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A Family of Knots Yielding Graph Manifolds by Dehn Surgery

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

1. Main Theorem

Let $P_{(l,r)}$ be an embedded once-punctured torus, $k_{(l,a;r,b)}$ a knot in $P_{(l,r)}$ in S^3 defined as in Figure 1, and

$$p_{(l,a;r,b)} := la^2 + ab + rb^2,$$

where (a,b) is a coprime pair of integers a, b with $1 < a < b$ and where l and r are integers. We will study the knots $k_{(l,a;r,b)}$ themselves later. Our main theorem concerns Dehn surgery along $k_{(l,a;r,b)}$.

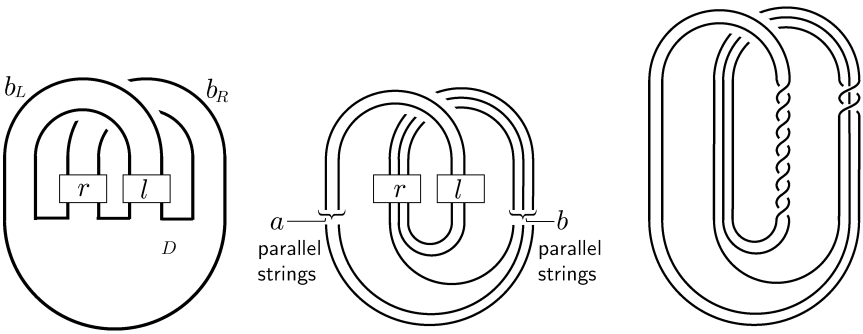

Figure 1 $k_{(l,a;r,b)}$ in $P_{(l,r)}$ (here, $k_{(4,2;1,3)}$)

Theorem 1.1. *For each $(l,a;r,b)$ as described previously, the resulting manifold $(k_{(l,a;r,b)}; p_{(l,a;r,b)})$ of $p_{(l,a;r,b)}$-surgery along the knot $k_{(l,a;r,b)}$ is "at most" a graph manifold obtained by splicing two Seifert manifolds over S^2 (possibly reduced to a Seifert manifold over S^2, a lens space, or a connected sum of two lens spaces in some cases).*

In fact, $(k_{(l,a;r,b)}; p_{(l,a;r,b)})$ bounds a plumbing manifold [O, p. 22] corresponding to the weighted graph in Figure 2; that is, $(k_{(l,a;r,b)}; p_{(l,a;r,b)})$ is described by the framed link in the figure. We will give an algorithm to decide the integers n_L, n_R

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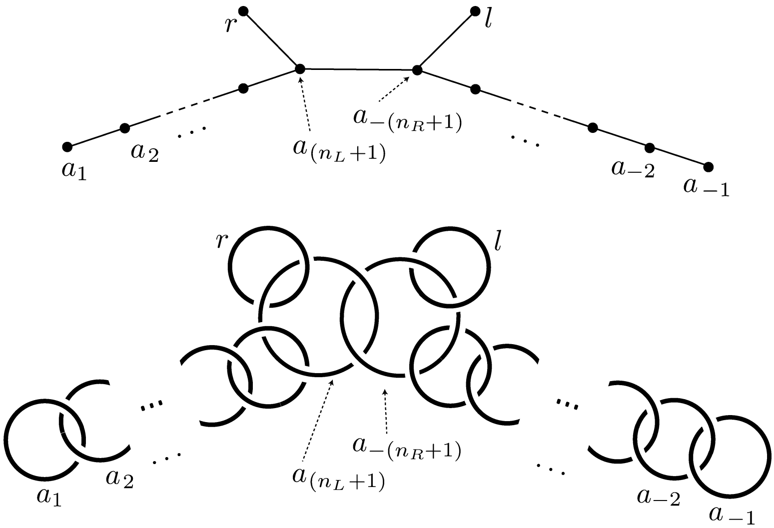

Figure 2 $(k_{(l,a;r,b)}, p_{(l,a;r,b)})$

and the weights (i.e., framings) $\{a_j\}$ in Section 2, where $a_{-(n_R+1)} = -1$. Each vertex with weight a_j corresponds to a disk bundle over S^2 whose self-intersection number of the zero-section is a_j, and each edge corresponds to a plumbing. The reason why the weight r (or l, respectively) is in the left (or right) half of the figure will become clear in Sections 2 and 3.

Theorem 1.1 includes the following Dehn surgeries, which were discovered one by one.

(1) ab-surgery along $T(a, b)$ is a connected sum of two lens spaces as the cases $(l, a; r, b) = (0, a; 0, b)$; see [M].
(2) A subfamily of Berge's lens surgery [Be] (see also [Ba]; denoted by $k^{\pm}(a, b)$ in [Y3]) as the cases $(l, a; r, b) = (\pm 1, a; 1, b)$; it includes 19-surgery along the pretzel knot $Pr(-2, 3, 7)$ as the case $(l, a; r, b) = (1, 2; 1, 3)$.
(3) $(4l + 15)$-surgery on the pretzel knot $Pr(-2, 3, 2l + 5)$ is a Seifert manifold [BH, Prop. 16] as the case $(l, a; r, b) = (l, 2; 1, 3)$ with $l \geq 2$.

These surgeries may be alternatively proved by Theorem 1.1 and moves of graphs [FS] in Figure 3 or Kirby calculus [K; GS].

In Section 3, we will prove Theorem 1.1 by Kirby calculus on framed links. The process incorporates a Euclidean algorithm and the resolution [HKK; L] of the singularity of the complex curve of type $z^a - w^b = 0$ or the twisting sequence on torus knots. This method was also discussed in [Y3] for the special case (2) of lens surgery just listed. In order to extend this method to the more general case, in this paper we will arrange the suffixes (js) of the sequence $\{a_j\}$.

In Section 4 we will study the knots $k_{(l,a;r,b)}$ themselves. Each $k_{(l,a;r,b)}$ belongs to the class of *twisted torus knots* studied in [D] and to the class of *A'Campo's*

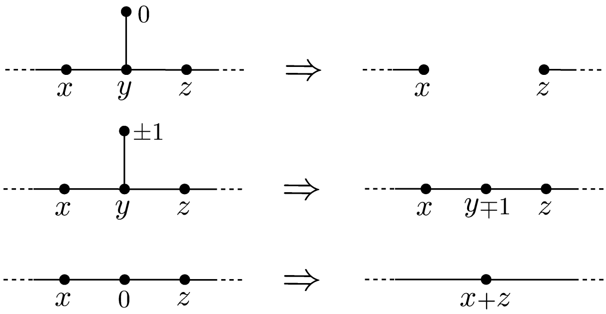

Figure 3 Moves on graphs

divide knots if l and r are nonnegative; see [A1; A2; A3] (and also [GHY; Hi; Y1; Y2]) for A'Campo's divide knots.

2. Algorithm

Here we present the algorithm for defining the integers n_R and n_L as well as the sequences

$$a_1, a_2, \ldots, a_{n_L}, a_{(n_L+1)} \quad \text{and} \quad a_{-(n_R+1)}, a_{-n_R}, \ldots, a_{-2}, a_{-1}$$

of weights (framings) in Figure 2, where $a_{-(n_R+1)} = -1$. The algorithm depends only on (a, b) and is independent of l and r.

ALGORITHM—from (a, b) to the sequence $\{a_j\}$.

(1) *Euclidean algorithm:* Get a word $w(a, b) = w_1 w_2 \cdots w_n$ of two letters L (left) and R (right) from the pair (a, b) (=: (a_0, b_0)) inductively by the following rule:

$$\text{if } a_i > b_i, \text{ then } w_{i+1} := L \text{ and } (a_{i+1}, b_{i+1}) := (a_i - b_i, b_i);$$

$$\text{if } a_i < b_i, \text{ then } w_{i+1} := R \text{ and } (a_{i+1}, b_{i+1}) := (a_i, b_i - a_i).$$

By the coprimeness of (a, b), after some n steps the pair (a_n, b_n) becomes $(1, 1)$, which is the end of this step. We define n_R (and n_L, respectively) as the number of R (and L) in the word $w(a, b)$.

(2) Next, starting with

$$\{a_*^{(0)}\} = (a_{-1}^{(0)}, a_0^{(0)}, a_1^{(0)}) := (-1, -1, -1),$$

we define the sequence $\{a_*^{(i)}\}$ $(i = 1, 2, \ldots, n)$ inductively as follows.

(a) For each i, $a_0^{(i)} = -1$.

(b) If $w_i = R$, then we define $\{a_*^{(i)}\}$ as

$$\begin{cases} a_j^{(i)} := a_j^{(i-1)} & \text{if } j > 1 \text{ and } a_j^{(i-1)} \text{ is defined,} \\ a_1^{(i)} := a_1^{(i-1)} - 1, & \\ a_{-1}^{(i)} := -2, & \\ a_j^{(i)} := a_{j+1}^{(i-1)} & \text{if } j < -1 \text{ and } a_{j+1}^{(i-1)} \text{ is defined.} \end{cases}$$

(c) If $w_i = L$, then we define $\{a_*^{(i)}\}$ as

$$\begin{cases} a_j^{(i)} := a_j^{(i-1)} & \text{if } j < -1 \text{ and } a_j^{(i-1)} \text{ is defined,} \\ a_{-1}^{(i)} := a_{-1}^{(i-1)} - 1, & \\ a_1^{(i)} := -2, & \\ a_j^{(i)} := a_{j-1}^{(i-1)} & \text{if } j > 1 \text{ and } a_{j-1}^{(i-1)} \text{ is defined.} \end{cases}$$

(3) For each integer j with $-(n_R+1) \le j \le (n_L+1)$, we define a_j as $a_j^{(n)}$ in the sequence $\{a_*^{(n)}\}$ obtained after the nth step, where n is the length of the word $w(a,b)$.

By the assumption $a < b$, we have $w_1 = R$ and $a_{-(n_R+1)} = -1$. The resulting sequence $\{a_j\}$ satisfies

$$[|a_{-(n_R+1)}|, |a_{-n_R}|, \ldots, |a_{-2}|, |a_{-1}|] = \frac{a}{b}, \qquad [|a_{(n_L+1)}|, |a_{n_L}|, \ldots, |a_2|, |a_1|] = \frac{b}{a},$$

where $[x_1, x_2, \ldots, x_n]$ is the continued fraction expansion

$$[x_1, x_2, \ldots, x_n] := x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_n}}}.$$

EXAMPLE. $(2,7) \to_R (2,5) \to_R (2,3) \to_R (2,1) \to_L (1,1)$, with $n_R = 3$ and $n_L = 1$.

i	$a_{-4}^{(i)}$	$a_{-3}^{(i)}$	$a_{-2}^{(i)}$	$a_{-1}^{(i)}$	$a_0^{(i)}$	$a_1^{(i)}$	$a_2^{(i)}$
0				−1	−1	−1	
1			−1	−2	−1	−2	
2		−1	−2	−2	−1	−3	
3	−1	−2	−2	−2	−1	−4	
4	−1	−2	−2	−3	−1	−2	−4

See Figure 4.

3. Proof of Main Theorem

Let $P := P_{(0,0)}$ be a standardly embedded once-punctured torus in the position S^3 (cf. Figure 1); it consists of a disk D and two bands b_L and b_R. We take a simple closed curve $k^0(a,b) := k_{(0,a;0,b)}$ in P as in Figure 1. The framing of $k^0(a,b)$ defined by the surface P is ab. From now on, we call such a framing *P-framing* ("surface framing").

Twisting the bands b_L right-handed l-fully, b_R r-fully, and the curve $k^0(a,b)$ in it simultaneously, we have the knot $k_{(l,a;r,b)}$ in the surface $P_{(l,r)}$. This operation is realized by the framed link in the complement of P in S^3; see Figure 5. Observe that $P_{(l,r)}$-framing of $k_{(l,a;r,b)}$ is $p_{(l,a;r,b)}$.

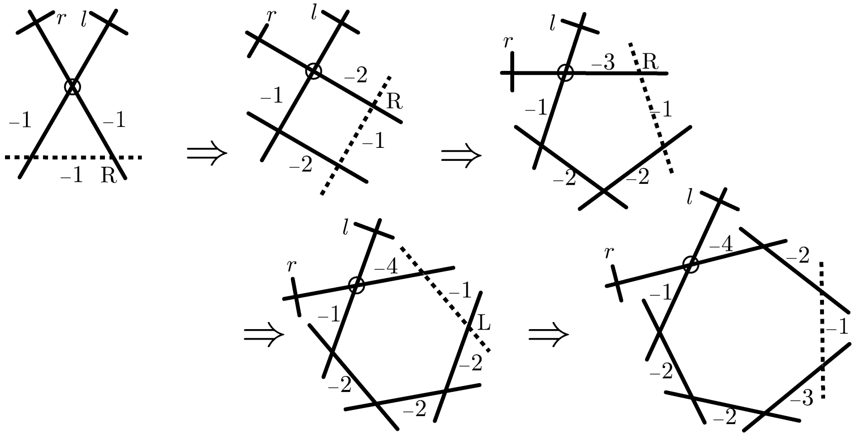

Figure 4 Blow-ups

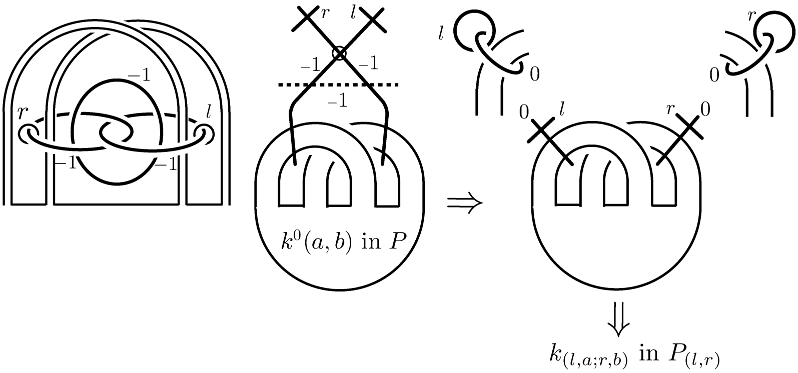

Figure 5 From $(P, k^0(a,b))$ to $(P_{(l,r)}, k_{(l,a;r,b)})$

Next, we move P and the curve $k^0(a,b)$ simultaneously in the total space S^3 in another way, according to each step of (2) in the Algorithm: if $w_{i+1} = R$ (i.e., $a_i < b_i$), we move the *left* band b_L over the central (-1)-component and slide over b_R as in Figure 6. In each black box of the figure, we take a tangle T $(x = y = -1)$ for the first step and take the tangle that appeared in the gray box at the end of the previous step, inductively. If $w_{i+1} = L$, the operation is similar by symmetry. Note that, after each operation in Figure 6: P comes back to the starting position; and $k^0(a_i, b_i)$ is changed to $k^0(a_i, b_i - a_i)$ in the R case or to $k^0(a_i - b_i, b_i)$ in the L case—that is, to $k^0(a_{i+1}, b_{i+1})$ in either case—and a new (-1)-component appears for the next step. Note that the relation "P-framing of $k^0(a_i, b_i)$ is $a_i b_i$" is kept during the process.

After n steps (n is the length of the word $w(a,b)$ in step (1) of the Algorithm), we have the framed link we seek: the final (-1)-curve γ and a $(+1)$-framed curve $\gamma' := k^0(1,1)$ in P. Sliding γ' over γ, we can cancel them. The proof of Theorem 1.1 is completed.

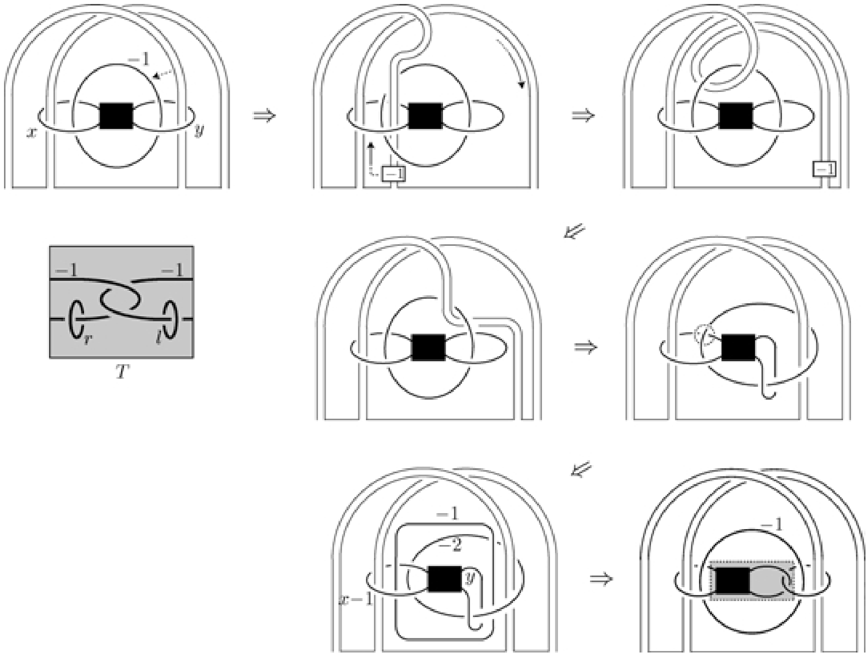

Figure 6 Operation (R case)

4. Knots $k_{(l,a;r,b)}$

Here we describe the knots $k_{(l,a;r,b)}$ themselves, but we do not give complete proofs because these can be established by method(s) already reported by the author [Y1; Y2; Y3].

THEOREM 4.1. *If* $l \geq 1$ *and* $r \geq 1$, *then the knot* $k_{(l,a;r,b)}$ *is equal to a twisted torus knot* $T(la+b,a;b,r)$ *and also to* $T(a+rb,b;a,l)$, *where* $T(p,q;x,y)$ *is a knot obtained from a torus knot* $T(p,q)$ *by* y *fully twisting of* x *strings in* p *parallel strings of* $T(p,q)$ *in the standard position.*

Outline of Proof. From $k^0(a,b) = k_{(0,a;0,b)}$ in $P = P_{(0,0)}$, we have the knot $k_{(l,a;r,b)}$ in the surface $P_{(l,r)}$ by twisting the bands b_L l-fully and b_R r-fully (and the curve $k^0(a,b)$ in it simultaneously). Here, if we twist b_L first, we have $k_{(l,a;0,b)}$ in $P_{(l,0)}$ once; on the other hand, if we twist b_R first then we have $k_{(0,a;r,b)}$ in $P_{(0,r)}$. The once-punctured torus $P_{(l,0)}$ (and $P_{(0,r)}$ also) is isotopic to a subsurface of the standard torus in S^3, so both $k_{(l,a;0,b)}$ and $k_{(0,a;r,b)}$ are torus knots. Their indices are easily calculated to be $T(la+b,a)$ and $T(a+rb,b)$, respectively. The second twisting of b_R or b_L is easily checked to be the construction stated in the theorem. □

Next, we point out that $k_{(l,a;r,b)}$ belongs to A'Campo's divide knots if $l, r \geq 0$. Let $C_{(l,a;r,b)}$ be a plane curve obtained by cutting out from the lattice X in the plane

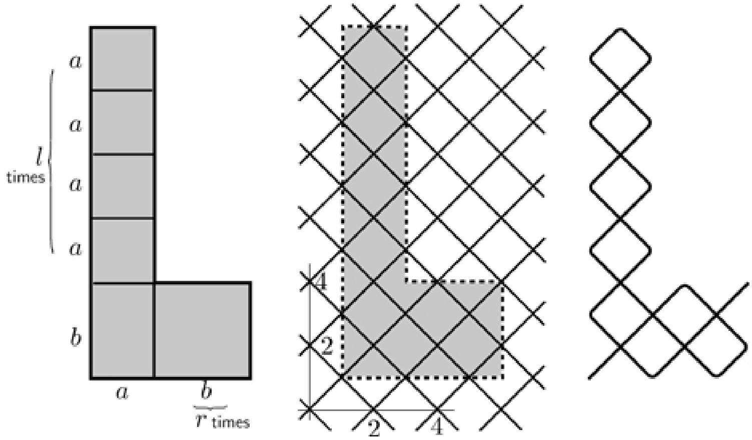

Figure 7 Curve $C_{(l,a;r,b)}$ (here, $C_{(4,2;1,3)}$)

as $X \cap \mathcal{R}_{(l,a;r,b)}$ (and by smoothing), where $\mathcal{R}_{(l,a;r,b)}$ is a region defined as in Figure 7. Note that $\mathcal{R}_{(l,a;r,b)}$ should be in the position such that $X \cap \mathcal{R}_{(l,a;r,b)}$ is an image of an immersion of an arc; see [Hi; Y2].

THEOREM 4.2. *For each $(l, a; r, b)$ with $l, r \geq 0$, the knot $k_{(l,a;r,b)}$ is A'Campo's divide knot $L(C_{(l,a;r,b)})$ of $C_{(l,a;r,b)}$. Hence the unknotting number, minimal Seifert genus, and 4-genus of $k_{(l,a;r,b)}$ are all equal to the number of double points in $C_{(l,a;r,b)}$:*

$$\frac{1}{2}\{la^2 + ab + rb^2 - (l+1)a - (r+1)b + 1\}.$$

Outline of Proof. Each torus knot $T(p,q)$ is A'Campo's divide knot of the "billiard curve" of a $p \times q$ rectangle region; see [GHY] (and [AGV; CP; GZ]). Adding $x \times x$ squares along an edge of length p $(x \leq p)$ corresponds to once twisting x strings among the p strings. □

Note that the area of the region $\mathcal{R}_{(l,a;r,b)}$ is equal to $p_{(l,a;r,b)} = la^2 + ab + rb^2$ (see [Y1; Y2; Y3]).

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