha \in L$. That is: $h \in \frac{1}{2}L^*$, where L^* is the dual lattice of L.

LEMMA 3.3. *The subgroup of* $\mathrm{Aut}(V_L^+)$ *that preserves* $M(1)_2^+$ *is just the LVOA*$^+$ *group.*

Proof. Let $n := \dim(H)$, and let $\sigma \in \mathrm{Aut}(V_L^+)$ be such that $\sigma M(1)_2^+ \subset M(1)_2^+$. Then $\sigma|_{M(1)_2^+} \in \mathrm{PO}(n,\mathbb{C})$ as in Proposition 3.2. Note that $M(1)^+$ is generated by $M(1)_2^+$ as $\mathrm{rank}(L) > 1$ (see the proof of Proposition 3.2). Hence σ preserves $M(1)^+$.

For any $\alpha \in L$, let $V_L^+(\alpha)$ be the $M(1)^+$-submodule generated by $v_\alpha := e^\alpha + e^{-\alpha}$. Then $V_L^+(\alpha)$ is an irreducible $M(1)^+$-module, and $V_L^+(\alpha)$, $V_L^+(\beta)$ are isomorphic $M(1)^+$-modules if and only if $\alpha = \pm\beta$ (cf. [AD]). Moreover, if $\alpha \neq 0$ then $V_L^+(\alpha)$ is isomorphic to $M(1) \otimes e^\alpha$ (cf. [AD]).

Note that $V_L^+ = \sum_{\alpha\in L} V_L^+(\alpha)$. Let S be a subset of L such that $|S \cap \{\pm\alpha\}| = 1$ for any $\alpha \in L$. Then, for any two different $\alpha, \beta \in S$, it follows that $V_L^+(\alpha)$, $V_L^+(\beta)$ are nonisomorphic $M(1)^+$-modules and that

$$V_L^+ = \bigoplus_{\alpha\in S} V_L^+(\alpha)$$

is a direct sum of nonisomorphic irreducible $M(1)^+$-modules.

Let $\alpha \in L$. Because σ preserves $M(1)^+$, it sends $V_L^+(\alpha)$ to $V_L^+(\beta)$ for some $\beta \in L$. The vector v_α is the unique lowest weight vector (up to a scalar) of $V_L^+(\alpha)$. This implies that $\sigma(v_\alpha) = \lambda v_\beta$ for some nonzero scalar $\lambda \in \mathbb{C}$ (depending on α and β).

For a vertex operator algebra V and a homogeneous $v \in V$, we set $o(v) = v_{\mathrm{wt}(v)-1}$ and extend to all of V linearly. Note that v_α is an eigenvector of $o(v)$ for $v \in M(1)_2^+$. In fact, $o(h_1(-1)h_2(-1))v_\alpha = (h_1,\alpha)(h_2,\alpha)v_\alpha$ for $h_i \in H$. Recall the proof of Proposition 3.2. We can regard the restriction of σ to $(V_L^+)_2 \cong M(1)_2^+$ as an element of $O(n,\mathbb{C})$ that is well-defined modulo ± 1. Then $\sigma(h_1(-1)h_2(-1)) = (\sigma h_1)(-1)(\sigma h_2)(-1)$. Note that σ^{-1} is the adjoint of σ. Therefore,

$$\begin{aligned}(h_1,\alpha)(h_2,\alpha)\lambda v_\beta &= \sigma((h_1,\alpha)(h_2,\alpha)v_\alpha) = \sigma(o(h_1(-1)h_2(-1))v_\alpha)\\ &= o(\sigma(h_1(-1)h_2(-1)))\lambda v_\beta = (\sigma h_1,\beta)(\sigma h_2,\beta)\lambda v_\beta.\end{aligned}$$

Since the h_i are arbitrary, we have $\sigma\alpha = \pm\beta$. Thus σ maps L onto L and so induces an isometry of L that is well-defined modulo $\langle\pm 1\rangle$.

Multiplying σ by an element from $\mathrm{LVOAG}^+(L)$ (which comes from $\mathbb{N}(\hat{L})$), we can assume that $\sigma|_{M(1)^+} = \mathrm{id}_{M(1)^+}$. Then $\sigma v_\alpha = \lambda_\alpha v_\alpha$ for some nonzero $\lambda_\alpha \in \mathbb{C}$.

Since $V_L^+(\alpha)$ is an irreducible $M(1)^+$-module, we see that σ acts as the scalar λ_α on $V_L^+(\alpha)$. Clearly, $\lambda_\alpha = \lambda_{-\alpha}$. Note that

$$Y(v_\alpha, z)v_\beta = E^-(-\alpha, z)\varepsilon(\alpha, \beta)e^{\alpha+\beta}z^{(\alpha,\beta)} + E^-(-\alpha, z)\varepsilon(\alpha, -\beta)e^{\alpha-\beta}z^{-(\alpha,\beta)} + E^-(\alpha, z)\varepsilon(-\alpha, \beta)e^{-\alpha+\beta}z^{-(\alpha,\beta)} + E^-(\alpha, z)\varepsilon(\alpha, \beta)e^{-\alpha-\beta}z^{(\alpha,\beta)},$$

where

$$E^-(\alpha, z) = \exp\left(\sum_{n<0} \frac{\alpha(n)z^{-n}}{n}\right).$$

Thus, if n is sufficiently negative, $(v_\alpha)_n(v_\beta) = u + v$ for some nonzero $u \in V_L^+(\alpha + \beta)$ and $v \in V_L^+(-\alpha + \beta)$. This gives $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta} = \lambda_{\alpha-\beta}$ by applying σ to $(v_\alpha)_n(v_\beta) = u + v$. So $\alpha \mapsto \lambda_\alpha$ defines a character of abelian group $L/2L$ of order 2^n. Clearly, any character $\lambda\colon L/2L \to \langle\pm 1\rangle$ defines an automorphism σ that acts on $V_L^+(\alpha)$ as λ_α. Therefore, the subgroup of $\mathrm{Aut}(V_L^+)$ that acts trivially on $M(1)^+$ is isomorphic to the dual group of $L/2L$ and is exactly the subgroup of $O(\hat{L})/\langle\pm 1\rangle$ that we identified as $\mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z})$. As a result, the subgroup of $\mathrm{Aut}(V_L^+)$ that preserves $M(1)_2^+$ is exactly the group $O(\hat{L})/\langle\pm 1\rangle$, as desired. □

PROPOSITION 3.4. *Let L be a positive definite even lattice such that $L_1 = L_2 = \emptyset$. Then $\mathrm{Aut}(V_L^+)$ is the inherited group—that is, the LVOA$^+$ group.*

Proof. In this case we have $(V_L^+)_2 = M(1)_2^+$. Thus, any automorphism of V_L^+ preserves $M(1)_2^+$. By Lemma 3.3, $\mathrm{Aut}(V_L^+)$ is the LVOA$^+$ group. □

4. Rank-2 Lattices

All lattices in this paper are positive definite; throughout, L denotes an even integral lattice. We recall a general result.

LEMMA 4.1. *Let L be a lattice and M a sublattice.*

(i) *If $|L : M|$ is finite, then $\det(M) = \det(L)|L : M|^2$.*
(ii) *If M is a direct summand of L, then $L/[M + \mathrm{ann}_L(M)]$ embeds in $\mathcal{D}(M)$.*

Proof. These are standard results. For example, see [G4]. □

We need to categorize rank-2 lattices by their configurations of norm-2 and norm-4 elements, because such elements contribute to low-degree terms of the lattice VOA. We shall use the notation described in 2.1.

LEMMA 4.2. *Suppose that $\mathrm{rank}(L_1) = 2$. Then L_1 spans L, and L is one of $L_{A_1^2}$ or L_{A_2}.*

Proof. The span of L_1 is isometric to $L_{A_1^2}$ or L_{A_2}, and each of these is a maximal even integral lattice under containment. □

LEMMA 4.3. *Suppose that* $\operatorname{rank}(L_1) = 1$. *Let* $r \in L_1$ *and let* s *generate* $\operatorname{ann}_L(r)$. *Then* $(s,s) \geq 4$, *and if* $L > \operatorname{span}\{r,s\}$ *then* $14 \leq (s,s) \in 6 + 8\mathbb{Z}$.

Proof. Note that $\mathbb{Z}r$ is a direct summand of L. We have $(s,s) \geq 4$. In case $L > N := \operatorname{span}\{r,s\}$, L/N has order 2 by Lemma 4.1. If x represents the nontrivial coset then $(x,x) \geq 4$ and $(2x,2x) \geq 16$; also, $(2x,2x) \in 8\mathbb{Z}$. Since (x,r) is odd, if we write $2x = pr + qs$ for integers $p, q \in \mathbb{Z}$ then p is odd, so $q^2(s,s) \in 6 + 8\mathbb{Z}$. It follows that q is odd and $(s,s) \in 6 + 8\mathbb{Z}$. □

LEMMA 4.4. *Suppose that* $L_1 = \emptyset$ *and* $\operatorname{rank}(L_2) = 2$. *If* r *and* s *are linearly independent norm-4 elements, then they span* L *and have Gram matrix* $G = \left(\begin{smallmatrix} 4 & b \\ b & 4 \end{smallmatrix}\right)$ *for some* $b \in \{0, \pm 1, \pm 2\}$.

Proof. If $L \neq N := \operatorname{span}\{r,s\}$, then $\det(N) = 16 - b^2$ is divisible by a perfect square, whence $b = 0$ or $b = \pm 2$ and the index is 2. Actually, $b = 0$ does not occur here because $\frac{1}{2}r, \frac{1}{2}s \notin L$ implies that $\frac{1}{2}(r+s) \in L_1$, a contradiction; thus $b = \pm 2$. Clearly, $\operatorname{span}\{r,s\} \cong \sqrt{2}L_{A_2}$. However, any integral lattice containing the latter with index 2 is odd, a contradiction. Therefore, $L = N$ and the Gram matrix is as shown. Positive definiteness implies that $|b| < 4$, and rootlessness implies that $b \neq \pm 3$. □

LEMMA 4.5. *Suppose that* $L_1 = \emptyset$ *and* $\operatorname{rank}(L_2) = 1$. *Let* $r \in L_2$ *and let* s *generate* $\operatorname{ann}_L(x)$. *Then* $(s,s) \geq 6$ *and* $L/\operatorname{span}\{r,s\}$ *is a subgroup of* $\mathbb{Z}_4$. *Moreover*:

(a) *if the order of* $L/\operatorname{span}\{r,s\}$ *is* 2, *then* $8 \leq (s,s) \in 4 + 8\mathbb{Z}$.
(b) *if the order of* $L/\operatorname{span}\{r,s\}$ *is* 4, *then* $28 \leq (s,s) \in 28 + 32\mathbb{Z}$.

Proof. Let x be in a nontrivial coset of $N := \operatorname{span}\{r,s\}$ in L.

If $(x,r) \in 2 + 4\mathbb{Z}$ then $2x = pr + qs$, where p is odd. We have $(x,x) \geq 6$ with p odd and $q \neq 0$. Therefore, $(2x,2x) \in 8\mathbb{Z}$ and $24 \leq 4p^2 + (s,s)q^2$, whence q is odd and $(s,s) \in 4 + 8\mathbb{Z}$.

If $(x,r) \in 1 + 2\mathbb{Z}$ then $(4x,4x) \in 32\mathbb{Z}$. We have $(x,x) \geq 6$, whence $(4x,4x) \geq 96$. Writing $4x = pr + qs$ yields $4p^2 + q^2(s,s) \in 32\mathbb{Z}$. Since p is odd, $p^2 \in 1 + 8\mathbb{Z}$ and $4p^2 \in 4 + 32\mathbb{Z}$. Since (s,s) is even and q is odd, we have $q^2 \in 1 + 8\mathbb{Z}$ and $(s,s) \in 4 + 8\mathbb{Z}$. It follows that $\frac{1}{4}q^2(s,s) \in 7 + 8\mathbb{Z}$, whence $\frac{1}{4}(s,s) \in 7 + 8\mathbb{Z}$. □

5. Idempotents in Small-Dimensional Algebras

We can derive a lot of information about the automorphism group of a vertex operator algebra by restricting to low-degree homogeneous pieces. For the V_L^+ problem, the degree-2 piece and its product $x, y \mapsto x_1 y$ give an algebra that is useful to study. Here, for $\operatorname{rank}(L) = 2$, we concentrate on some commutative algebras of dimension about 5. Commutativity of $(V_L^+, 1^{st})$ is implied if $L_1 = \emptyset$, which holds for $b \in \{0, \pm 1, \pm 2\}$ as in Lemma 4.4.

It does not seem advantageous to give particular values to b most of the time, so we keep it as an unspecified constant in case these arguments might be a model for future work. In this paper, we shall note any limits on b as needed.

NOTATION 5.1. Let S be the Jordan algebra of degree-2 symmetric matrices, and suppose that A is a commutative 5-dimensional algebra of the form $A = S \oplus \mathbb{C}v_r \oplus \mathbb{C}v_s$. Suppose further that $v_r \times v_s = 0$ and that the notation of Section 9 applies here, with the usual inner products and algebra product. Let $w = p + c_r v_r + c_s v_s$ be an idempotent, and suppose that t is a norm-4 vector orthogonal to r. Let a_1, a_2, a_3 be scalars such that $p = a_1 r^2 + a_2 rt + a_3 t^2$.

REMARK 5.2. We note that the basis r, s of H has dual basis r^*, s^*, where $r^* = (4r - bs)/(16 - b^2)$ and $s^* = (4s - br)/(16 - b^2)$. The identity of A is

$$\frac{1}{4}\frac{1}{16 - b^2}(rr^* + ss^*) = \frac{1}{4}\frac{1}{16 - b^2}(4r^2 + 4s^2 - 2brs).$$

NOTATION 5.3. If w is an element of A, write $w = p + q$ for $p \in S^2H$ and $q \in \mathbb{C}v_r \oplus \mathbb{C}v_s$. Call the element $\bar{w} := p - q$ the *conjugate element*. The components p and q are called (respectively) the *P-part* and the *Q-part* of w. Extend this notation to subscripted elements: $w_i = p_i + q_i$ and $\bar{w}_i = p_i - q_i$ for indices i.

REMARK 5.4. In 5.3, $q^2 \in S^2H$ because $v_r \times v_s = 0$. Also, $w = p + q$ is an idempotent if and only if $p = p^2 + q^2$ and $q = 2p \times q$. Hence $w = p + q$ is an idempotent if and only if the conjugate $p - q$ is an idempotent.

LEMMA 5.5. *Suppose that w_1 and w_2 are idempotents and that their sum is an idempotent. Then $w_1 \times w_2 = 0$ and $(w_1, w_2) = 0$.*

Proof. We have $(w_1 + w_2)^2 = w_1^2 + 2w_1 \times w_2 + w_2^2$, whence $w_1 \times w_2 = 0$. Also, $(w_1, w_2) = (w_1^2, w_2) = (w_1, w_1 \times w_2) = 0$. □

DEFINITION 5.6. Throughout this paper, an idempotent is neither zero nor the identity unless the context clearly allows the possibility. We call an idempotent w of *type 0, 1,* or *2* (respectively) if it has Q-part that is 0, is a multiple of v_r or v_s, or is not a multiple of either v_r or v_s.

LEMMA 5.7. (i) $r^2 \times s^2 = 4brs$, $r^2 \times r^2 = 16r^2$, $s^2 \times s^2 = 16s^2$, *and* $rs \times rs = 4r^2 + 4s^2 + 2brs$. *Moreover,* $x^2 \times v_r = (x, r)^2 v_r = \frac{1}{2}(x^2, r^2)v_r$, $r^2 \times rs = 8rs + 2b^2 r^2$, *and* $s^2 \times rs = 8rs + 2b^2 s^2$; *also,* $v_r \times v_r = r^2$, $v_r \times v_s = 0$, *and* $v_s \times v_s = s^2$.

(ii) $(r^2, r^2) = 32 = (s^2, s^2)$, $(r^2, s^2) = 2b^2$, $(rs, rs) = 16 + b^2$, *and* $(rs, r^2) = 8b = (rs, s^2)$; *also,* $(v_r, v_r) = 2 = (v_s, v_s)$ *and* $(v_r, v_s) = 0$.

Proof. See the Appendix (Section 9; take $a = d = 4$ and $b \neq 0, \pm 2$). □

5.1. Idempotents of Type 0

REMARK 5.8. Idempotents of type 0 are simply idempotents in the Jordan algebra of symmetric matrices; they are ordinary idempotent matrices that are symmetric. Up to conjugacy by orthogonal transformation, they are diagonal matrices whose diagonal entries are 0 and 1 only.

5.2. Idempotents of Type 1

NOTATION 5.9. The next few results apply to the case of an idempotent of type 1—that is, of the form $w = p + c_r v_r$, where $c_r \neq 0$. In such a case, $w = w^2 = p^2 + c_r^2 r^2 + c_r(p, r^2)v_r$ (see Section 9). From $c_r \neq 0$ it follows that $(p, r^2) = 1$. We continue to use the notation of 5.1.

LEMMA 5.10. *Suppose $c_r \neq 0$ and $c_s = 0$. Then we have $a_1 = 16a_1^2 + 4a_2^2 + c_r^2$, $a_2 = 16a_2(a_1 + a_3)$, $a_3 = 16a_3^2 + 4a_2^2$, and $(p, r^2) = 1$.*

Proof. Compute $p + c_r v_r = w = w^2 = p^2 + c_r^2 r^2 + c_r(p, r^2)v_r$ (see Lemma 5.7) and expand in the basis r^2, rt, t^2, v_r. □

COROLLARY 5.11. $a_1 = \frac{1}{32}$.

Proof. We have $1 = (p, r^2) = a_1(r^2, r^2) = 32a_1$, whence $a_1 = \frac{1}{32}$. □

LEMMA 5.12. *Suppose that $c_r \neq 0$ and $c_s = 0$. Then*:

(A1) $a_1 = 16a_1^2 + 4a_2^2 + c_r^2$;
(A2) $a_2 = 16a_2(a_1 + a_3)$; *and*
(A3) $a_3 = 16a_3^2 + 4a_2^2$.

Proof. Compute $p^2 = (16a_1^2 + 4a_2^2)r^2 + 16(a_1a_2 + a_3a_2)rt + (16a_3^2 + 4a_2^2)t^2$ and use $w = w^2 = p^2 + c_r^2 r^2 + c_r v_r$. □

LEMMA 5.13. *Suppose that $c_r \neq 0$ and $c_s = 0$. If $a_2 = 0$, then $a_3 \in \{0, \frac{1}{16}\}$ and $c_r = \pm\frac{1}{8}$.*

Proof. We deduce from (A3) that $a_3 = 16a_3^2$; hence $c_r = \pm\frac{1}{8}$. □

LEMMA 5.14. *Suppose that $c_r \neq 0$ and $c_s = 0$. Then $a_2 = 0$.*

Proof. If $a_2 \neq 0$, then from (A2) we have $1 = 16(a_1 + a_3)$ and so $a_3 = \frac{1}{32}$. Next, use (A3) to obtain $\frac{1}{64} = 4a_2^2$. Finally use (A1) to get $c_r = 0$. □

THEOREM 5.15. *Assume that $c_r \neq 0$ and $c_s = 0$. Then*:

(i) $a_1 = \frac{1}{32}$, $a_2 = 0$, *and* $c_r = \pm\frac{1}{8}$; *and*
(ii) *either* $(w, w) = \frac{1}{16}$ *and* $a_3 = 0$ *or* $(w, w) = \frac{3}{16}$ *and* $a_3 = \frac{1}{16}$.

All of these cases occur. If an idempotent occurs, so does its complementary idempotent.

Proof. This is a summary of preceding results. □

LEMMA 5.16. *If w is an idempotent of type 1, then the following statements hold.*

(i) *If $(w, w) = \frac{1}{16}$, the eigenvalues for* $\mathrm{ad}(w)$ *are* $1, 0, 0, \frac{1}{4}, \frac{1}{32}b^2$; *eigenvectors for these respective eigenspaces are $w, 1 - w, t^2, rt, v_s$.*

(ii) *If* $(w,w) = \frac{3}{16}$, *the eigenvalues for* $\mathrm{ad}(w)$ *are* $0,1,1,\frac{3}{4},1-\frac{1}{32}b^2$; *eigenvectors for these respective eigenspaces are* $w, 1-w, t^2, rt, v_s$.

If $\frac{1}{32}b^2 \neq 0,1,\frac{1}{4},\frac{3}{4}$, *then the multiplicities of* 0 *and* 1 *are* (*respectively*) 2 *and* 1 *in part* (i) *and* 1 *and* 2 *in part* (ii).

Proof. This can be shown by straightforward calculation. Note that $\frac{1}{32}b^2 \neq 0,1,\frac{1}{4},\frac{3}{4}$ follows if b is rational. □

COROLLARY 5.17. *If* w *is a type-1 idempotent and is the sum of two nonzero idempotents* w_1, w_2, *then*: w *has the form* $\frac{1}{32}r^2 + \frac{1}{16}t^2 \pm \frac{1}{8}v_r$; *and* w_1 *and* w_2 *are, up to order,* $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ *and* $\frac{1}{16}t^2$.

Proof. If w is such a sum then each w_i is in the 1-eigenspace of $\mathrm{ad}(w)$, which must be more than 1-dimensional. This means that w has norm $\frac{3}{16}$ and one of the w_i (say, for $i = 1$) has type 1 and Q-part $\pm\frac{1}{8}v_r$. Therefore, w_2 has type 0 and hence norm $\frac{1}{8}$. This means that w_1 has norm $\frac{1}{16}$, so we know that w_1 has shape $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ and that $w = \frac{1}{16}t^2$. □

5.3. *Idempotents of Type 2*

HYPOTHESIS 5.18. We assume in this section that the parameter $b \neq 0, \pm 2, \pm 3$ (which means $b = \pm 1$). Then the algebra $(V_2, 1^{st})$ is commutative because $V_1 = 0$.

NOTATION 5.19. Let $p = c(r^2 + s^2) + drs$ and $v = c_r v_r + c_s v_s$.

LEMMA 5.20. *If* c_r *and* c_s *are nonzero, then there are at most eight possibilities for* w. *In more detail, there are at most two values of* c (*and, correspondingly, of* d). *We have* $c_r^2 = c_s^2$, *and this common value depends on* c (*or on* d).

Proof. Compute $p + c_r v_r + c_s v_s = w = w^2 = p^2 + c_r^2 r^2 + c_s^2 s^2 + c_r(p, r^2)v_r + c_s(p, s^2)v_s$. Since c_r and c_s are nonzero, $(p, r^2) = 1 = (p, s^2)$.

Since $(r^2, r^2) = 32 = (s^2, s^2)$, $(rs, r^2) = 8b = (rs, s^2)$, and $(r^2, s^2) = 16 + b^2$, we have $p = c(r^2 + s^2) + drs$ for some scalars c, d. The previous paragraph then implies that $1 = (32 + 2b^2)c + 8bd$. Since $b \neq 0$,

$$d = \frac{1}{8b}(2c(32 + 2b^2) - 1) \tag{5.1}$$

is a linear expression in c.

Now, $p^2 = (16c^2 + 4d^2)(r^2 + s^2) + (2bc^2 + 2bd^2)rs$ and so

$$w^2 = (16c^2 + 4d^2 + c_r^2)r^2 + (16c^2 + 4d^2 + c_s^2)s^2 + (2bc^2 + 2bd^2)rs + c_r v_r + c_s v_s.$$

It follows that $c_r^2 = c_s^2$.

Comparing the coefficients of r^2, we get

$$c = 16c^2 + 4d^2 + c_r^2; \tag{5.2}$$

comparing the coefficients of rs yields

$$d = 2bc^2 + 2bd^2. \tag{5.3}$$

Since d is a linear expression in c, we know that c satisfies a quadratic equation that depends on b but not on $c_r^2 = c_s^2$. The degree of this equation really is 2, since $b \neq 0$ real implies that the top coefficient is nonzero.

It follows that the ordered pair d, c has at most two possible values. For each, there is a unique value for c_r^2 and hence at most two possible values for c_r (with the same two for c_s). Hence there are at most eight idempotents of type 2. □

LEMMA 5.21. *$c \neq 0$ and $d \neq 0$.*

Proof. Suppose that $c = 0$. We then have $p = drs$ and $d = -\frac{1}{8}b$. On the other hand, since $w = drs + v$ is an idempotent, the coefficient for w^2 at rs is $d = 8bc^2 + 2bd^2 = 2bd^2$, which implies that $1 = 2bd$. This is incompatible with $d = -\frac{1}{8}b$.

If $d = 0$ then equation (5.3) implies that $c = 0$, which is false. □

LEMMA 5.22. *If w is a type-2 idempotent with $w = c(r^2+s^2)+drs+c_rv_r+c_sv_s$ and if $1 - w$ is the complementary idempotent, expanded similarly as $1 - w = c'(r^2 + s^2) + d'rs + c'_rv_r + c'_sv_s$, then $c'_r = -c_r \neq c_r$, $c'_s = -c_s$, $c \neq c'$, and $d \neq d'$. In particular, in the notation of Lemma 5.20, the function $c \mapsto c_r^2$ is two-to-one and so only one value of c_r^2 occurs for type-2 idempotents.*

Proof. If it were true that $c = c'$, then $w = \frac{1}{2}\mathbb{I} + v$ and $1 - w = \frac{1}{2}\mathbb{I} - v$. Since these are idempotents, $v^2 = \frac{1}{2}\mathbb{I}$. But this is impossible, since $b \neq \pm 2$ implies that v^2 is a multiple of $r^2 + s^2$ and since $\mathbb{I}$ is not a linear combination of r^2 and s^2 for $b \neq 0$ (see Remark 5.2). □

5.4. Sums of Idempotents

HYPOTHESIS 5.23. We continue to take $b = \pm 1$. Results of Section 5.3 still apply.

In the arguments of this section, we allow the symbol b to be any odd integer, though the lattice is positive definite only for $b = \pm 1$.

LEMMA 5.24. *Suppose that w_1, w_2 are two idempotents of type 1. If $w_1 + w_2$ is an idempotent, then $w_1 + w_2$ does not have type 1 or type 2.*

Proof. We use Corollary 5.17 to eliminate the sum having type 1. To eliminate a sum having type 2, we note that for type-1 idempotents $a_2 = 0$ by Theorem 5.15, whereas $d \neq 0$ for type-2 idempotents by Lemma 5.21. □

LEMMA 5.25. *If w_1, w_2 are idempotents of type 2 and are not complementary, then their sum is not an idempotent.*

Proof. Assume that the sum w is an idempotent. From Lemma 5.24, the sum has type 0 and so has the form $\frac{1}{16}u^2$ for some vector $u \in H$ of norm 4. The

eigenvalues of $\mathrm{ad}(u)$ are $1, 0, \frac{1}{2}, \frac{1}{16}(u,r)^2, \frac{1}{16}(u,s)^2$ with respective eigenvectors $u^2, 1-u^2, \frac{1}{2}uu', v_r, v_s$, where u' spans the orthogonal of u in H.

Now w_1 and w_2 are linearly independent (or else they are equal, which is impossible). This means that the eigenvalue 1 has multiplicity at least 2. Therefore, at least one of $(u,r)^2$ and $(u,s)^2$ is 16. Since w_1, w_2 lie in the 1-eigenspace of $\mathrm{ad}(u)$ and since both w_i have type 2, both these square norms must be 16; that is, $m := (u,r) = \pm 4$ and $n := (u,s) = \pm 4$. Since r, s form a basis and the form is nonsingular, this forces $u = mr^* + ns^*$, where r^*, s^* is the dual basis. We have $4 = (u,u) = 16(r^*, r^*) + 2mn(r^*, s^*) + 16(s^*, s^*)$. The right side is

$$\frac{1}{16 - b^2}[16(4r - bs, 4r - bs) + 2mn(4r - bs, 4s - br) + 16(4s - br, 4s - br)].$$

Since b is an odd integer, the above rational number in reduced form clearly has numerator divisible by 16 and so does not equal 4, a contradiction. □

LEMMA 5.26. *The sum of a type-1 and a type-2 idempotent is not an idempotent.*

Proof. Assume that $w := w_1 + w_2$ is an idempotent. Obviously it does not have type 0. By Corollary 5.17, it does not have type 1.

We conclude that w has type 2. However, the coefficients of w at r^2 and s^2 must be equal for type 2, a contradiction since this forces the P-part of the type-1 idempotent to be zero. □

COROLLARY 5.27. *The only idempotents that are a proper summand of some nontrivial idempotent are the ones of type 1 and norm* $\frac{1}{16}$. *There exist four such and they come in orthogonal pairs, which are just pairs of idempotents and their conjugates.*

COROLLARY 5.28. $\mathrm{Aut}(A)$ *is a dihedral group of order* 8.

Proof. The automorphism group preserves and acts faithfully on the set J of type-1 idempotents of norm $\frac{1}{16}$ (i.e., the complete set of idempotents that are proper summands of proper idempotents) and furthermore preserves the partition defined by orthogonality. The orthogonal in A of the nonsingular subspace $\mathrm{span}(J)$ is spanned by $v := r^2 + s^2 - \frac{16+b^2}{4b}rs$. We claim that if an automorphism acts trivially on $\mathrm{span}(J)$, it acts trivially on A. This is so because $rs \in \mathrm{span}\{r^2 \times s^2, r^2, s^2\}$ and $\{rs\} \cup J$ spans A.

Thus we have shown that the automorphism group of A embeds in a dihedral group of order 8. This embedding is an isomorphism that is onto, since the LVOA^+ group embeds in $\mathrm{Aut}(A)$. □

PROPOSITION 5.29. $\mathrm{Aut}(V_L^+)$ *is just the LVOA*$^+$ *group, isomorphic to* Dih_8.

Proof. In this case we have $(V_L^+)_2 = M(1)_2^+$. Thus, any automorphism of V_L^+ preserves $M(1)_2^+$. Now use Lemma 3.3. □

6. Automorphism Group of V_L^+ with rank$(L) = 2$

In this section we assume that the rank of L is equal to 2. If $L_1 = L_2 = \emptyset$, then the automorphism group of V_L^+ was determined in Proposition 3.4. Hence, in this section we assume that either L_1 or L_2 is not empty.

6.1. $L_1 = \emptyset$ *and* rank$(L_2) = 2$; $b = 0$

Note that L is generated by L_2. We will discuss the automorphism group according to the value b in the Gram matrix G (see Lemma 4.4).

First we assume that $b = 0$ in the Gram matrix G. Then $L \cong \sqrt{2}L_{A_1} \perp \sqrt{2}L_{A_1}$, where L_{A_1} is the root lattice of type A_1. Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ with $(\alpha_i, \alpha_j) = 4\delta_{ij}$ for $i, j = 1, 2$. Set

$$\omega_1 = \tfrac{1}{16}\alpha_1(-1)^2 + \tfrac{1}{4}(e^{\alpha_1} + e^{-\alpha_1}),$$
$$\omega_2 = \tfrac{1}{16}\alpha_2(-1)^2 - \tfrac{1}{4}(e^{\alpha_2} + e^{-\alpha_2}).$$

We also use α_2 to define ω_3 and ω_4 in the same fashion. Then ω_i for $i = 1, 2, 3, 4$ are commutative Virasoro vectors of central charge $\frac{1}{2}$ (see [DGH] and [DMZ]). It is well known that $(V_L^+)_2$ is a commutative (nonassociative) algebra under $u \times v = u_1 v$, since the degree-1 part is zero (cf. [FLMe]). Let X be the span of ω_i for all i.

LEMMA 6.1. *If $u \in (V_L)_2$ is a Virasoro vector of central charge $\frac{1}{2}$, then $u = \omega_i$ for some i.*

Proof. The space $(V_L)_2$ is 5-dimensional with a basis

$$\{\omega_1, \omega_2, \omega_3, \omega_4, \alpha_1(-1)\alpha_2(-1)\}.$$

Let $u = \sum_{i=1}^4 c_i\omega_i + x\alpha_1(-1)\alpha_2(-1) \in (V_L)_2$ be a Virasoro vector of central charge $\frac{1}{2}$. Then $u \times u = 2u$. Note that $\omega_i \times \omega_j = \delta_{i,j}2\omega_i$ for $i, j = 1, 2, 3, 4$, $\omega_i \times \alpha_1(-1)\alpha_2(-1) = \frac{1}{2}\alpha_1(-1)\alpha_2(-1)$, and $\alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = 4\alpha_1(-1)^2 + 4\alpha_2(-1)^2$. Thus we have a nonlinear system:

$$2c_i = 2c_i^2 + 32x^2, \quad i = 1, 2, 3, 4,$$
$$2x = \sum_{i=1}^4 xc_i.$$

If $x \neq 0$, then $\sum_{i=1}^4 c_i = 2$ and $2 = \sum_{i=1}^4 c_i^2 + 64x^2$.

Since the central charge of u is $\frac{1}{2}$, we have

$$\frac{1}{4} = u_3 u = \sum_{i=1}^4 \frac{c_i^2}{4} + 16x^2 \quad \text{and} \quad 1 = \sum_{i=1}^4 c_i^2 + 64x^2.$$

This is a contradiction. So $x = 0$, which implies that $c_i = 0, 1$ and $u = \omega_i$ for some i. □

By Lemma 6.1, any automorphism σ of V_L^+ induces a permutation of the four ω_i.

It is known from [FLMe] that $(V_L^+)_2$ has a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ given by $(u,v) = u_3 v$ for $u, v \in (V_L^+)_2$. The orthogonal complement of X in $(V_L^+)_2$ with respect to the form is spanned by $\alpha_1(-1)\alpha_2(-1)$. Thus $\sigma\alpha_1(-1)\alpha_2(-1) = \lambda\alpha_1(-1)\alpha_2(-1)$ for some nonzero constant λ. Hence $\alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = \alpha_1(-1)^2 + \alpha_2(-1)^2$, which is a multiple of the Virasoro element ω. This shows that $\lambda = \pm 1$.

On the other hand,

$$V_L^+ \cong V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}} \oplus V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}.$$

By [DGH, Cor. 3.3],

$$V_L^+ \cong L\left(\tfrac{1}{2}, 0\right)^{\otimes 4} \oplus L\left(\tfrac{1}{2}, \tfrac{1}{2}\right)^{\otimes 4}.$$

So if the restriction of σ to X is the identity, then the action of σ on $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ is trivial and on $V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}$ is ± 1. Indeed, there is automorphism τ of V_L^+ such that τ acts trivially on $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ and acts as -1 on $V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}$ by the fusion rule for $V^+_{\sqrt{2}L_{A_1}}$ (see [ADLi]). Since $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ is generated by ω_i for $i = 1, 2, 3, 4$, it follows that any automorphism preserves $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ and its irreducible module $V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}$ (cf. [DM1]). As a result, $\langle\tau\rangle$ is a normal subgroup of $\mathrm{Aut}(V_L^+)$ that is isomorphic to $\mathbb{Z}_2$.

Next we describe how Sym_4 can be realized as a subgroup of $\mathrm{Aut}(V_L^+)$ by showing that any permutation $\sigma \in \mathrm{Sym}_4$ gives rise to an automorphism of V_L^+. But it is clear that Sym_4 acts on V_L^+ by permuting the tensor factors. In order to see that Sym_4 acts on V_L^+ as automorphisms, it is enough to show that $\sigma(Y(u,z)v) = Y(\sigma u, z)\sigma v$ for $\sigma \in \mathrm{Sym}_4$ and $u, v \in V_L^+$. There are four different ways to choose u, v. We discuss only the case $u, v \in L\left(\frac{1}{2}, \frac{1}{2}\right)^{\otimes 4}$, since the other cases can be dealt with in a similar fashion. Let $u = u^1 \otimes u^2 \otimes u^3 \otimes u^4$ and $v = v^1 \otimes v^2 \otimes v^3 \otimes v^4$, where u_i, v_i are tensor factors in the ith $L\left(\frac{1}{2}, \frac{1}{2}\right)$. Let $\mathcal{Y}$ be a nonzero intertwining operator of type $\binom{L(1/2,\,0)}{L(1/2,\,1/2)\,L(1/2,\,1/2)}$. Then, up to a constant,

$$Y(u,z)v = \mathcal{Y}(u_1,z)v_1 \otimes \mathcal{Y}(u_2,z)v_2 \otimes \mathcal{Y}(u_3,z)v_3 \otimes \mathcal{Y}(u_4,z)v_4$$

(see [DMZ]). Since σ is a permuation, it is trivial to verify that $\sigma(Y(u,z)v) = Y(\sigma u, z)\sigma v$.

Thus we have proved the following result.

PROPOSITION 6.2. *If $b = 0$ in the Gram matrix G, then $L \cong \sqrt{2}L_{A_1} \perp \sqrt{2}L_{A_1}$ and $\mathrm{Aut}(V_L^+) \cong \mathrm{Sym}_4 \times \mathbb{Z}_2$.*

REMARK 6.3. Here is a different proof that $\mathrm{Aut}(V_L^+)$ contains a copy of $\mathrm{Sym}_4 \times \mathbb{Z}_2$, using the theory of finite subgroups of Lie groups. Our lattice L lies in $M \cong L_{A_1^2}$. Take V_M, which is a lattice VOA. By [DN1], V_M has automorphism group that is isomorphic to $\mathrm{PSL}(2,\mathbb{C}) \wr 2$. In $\mathrm{PSL}(2,\mathbb{C})$, there is (up to conjugacy) a

unique 4-group, and its normalizer is isomorphic to Sym_4. Correspondingly, in $\mathrm{PSL}(2,\mathbb{C})\wr 2$ there is a subgroup isomorphic to $\mathrm{Sym}_4\wr 2$. In this, take a subgroup H of the form $2^4:[\mathrm{Sym}_3\times 2]$. Let t be an involution of H that maps to the central involution of $H/O_2(H)\cong \mathrm{Sym}_3\times 2$, and take $R:=C_{O_2(H)}(t)\cong 2^2$. Take the fixed points V_M^R. We have that V_M^R is isomorphic to our V_L^+. Therefore, V_L^+ gets an action of $H/R\cong 2^2:[\mathrm{Sym}_3\times 2]\cong \mathrm{Sym}_4\times 2$.

6.2. $L_1=\emptyset$ *and* $\mathrm{rank}(L_2)=2$; $b=2$

Next we assume that b in the Gram matrix is 2. Then $L\cong\sqrt{2}L_{A_2}$, and $L=\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$ with $(\alpha_i,\alpha_i)=4$ and $(\alpha_1,\alpha_2)=2$. As before, we define ω_1,ω_2 by using α_1, ω_3,ω_4 by using α_2, and ω_5,ω_6 by using $\alpha_1+\alpha_2$. Then the ω_i $(i=1,\dots,6)$ form a basis of $(V_L^+)_2$.

LEMMA 6.4. *If $u\in(V_L^+)_2$ is a Virasoro vector of central charge $\frac{1}{2}$, then $u=\omega_i$ for some i.*

Proof (see Section 6.3 for an alternate proof). Let $u=\sum_{i=1}^6 c_i\omega_i$ for some $c_i\in\mathbb{C}$. Then u is a Virasoro vector of central charge $\frac{1}{2}$ if and only if $(u,u)=\frac{1}{4}$ and $u\times u=2u$. Note that

$$(\omega_i,\omega_i)=\tfrac{1}{4}\quad\text{and}\quad(\omega_{2j-1},\omega_{2j})=0,\quad 1\le i\le 6,\ j=1,2,3;$$
$$(\omega_1,\omega_k)=(\omega_2,\omega_k)=\tfrac{1}{32},\quad k=3,4,5,6.$$

Therefore,

$$\begin{aligned}(u,u)&=\frac{1}{4}\sum_{i=1}^6 c_i^2+\frac{1}{16}\sum_{j=1,2}\sum_{j<k\le 3}(c_{2j-1}c_{2k-1}+c_{2j-1}c_{2k}+c_{2j}c_{2k-1}+c_{2j}c_{2k})\\&=\frac{1}{4}.\end{aligned}$$

In order to compute $u\times u$, we need the following multiplication table in $(V_L^+)_2$:

$$\omega_{2i-1}\times\omega_{2i}=0,\quad i=1,2,3;$$
$$\begin{aligned}\omega_1\times\omega_3&=\tfrac{1}{4}(\omega_1+\omega_3-\omega_6),&\omega_2\times\omega_3&=\tfrac{1}{4}(\omega_2+\omega_3-\omega_5),\\\omega_1\times\omega_4&=\tfrac{1}{4}(\omega_1+\omega_4-\omega_5),&\omega_2\times\omega_4&=\tfrac{1}{4}(\omega_2+\omega_4-\omega_6),\\\omega_1\times\omega_5&=\tfrac{1}{4}(\omega_1+\omega_5-\omega_4),&\omega_2\times\omega_5&=\tfrac{1}{4}(\omega_2+\omega_5-\omega_3),\\\omega_1\times\omega_6&=\tfrac{1}{4}(\omega_1+\omega_6-\omega_3),&\omega_2\times\omega_6&=\tfrac{1}{4}(\omega_2+\omega_6-\omega_4),\\\omega_3\times\omega_5&=\tfrac{1}{4}(\omega_3+\omega_5-\omega_2),&\omega_4\times\omega_5&=\tfrac{1}{4}(\omega_4+\omega_5-\omega_1),\\\omega_3\times\omega_6&=\tfrac{1}{4}(\omega_3+\omega_6-\omega_1),&\omega_2\times\omega_6&=\tfrac{1}{4}(\omega_2+\omega_6-\omega_2).\end{aligned}$$

Then $u\times u=2u$ if and only if

$$c_1^2+\tfrac{1}{4}(c_1c_3+c_1c_4+c_1c_5+c_1c_6-c_3c_6-c_4c_5)=c_1,$$
$$c_2^2+\tfrac{1}{4}(c_2c_3+c_2c_4+c_2c_5+c_2c_6-c_3c_5-c_4c_6)=c_2,$$

$$c_3^2 + \tfrac{1}{4}(c_1c_3 + c_2c_3 + c_3c_5 + c_3c_6 - c_1c_6 - c_2c_5) = c_3,$$
$$c_4^2 + \tfrac{1}{4}(c_1c_4 + c_2c_4 + c_4c_5 + c_4c_6 - c_1c_5 - c_2c_6) = c_4,$$
$$c_5^2 + \tfrac{1}{4}(c_1c_5 + c_2c_5 + c_3c_5 + c_4c_6 - c_1c_4 - c_2c_3) = c_5,$$
$$c_6^2 + \tfrac{1}{4}(c_1c_6 + c_2c_6 + c_3c_6 + c_4c_6 - c_1c_3 - c_2c_4) = c_6.$$

There are exactly six solutions to this linear system: $c_i = 1$ and $c_j = 0$ if $j \neq i$ where $i = 1, \ldots, 6$. We thank Harm Derksen for obtaining this result with the MacCauley software package. This finishes our first proof of Lemma 6.4. □

PROPOSITION 6.5. *If $b = 2$ in the Gram matrix, then $L \cong \sqrt{2}L_{A_2}$ and $\mathrm{Aut}(V_L^+)$ is the LVOA$^+$ group.*

Proof. First note that the Weyl group acts on L, preserving and acting as Sym_3 on the set

$$\{\{\pm\alpha_1\}, \{\pm\alpha_2\}, \{\pm(\alpha_1 + \alpha_2)\}\}.$$

Now let $\sigma \in \mathrm{Aut}(V_L^+)$. Set $X_i = \{\omega_{2i-1}, \omega_{2i}\}$ for $i = 1, 2, 3$. Then X_i are the only orthogonal pairs in $X = X_1 \cup X_2 \cup X_3$. Since $\sigma X = X$, we see that σ induces a permutation on the set $\{X_1, X_2, X_3\}$.

The foregoing shows that $\mathbb{O}(\hat{L})$ induces Sym_3 on this 3-set. We may therefore assume that σ preserves each X_i. In this case σ acts trivially on $\alpha_1(-1)^2, \alpha_2(-1)^2$, $(\alpha_1+\alpha_2)(-1)^2$; that is, σ acts trivially on the subVOA they generate, which is isomorphic to $M(1)^+$. As a result, σ is in the LVOA$^+$ group. □

6.3. *Alternate Proof for $b = 2$*

The system of equations in the variables c_i that occurred in the proof of Lemma 6.4 can be replaced by an equivalent system (Lemma 6.7) that looks more symmetric. The old system was solved with the MacCauley software but not with Maple; the new system was solved with Maple and gives the same result as before.

NOTATION 6.6. Let r and s be independent norm-4 elements such that $t := -r-s$ has norm 4. Let w be an idempotent $w = p + q$, where $p = ar^2 + bs^2 + ct^2$ and $q = dv_r + ev_s + ev_t$, that satisfies $(w, w) = \frac{1}{16}$. Since $(L, L) \leq 2\mathbb{Z}$, we may assume that the epsilon-function is identically 1. It follows that $v_r \times v_s = v_t$ and similarly for all permutations of $\{r, s, t\}$.

LEMMA 6.7. *From $w^2 = w$, we have the equations*

$$a = 16a^2 + 4ab + 4ac - 4bc + d^2, \tag{6.1}$$
$$b = 16b^2 + 4bc + 4ba - 4ac + e^2, \tag{6.2}$$
$$c = 16c^2 + 4cq + 4cb - 4ab + f^2, \tag{6.3}$$
$$d = 2d(16a + 4b + 4c) + 2ef, \tag{6.4}$$
$$e = 2e(4a + 16b + 4c) + 2df, \tag{6.5}$$
$$f = 2f(4a + 4b + 16c) + 2de; \tag{6.6}$$

and from $(w, w) = \frac{1}{16}$ *we obtain the equation*

$$\tfrac{1}{16} = 32(a^2 + b^2 + c^2) + 16(ab + ac + bc) + 2(d^2 + e^2 + f^2). \tag{6.7}$$

Proof. This follows in a straightforward way from material in the Appendix. □

PROPOSITION 6.8. *There are just six solutions* $(a, b, c, d, e, f) \in \mathbb{C}^6$ *to the equations* (6.1)–(6.7). *They are* $\left(\frac{1}{32}, 0, 0, \frac{1}{8}, 0, 00\right)$, $\left(\frac{1}{32}, 0, 0, -\frac{1}{8}, 0, 00\right)$, *and the solutions obtained from these by powers of the permutation* $(abc)(def)$.

Proof. This follows from use of the `solve` command in the software package Maple. □

REMARK 6.9. If we omit (6.7), then there are infinitely many solutions with $d = e = f = 0$. The reason is that the Jordan algebra of symmetric degree-2 matrices has infinitely many idempotents. It seems possible that the system in Lemma 6.7 could be solved by hand.

6.4. $L_1 = \emptyset$ *and* $\operatorname{rank}(L_2) = 2$; $b = 1$

We now deal with the cases $b = 1$ in the Gram matrix.

PROPOSITION 6.10. *If* $b = 1$ *in the Gram matrix, then* $\operatorname{Aut}(V_L^+)$ *is the LVOA*$^+$ *group.*

Proof. By Corollary 5.27, any automorphism of V_L^+ preserves $M(1)_2^+$. The result now follows from Lemma 3.3. □

7. Automorphism Group of V_L^+ with $L_1 = \emptyset$ and $\operatorname{rank}(L_2) = 1$

Here we can assume that $L_2 = \{2\alpha_1, -2\alpha_1\}$. Let $\alpha_2 \in H$ such that $(\alpha_i, \alpha_j) = \delta_{i,j}$. Then $(V_L^+)_2$ is 4-dimensional with basis $v_{2\alpha_1}$, $\frac{1}{2}\alpha_1(-1)^2$, $\frac{1}{2}\alpha_2(-1)^2$, $\alpha_1(-1)\alpha_2(-1)$.

LEMMA 7.1. *Any automorphism of* V_L^+ *preserves the subspace* S^2H *of* $(V_L^+)_2$ *spanned by* $\frac{1}{2}\alpha_1(-1)^2$, $\frac{1}{2}\alpha_2(-1)^2$, $\alpha_1(-1)\alpha_2(-1)$.

Proof. Since Virasoro vectors of central charge 1 in S^2H span S^2H, it is enough to show that any Virasoro vector of central charge 1 lies in S^2H.

Let $t = d_1(\alpha_1^2/2) + d_2(\alpha_2^2/2) + d_3 v_{2\alpha_1} + d_4\alpha_1\alpha_2$ be a Virasoro vector of central charge 1 with $d_3 \neq 0$. Then we must have $t \times t = 2t$ and $(t, t) = \frac{1}{2}$. A straightforward computation shows that

$$\begin{aligned} t \times t = {} & d_1^2\alpha_1^2 + d_2^2\alpha_2^2 + d_3^2(2\alpha_1)^2 + d_4^2(\alpha_1^2 + \alpha_2^2) \\ & + 4d_1d_3v_{2\alpha_1} + 2d_1d_4\alpha_1\alpha_2 + 2d_2d_4\alpha_1\alpha_2. \end{aligned}$$

This yields four equations:

$$d_1 = d_1^2 + 4d_3^2 + d_4^2,$$
$$d_2 = d_2^2 + d_4^2,$$
$$d_3 = 2d_1d_3,$$
$$d_4 = d_1d_4 + d_2d_4.$$

The relation $(t,t) = \frac{1}{2}$ gives one more equation,

$$\frac{1}{2} = \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2 + d_4^2 + 2d_3^2.$$

Thus,

$$1 = d_1 + d_2.$$

Since $d_3 \neq 0$, it follows that $d_1 = \frac{1}{2}$ and $d_2 = \frac{1}{2}$. We therefore have

$$\frac{1}{4} = 4d_3^2 + d_4^2 \quad \text{and} \quad \frac{1}{4} = 2d_3^2 + d_4^2.$$

This forces $d_3 = 0$, a contradiction. □

PROPOSITION 7.2. *In this case,* $\mathrm{Aut}(V_L^+)$ *is the LVOA*$^+$ *group.*

Proof. By Corollary 5.27, any automorphism of V_L^+ preserves $M(1)_2^+$; the result then follows from Lemma 3.3. □

8. Automorphism Group of V_L^+ with $L_1 \neq \emptyset$

Now we are finally ready to deal with matters when $L_1 \neq \emptyset$. There are two cases: $\mathrm{rank}(L_1) = 2$ or $\mathrm{rank}(L_1) = 1$.

8.1. $\mathrm{rank}(L_1) = 2$

In this case either $L = L_{A_1^2}$ or $L = L_{A_2}$, because these are the only rank-2 root lattices possible and each is a maximal even integral lattice in its rational span.

8.1.1. L of Type A_1^2
If $L \cong L_{A_1^2}$ then $\mathrm{Aut}(V_L) \cong \mathrm{PSL}(2,\mathbb{C}) \wr 2$ and

$$V_L^+ \cong V_{L_{A_1}}^+ \otimes V_{L_{A_1}}^+ \oplus V_{L_{A_1}}^- \otimes V_{L_{A_1}}^-.$$

Since the connected component of the identity in $\mathrm{Aut}(V_L)$ contains a lift of -1_L, we may assume that such a lift is in a given maximal torus and so is equal to the automorphism $e^{\pi i\beta(0)/2}$, where β is a sum of orthogonal roots.

It follows that $V_L^+ \cong V_K$, where $K = 2L + \mathbb{Z}\beta$. The result implies (see [DN1]) that $\mathrm{Aut}(V_L^+) \cong \mathrm{Aut}(V_K)$, which is the LVOA group $\mathbb{T}_2.\mathrm{Dih}_8$.

8.1.2. L of Type A_2
Here, $(V_L^+)_1$ is a 3-dimensional Lie algebra isomorphic to $\mathrm{sl}(2,\mathbb{C})$. The difficult part in this case is determining the vertex operator subalgebra generated by $(V_L^+)_1$.

Let $L_{A_2} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ such that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_1, \alpha_2) = -1$. The set of roots in L is $L_1 = \{\pm\alpha_i \mid i = 1, 2, 3\}$, where $\alpha_3 = \alpha_1 + \alpha_2$. The positive roots are $\{\alpha_i \mid i = 1, 2, 3\}$. The space $(V_L^+)_1$ is 3-dimensional with a basis v_{α_i} for $i = 1, 2, 3$, and $(V_L^-)_1$ is 5-dimensional with a basis $\alpha_1(-1), \alpha_2(-1), e^{\alpha_i} - e^{-\alpha_i}$ for $i = 1, 2, 3$. It is a straightforward to verify that $(v_{\alpha_i})_{-1}v_{\alpha_i}$ for $i = 1, 2, 3$ and $\alpha_i(-1)^2$ for $i = 1, 2, 3$ span the same space. Thus $\omega = \frac{1}{4}\alpha_1(-1)^2 + \frac{1}{12}(\alpha_1(-1) + 2\alpha_2(-1))^2$ lies in the VOA generated by $(V_L^+)_1$.

In order to determine the vertex operator algebra generated by $(V_L^+)_1$, we recall the standard modules for affine algebra:

$$A_1^{(1)} = \hat{\mathrm{sl}}(2, \mathbb{C}) = \mathrm{sl}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

(cf. [DL]). We use the standard basis $\{\alpha, x_\alpha, x_{-\alpha}\}$ for $\mathrm{sl}(2, \mathbb{C})$ such that

$$[\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}, \qquad [x_\alpha, x_{-\alpha}] = \alpha.$$

Fix an invariant symmetric nondegenerate bilinear form on $\mathrm{sl}(2, \mathbb{C})$ such that $(\alpha, \alpha) = 2$. The level-k standard $A_1^{(1)}$-modules are parameterized by dominant integral linear weights $\frac{i}{2}\alpha$ for $i = 0, \ldots, l$ such that the highest weight of the $A_1^{(1)}$-module, viewed as a linear form on $\mathbb{C}\alpha \oplus \mathbb{C}K \subset \hat{\mathrm{sl}}(2, \mathbb{C})$, is given by $\frac{i}{2}\alpha$ and the correspondence $K \mapsto k$. Let us denote the corresponding standard $A_1^{(1)}$-module by $L\left(k, \frac{i}{2}\alpha\right)$. It is well known that $L(k, 0)$ is a simple rational vertex operator algebra and that $L\left(k, \frac{i}{2}\alpha\right)$ for $i = 0, \ldots, k$ is a complete list of irreducible $L(k, 0)$-modules (cf. [DL; FZ; Li2]). Note that

$$L\left(k, \frac{i}{2}\alpha\right) = \bigoplus_{n=0}^{\infty} L\left(k, \frac{i}{2}\alpha\right)_{\lambda_i + n},$$

where $\lambda_i = i(i+2)/4(k+2)$ and $L\left(k, \frac{i}{2}\alpha\right)_{\lambda_i+n}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda_i + n$ (cf. [DL]). In fact, the lowest weight space $L\left(k, \frac{i}{2}\alpha\right)_{\lambda_i}$ of $L\left(k, \frac{i}{2}\alpha\right)$ is an irreducible $\mathrm{sl}(2, \mathbb{C})$-module of dimension $i + 1$.

Since $V_{L_{A_2}}$ is a unitary module for the affine algebra $A_2^{(1)}$ (cf. [FK]), the vertex operator algebra V generated by $(V_L^+)_1$ is isomorphic to the standard level-k $A_1^{(1)}$-module $L(k, 0)$ for some nonnegative integer k. Let $\{v_1, v_2, v_3\}$ be an orthonormal basis of $\mathrm{sl}(2, \mathbb{C})$ with respect to the standard bilinear form. Then $\omega' = \frac{1}{2(k+2)} \sum_{i=1}^{3} v_i(-1)^2\mathbf{1} \in V$ is the Segal–Sugawara Virasoro vector. Let

$$Y(\omega', z) = \sum_{n \in \mathbb{Z}} L(n)' z^{-n-2}.$$

Then

$$[L(n) - L(n)', u_m] = 0$$

for $m, n \in \mathbb{Z}$ and $u \in V$. Hence $L(-2) - L(-2)'$ acts as a constant on V because V is a simple vertex operator algebra. As a result, $L(-2) - L(-2)' = 0$, since the left side is both a constant and an operator that shifts degree by 2. The creation axiom for VOAs implies that $\omega' = \omega$. Since the central charge of ω is 2, the central charge $3k/2(k+2)$ of ω' is also 2. This implies that $k = 4$ and $V \cong L(4, 0)$.

Now V_L^+ is a $L(4,0)$-module, and the quotient module V_L^+/V has minimal weight (as inherited from V_L^+) greater than 1. On the other hand, the minimal weight of the irreducible $L(4,0)$-module $L\left(4, \frac{i}{2}\alpha\right)$ is $i(i+2)/4(4+2)$, which is less than 2 for $0 \leq i \leq 4$. Since every irreducible module is one of these, we conclude that $V_L^+ = V = L(4,0)$. Since V_L^- is an irreducible V_L^+-module with minimal weight 1, we immediately see that $V_L^- = L(4, 2\alpha)$.

Thus we have proved the following proposition.

PROPOSITION 8.1. *Suppose* $\operatorname{rank}(L_1) = 2$.

(1) *If* $L = L_{A_1^2}$ *then* V_L^+ *is a lattice vertex operator algebra* V_K, *where* K *is generated by* β_1, β_2 *with* $(\beta_i, \beta_i) = 4$ *and* $(\beta_1, \beta_2) = 0$. *The automorphism group of* V_L^+ *is the LVOA*$^+$ *group, which is isomorphic to the LVOA group for lattice* K.

(2) *If* $L = L_{A_2}$, *then* V_L^+ *is isomorphic to the vertex operator algebra* $L(4,0)$ *and* $\operatorname{Aut}(V_L^+)$ *is isomorphic to* $\operatorname{PSL}(2,\mathbb{C})$, *which is the automorphism group of* $\mathrm{sl}(2,\mathbb{C})$.

8.2. $\operatorname{rank}(L_1) = 1$

8.2.1. L Rectangular

We first assume $L = \mathbb{Z}r + \mathbb{Z}s$ such that $(r,r) = 2$, $(s,s) \in 6 + 8\mathbb{Z}$, and $(r,s) = 0$. Then $V_L = V_{L_{A_1}} \otimes V_{\mathbb{Z}s}$ and

$$V_L^+ = V_{L_{A_1}}^+ \otimes V_{\mathbb{Z}s}^+ \oplus V_{L_{A_1}}^- \otimes V_{\mathbb{Z}s}^-.$$

LEMMA 8.2. *A group of shape* $\left[\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta\right)\cdot\mathbb{Z}_2\right]\times\mathbb{Z}_2$ *acts on* V_L^+ *as automorphisms.*

Proof. We have already mentioned that $V_{L_{A_1}}^+$ is isomorphic to $V_{\mathbb{Z}\beta}$ for $(\beta,\beta) = 8$ and that $V_{L_{A_1}}^-$ is isomorphic to $V_{\mathbb{Z}\beta+\beta/2}$ as $V_{L_{A_1}}^+$-modules. We also know from [DN1] that $\operatorname{Aut}(V_{\mathbb{Z}\beta})$ is isomorphic to $\mathbb{C}\beta/\left(\mathbb{Z}\frac{1}{8}\beta\right)\cdot\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ is induced from the -1 isometry of the lattice $\mathbb{Z}\beta$. The action of $\lambda\beta \in \mathbb{C}\beta$ is given by the operator $e^{2\pi i\lambda\beta(0)}$. Note that $\mathbb{C}\beta$ acts on $V_{\mathbb{Z}\beta+\beta/2}$ in the same way. But the kernel of the action of $\mathbb{C}\beta$ on $V_{\mathbb{Z}\beta+\beta/2}$ is $\mathbb{Z}\frac{1}{4}\beta$ instead of $\mathbb{Z}\frac{1}{8}\beta$. As a result, the torus $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$ acts on both $V_{\mathbb{Z}\beta}$ and $V_{\mathbb{Z}\beta+\beta/2}$. By [DG], $\operatorname{Aut}(V_{\mathbb{Z}s}^+)$ is isomorphic to $\frac{1}{2}\mathbb{Z}s/\mathbb{Z}s \cong \mathbb{Z}_2$, which also acts on $V_{\mathbb{Z}s}^-$. So the group $\left[\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta\right)\cdot\mathbb{Z}_2\right]\times\mathbb{Z}_2$ acts on V_L^+ as automorphisms. □

In order to determine $\operatorname{Aut}(V_L^+)$ in this case, we recall the notion of commutant from [FZ].

DEFINITION 8.3. Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and let $U = (U, Y, \mathbf{1}, \omega')$ be a vertex operator subalgebra with a different Virasoro vector ω'. The *commutant* U^c of U in V is defined by

$$U^c := \{v \in V \mid u_n v = 0,\ u \in U,\ n \geq 0\}.$$

REMARK 8.4. The space U^c is the space of vacuum-like vectors for U (see [Li1]).

LEMMA 8.5. *Let V be a vertex operator algebra. Let $U^i = (U^i, Y, 1, \omega^i)$ be simple vertex operator subalgebras of V with Virasoro vector ω^i for $i = 1, 2$ such that $\omega = \omega^1 + \omega^2$. We assume that V has a decomposition*

$$V \cong \bigoplus_{i=0}^{p} P^i \otimes Q^i$$

as a $(U^1 \otimes U^2)$-module such that $P^0 \cong U^1$, $Q^0 \cong U^2$, the P^i are inequivalent U^1-modules, and the Q^i are inequivalent U^2-modules. Then $(U^1)^c = U^2$ and $(U^2)^c = U^1$.

Proof. It is enough to prove that $(U^2)^c \subset U^1$. Let $v \in (U^2)^c$. Then v is a vacuum-like vector for U^2. As a result, the U^2-submodule generated by v is isomorphic to U^2 (see [Li1]). Since V is a completely reducible U^2-module, it follows that any U^2-submodule isomorphic to U^2 is contained in $U^1 \otimes U^2$; in particular, $v \in U^1 \otimes U^2$. This forces $v \in U^1$. □

PROPOSITION 8.6. *The group $\mathrm{Aut}(V_L^+)$ is isomorphic to $\left(\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta\right)\cdot\mathbb{Z}_2\right)\times\mathbb{Z}_2$. This can be interpreted as an action of $\mathbb{N}(\widehat{\mathbb{Z}\beta})\times\mathbb{Z}_2$, where $(\beta,\beta) = 8$.*

Proof. We have already shown (Lemma 8.2) that the group $\left(\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta\right)\cdot\mathbb{Z}_2\right)\times\mathbb{Z}_2$ acts on V_L^+ as automorphisms.

Let σ be an automorphism of V_L^+. Then $\sigma\beta(-1) = \lambda\beta(-1)$ for some nonzero $\lambda \in \mathbb{C}$ as $(V_L^+)_1$ is spanned by $\beta(-1)$. This implies that $\sigma\beta(n)\sigma^{-1} = \lambda\beta(n)$ for $n \in \mathbb{Z}$. Since $V_{\mathbb{Z}s}^+$ is precisely the subspace of V_L^+ consisting of vectors killed by $\beta(n)$ for $n \geq 0$, we see that $\sigma V_{\mathbb{Z}s}^+ \subset V_{\mathbb{Z}s}^+$. Thus $\sigma|_{V_{\mathbb{Z}s}^+}$ is an automorphism of $V_{\mathbb{Z}s}^+$. On the other hand, $V_{L_{A_1}}^+$ is the commutant of $V_{\mathbb{Z}s}^+$ in V_L^+ by Lemma 8.5.

The foregoing shows that σ induces an automorphism of the tensor factor $V_{L_{A_1}}^+$. The restriction of σ to $V_{L_{A_1}}^+ \otimes V_{\mathbb{Z}s}^+$ is a product $\sigma_1 \otimes \sigma_2$ for some $\sigma_1 \in \mathrm{Aut}(V_{L_{A_1}}^+)$ and $\sigma_2 \in \mathrm{Aut}(V_{\mathbb{Z}s}^+)$. Multiplying σ by σ_2, we can assume that $\sigma = 1$ on $V_{\mathbb{Z}s}^+$. As we have already mentioned, $\mathrm{Aut}(V_{L_{A_1}}^+)$ is isomorphic to $\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{8}\beta\right)\cdot\mathbb{Z}_2$. Since $\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{8}\beta\right)$ acts trivially on $\beta(-1)$ and the outer factor $\mathbb{Z}_2$ is represented in $\mathrm{Aut}(V_L^+)$ by action of ± 1 on $\beta(-1)$, $\sigma\beta(-1) = \pm\beta(-1)$. Now multiplying σ by an outer element of $\left(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta\right)\cdot\mathbb{Z}_2$, we can assume that $\sigma\beta(-1) = \beta(-1)$.

Set $W_{n\beta} = M(1)\otimes e^{n\beta}\otimes V_{\mathbb{Z}s}^+$ and $W_{n\beta+\beta/2} = M(1)\otimes e^{n\beta+\beta/2}\otimes V_{\mathbb{Z}s}^-$ for $n \in \mathbb{Z}$, where $M(1) = \mathbb{C}[\beta(-n) \mid n > 0]$. Then $V_L^+ = \bigoplus_{n\in\mathbb{Z}/2} W_{n\beta}$ and $u_m v \in W_{\mu+\nu}$ for $u \in W_\mu$, $v \in W_\nu$, and $m \in \mathbb{Z}$. Note that W_μ is the eigenspace of $\beta(0)$ with eigenvalue (β,μ). Since $\sigma\beta(-1) = \beta(-1)$, we see that σ acts on each W_μ as a constant λ_μ and that $\lambda_\mu\lambda_\nu = \lambda_{\mu+\nu}$. As a result, $\sigma = e^{2\pi i\gamma(0)}$ for some $\gamma \in \mathbb{C}\beta$; that is, σ lies in $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$. This completes the proof. □

8.2.2. L Not Rectangular

Next we assume that $L \neq \mathbb{Z}r \perp \mathbb{Z}s$. Then $L = \mathbb{Z}r \oplus \mathbb{Z}\frac{1}{2}(s+t)$, where $(s,s) \in 6 + 8\mathbb{Z}$ and $(s,s) \geq 14$ (see Lemma 4.3). Let $K = \mathbb{Z}r \oplus \mathbb{Z}s$. Then $L = K \cup \left(K + \frac{1}{2}(r+s)\right)$ and $V_L = V_{\mathbb{Z}r}\otimes V_{\mathbb{Z}s} \oplus V_{(\mathbb{Z}+1/2)r}\otimes V_{(\mathbb{Z}+1/2)s}$. Thus

$$V_L^+ = V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+ \oplus V_{\mathbb{Z}r}^- \otimes V_{\mathbb{Z}s}^- \oplus V_{(\mathbb{Z}+1/2)r}^+ \otimes V_{(\mathbb{Z}+1/2)s}^+ \oplus V_{(\mathbb{Z}+1/2)r}^- \otimes V_{(\mathbb{Z}+1/2)s}^-$$

and

$$V_L^+ = V_K^+ \oplus V_{K+(s+t)/2}^+.$$

As before, we note that $V_{\mathbb{Z}r}^+$ is isomorphic to $V_{\mathbb{Z}\beta}$ with $(\beta,\beta) = 8$.

PROPOSITION 8.7. *Assume that* $\operatorname{rank}(L_1) = 1$ *and* $L \neq \mathbb{Z}r + \mathbb{Z}s$ *with* r, s *as before. Then* $\operatorname{Aut}(V_L^+) \cong \left(\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta\right)\cdot\mathbb{Z}_2$, *where* $(\beta,\beta) = 8$. *The action is trivial on the subVOA* $V_{\mathbb{Z}s}^+$ *and leaves* $V_{\mathbb{Z}r}^+$ *invariant. A generator of the quotient* $\mathbb{Z}_2$ *comes from the* -1 *isometry of* $\frac{1}{4}\mathbb{Z}\beta$, *and* $\alpha \in \mathbb{C}\beta$ *acts as* $e^{2\pi i\alpha(0)}$.

Proof. Note that V_K^+ is a subalgebra of V_L^+ and that $V_{K+(r+s)/2}^+$ is an irreducible V_K^+-module. By Proposition 8.6,

$$\operatorname{Aut}(V_K^+) = \left(\left(\mathbb{C}\beta/\tfrac{1}{4}\mathbb{Z}\beta\right)\cdot\mathbb{Z}_2\right)\times\mathbb{Z}_2.$$

We have already mentioned that $V_{\mathbb{Z}r}^+$ is isomorphic to $V_{\mathbb{Z}\beta}$ with $(\beta,\beta) = 8$ and that $V_{\mathbb{Z}r}^-$ is isomorphic to $V_{\mathbb{Z}\beta+\beta/2}$ as a $V_{\mathbb{Z}\beta}$-module. It is easy to see that $V_{(\mathbb{Z}\pm1/4)\beta}^{\pm}$ is isomorphic to $V_{(\mathbb{Z}\pm1/4)\beta}$ as a $V_{\mathbb{Z}\beta}$-module. So the action of $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$ on V_K^+ cannot be extended to an action of V_L^+. But the torus $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta$ does act on V_L^+. As a result, $\mathbb{N}\widehat{\left(\mathbb{Z}\frac{1}{2}\beta\right)} \cong \left(\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta\right)\cdot\mathbb{Z}_2$ is a subgroup of $\operatorname{Aut}(V_L^+)$.

The same argument as used in the proof of Proposition 8.6 shows that any automorphism σ of V_L^+ preserves $V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+$. Since $V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+$, $V_{\mathbb{Z}r}^- \otimes V_{\mathbb{Z}s}^-$, $V_{(\mathbb{Z}+1/2)r}^+ \otimes V_{(\mathbb{Z}+1/2)s}^+$, and $V_{(\mathbb{Z}+1/2)r}^- \otimes V_{(\mathbb{Z}+1/2)s}^-$ are inequivalent irreducible $(V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+)$-modules (see [DLiM; DM1]), we see that σ preserves

$$V_K^+ = V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+ \oplus V_{\mathbb{Z}r}^- \otimes V_{\mathbb{Z}s}^-.$$

Since $\mathbb{C}\beta/\frac{1}{4}\mathbb{Z}\beta\cdot\mathbb{Z}_2$ is a quotient group of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta\cdot\mathbb{Z}_2$, we can multiply σ by an element of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta\cdot\mathbb{Z}_2$ and assume that σ acts trivially on the first tensor factor of V_K^+. If σ is the identity on V_K^+, then σ is either 1 or -1 on $V_{K+(r+s)/2}^+$. If σ is -1 on $V_{K+(r+s)/2}^+$ then $\sigma = e^{\pi i\beta(0)/2}$ is an element of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta\cdot\mathbb{Z}_2$.

If σ is not the identity on V_K^+ then we must have $\sigma = e^{\pi i s(0)/(s,s)}$ on V_K^+; we will get a contradiction in this case. Notice that the lowest weight space of $V_{(\mathbb{Z}+1/2)r}^+ \otimes V_{(\mathbb{Z}+1/2)s}^+$ is 1-dimensional and spanned by $u = (e^{r/2} + e^{-r/2}) \otimes (e^{s/2} + e^{-s/2})$. Since σ preserves $V_{(\mathbb{Z}+1/2)r}^+ \otimes V_{(\mathbb{Z}+1/2)s}^+$, it must map u to λu for some nonzero constant λ. Observe that $u_{(r+s,r+s)/4-1}u = 4$; this forces $\lambda = \pm 1$. On the other hand,

$$u_{-(r+s,r+s)/4-1}u = (e^r + e^{-r}) \otimes (e^s + e^{-s}) + \cdots$$

has nontrivial projection to the -1 eigenspace of σ in V_K^+. This forces $\lambda = \pm i$, a contradiction. □

9. Appendix: Algebraic Rules

For the symmetric matrices of degree n, there is a widely used basis, Jordan product, and inner product that we review here. (This section is taken almost verbatim from [G1].)

PROPOSITION 9.1. *Let H be a vector space of finite dimension n and with non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Let $r, s, \ldots$ stand for elements of H and let rs denote the symmetric tensor $r \otimes s + s \otimes r$. Then*:

$$rs \times pq = (r,p)sq + (r,q)sp + (s,p)rq + (s,q)rp;$$
$$(rs, pq) = (r,p)(s,q) + (r,q)(s,p);$$
$$rs \times v_t = (r,t)(s,t)v_t.$$

DEFINITION 9.2 (Symmetric Bilinear Form) [FLMe, p. 217]. This form is associative with respect to the product (see Section 3). We write H for H_1. The set of all g^2 and x_α^+ spans V_2, and

$$\langle g^2, h^2\rangle = 2\langle g, h\rangle^2,$$

whence

$$\langle pq, rs\rangle = \langle p, r\rangle\langle q, s\rangle + \langle p, s\rangle\langle q, r\rangle \quad \text{for } p, q, r, s \in H.$$

Also,

$$\langle x_\alpha^+, x_\beta^+\rangle = \begin{cases} 2 & \text{if } \alpha = \pm\beta, \\ 0 & \text{else;} \end{cases}$$
$$\langle g^2, x_\beta^+\rangle = 0.$$

DEFINITION 9.3. In addition, we have the distinguished Virasoro element ω and identity $\mathbb{I} := \frac{1}{2}\omega$ on V_2 (see Section 3). If h_i is a basis for H and if h_i^* is the dual basis, then $\omega = \frac{1}{2}\sum_i h_i h_i^*$.

REMARK 9.4. We have:

$$\langle g^2, \omega\rangle = \langle g, g\rangle;$$
$$\langle g^2, \mathbb{I}\rangle = \langle g, g\rangle/2;$$
$$\langle \mathbb{I}, \mathbb{I}\rangle = \dim(H)/8;$$
$$\langle \omega, \omega\rangle = \dim(H)/2.$$

If $\{x_i \mid i = 1, \ldots, l\}$ is an orthonormal basis, then

$$\mathbb{I} = \frac{1}{4}\sum_{i=0}^{l} x_i^2,$$

$$\omega = \frac{1}{2}\sum_{i=0}^{l} x_i^2.$$

DEFINITION 9.5. The *product on* V_2^F comes from the vertex operations. We give it on standard basis vectors, namely, $xy \in S^2H_1$ for $x, y \in H_1$ and $v_\lambda := e^\lambda + e^{-\lambda}$ for $\lambda \in L_2$. (This is the same as x_λ^+, used in [FLMe].) Note that equations (9.1) give the Jordan algebra structure on S^2H_1 that is identified with the space of symmetric 8×8 matrices and with $\langle x, y\rangle = \frac{1}{8}\operatorname{tr}(xy)$. The function ε in equation (9.3) is a standard part of notation for lattice VOAs.

$$\begin{aligned} x^2 \times y^2 &= 4\langle x,y\rangle xy, \qquad pq \times y^2 = 2\langle p,y\rangle qy + 2\langle q,y\rangle py, \\ pq \times rs &= \langle p,r\rangle qs + \langle p,s\rangle qr + \langle q,r\rangle ps + \langle q,s\rangle pr; \end{aligned} \tag{9.1}$$

$$x^2 \times v_\lambda = \langle x,\lambda\rangle^2 v_\lambda, \qquad xy \times v_\lambda = \langle x,\lambda\rangle\langle y,\lambda\rangle v_\lambda; \tag{9.2}$$

$$v_\lambda \times v_\mu = \begin{cases} 0 & \text{if } \langle\lambda,\mu\rangle \in \{0,\pm 1,\pm 3\}, \\ \varepsilon\langle\lambda,\mu\rangle v_{\lambda+\mu} & \text{if } \langle\lambda,\mu\rangle = -2, \\ \lambda^2 & \text{if } \lambda = \mu. \end{cases} \tag{9.3}$$

Some consequences of the foregoing may be summarized as follows.

COROLLARY 9.6. *If $x_1, \ldots$ is a basis and $y_1, \ldots$ is the dual basis, then $\mathbb{I} := \frac{1}{4}\sum_{i=1}^{n} x_i y_i$ is the identity of the algebra S^2H. Also, $(\mathbb{I},\mathbb{I}) = n/8$.*

References

[AD] T. Abe and C. Dong, *Classification of irreducible modules for vertex operator algebra V_L^+: General case,* J. Algebra 273 (2004), 657–685.

[ADLi] T. Abe, C. Dong, and H. Li, *Fusion rules for the vertex operator algebras $M(1)^+$ and V_L^+,* Comm. Math. Phys. 253 (2005), 171–219.

[B] R. E. Borcherds, *Vertex algebras, Kac–Moody algebras, and the Monster,* Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 3068–3071.

[D] C. Dong, *Vertex algebras associated with even lattices,* J. Algebra 161 (1993), 245–265.

[DG1] C. Dong and R. L. Griess, Jr., *Rank one lattice type vertex operator algebras and their automorphism groups,* J. Algebra 208 (1998), 262–275.

[DG2] ———, *Automorphism groups and derivation algebras of finitely generated vertex operator algebras,* Michigan Math. J. 50 (2002), 227–239.

[DGH] C. Dong, R. Griess, Jr., and G. Hoehn, *Framed vertex operator algebras, codes and the moonshine module,* Comm. Math. Phys. 193 (1998), 407–448.

[DGR] C. Dong, R. L. Griess, Jr., and A. J. E. Ryba, *Rank one lattice type vertex operator algebras and their automorphism groups, II. E-series,* J. Algebra 217 (1999), 701–710.

[DL] C. Dong and J. Lepowsky, *Generalized vertex algebras and relative vertex operators,* Progr. Math., 112, Birkhäuser, Boston, 1993.

[DLiM] C. Dong, H. Li, and G. Mason, *Compact automorphism groups of vertex operator algebras,* Internat. Math. Res. Notices 18 (1996), 913–921.

[DM1] C. Dong and G. Mason, *On quantum Galois theory,* Duke Math. J. 86 (1997), 305–321.

[DM2] ———, *Vertex operator algebras and their automorphism groups,* Representations and quantizations (Shanghai, 1998), pp. 145–166, China High. Educ. Press, Beijing, 2000.

[DN1] C. Dong and K. Nagatomo, *Automorphism groups and twisted modules for lattice vertex operator algebras,* Recent developments in quantum affine algebras and related topics (Raleigh, 1998), Contemp. Math., 248, pp. 117–133, Amer. Math. Soc., Providence, RI, 1999.

[DN2] ———, *Classification of irreducible modules for the vertex operator algebra $M(1)^+$, II. Higher rank,* J. Algebra 240 (2001), 289–325.

[FLMe] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster,* Pure Appl. Math., 134, Academic Press, Boston, 1988.

[FZ] I. Frenkel and Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras,* Duke Math. J. 66 (1992), 123–168.

[G1] R. Griess, Jr., *A vertex operator algebra related to E_8 with automorphism group $O^+(10,2)$*, The Monster and Lie algebras (Columbus, 1996), Ohio State Univ. Math. Res. Inst. Publ., 7, pp. 43–58, de Gruyter, Berlin, 1998.

[G2] ———, *Automorphisms of vertex operator algebras and variations, a survey,* Recent developments in quantum affine algebras and related topics (Raleigh, 1998), Contemp. Math., 248, pp. 235–241, Amer. Math. Soc., Providence, RI, 1999.

[G3] ———, *GNAVOA, I. Studies in groups, nonassociative algebras and vertex operator algebras,* Vertex operator algebras in mathematics and physics (Toronto, 2000), Fields Inst. Commun., 39, pp. 71–88, Amer. Math. Soc., Providence, RI, 2003.

[G4] ———, *Positive definite lattices of rank at most 8,* J. Number Theory 103 (2003), 77–84.

[GH] R. Griess, Jr., and G. Hoehn, *Virasoro frames and their stabilizers for the E_8 lattice type vertex operator algebra,* J. Reine Angew. Math. 561 (2003), 1–37.

[J] N. Jacobson, *Isomorphism of Jordan rings,* Amer. J. Math. 70 (1948), 317–326.

[Li1] H. Li, *Symmetric invariant bilinear forms on vertex operator algebras,* J. Pure Appl. Algebra 96 (1994), 279–297.

[Li2] ———, *Local systems of vertex operators, vertex superalgebras and modules,* J. Pure Appl. Algebra 109 (1996), 143–195.

[S] H. Shimakura, *The automorphism group of the vertex operator algebra V_L^+ for an even lattice L without roots,* J. Algebra 280 (2004), 29–57.

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