Michigan Math. J. 53 (2005)

Kähler Submanifolds with Ricci Curvature Bounded from Below

Young Jin Suh & Hae Young Yang

1. Introduction

A complex n-dimensional ($n \geq 2$) Kähler manifold of constant holomorpic sectional curvature $c > 0$ is called a *complex projective space* and is denoted by $\mathbb{C}P^{n+p}(c)$. In this paper we want to study some complete Kähler submanifolds in a complex projective space $\mathbb{C}P^{n+p}(1)$ concerned with the Ricci curvatures.

The theory of Kähler submanifolds was systematically studied by Ogiue [5; 6; 7]; in [6], some pinching problems concerned with the Ricci curvatures were studied. Specifically, the following theorem was proved.

Theorem A. *Let M be an n-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex projective space $\mathbb{C}P^{n+p}(1)$ of constant holomorphic sectional curvature* 1. *If the Ricci curvatures are greater than $n/2$, then M is totally geodesic.*

Now let us consider a generalization of this theorem to the case where the Ricci curvature is greater than *or equal* to $n/2$. Before giving such a classification problem concerned with the Ricci curvature, we introduce the following theorem (due to Nakagawa and Takagi [4]) related to the parallel second fundamental form.

Theorem B. *Let M^n be a compact Kähler submanifold immersed in a complex projective space $\mathbb{C}P^m(1)$ with parallel second fundamental form. Then M is an imbedded submanifold congruent to the standard imbedding of one of the following submanifolds.*

Submanifold	*Dim*	*Codim*	*Scalar*
$M_1 = \mathbb{C}P^n(1)$	n	0	$n(n+1)$
$M_2 = \mathbb{C}P^n(\frac{1}{2})$	n	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n+1)$
$M_3 = \mathbb{C}P^{n-s}(1) \times \mathbb{C}P^s(1)$	n	$s(n-s)$	$s^2 + (n-s)^2 + n$
$M_4 = Q^n, n \geq 3$	n	1	n^2
$M_5 = U(s+2)/U(2) \times U(s), s \geq 3$	n	$\frac{1}{2}s(s-1)$	$2s(s+2)$
$M_6 = \mathrm{SO}(10)/U(5)$	10	5	80
$M_7 = E_6/\mathrm{Spin}(10) \times T$	16	10	192

Received June 7, 2004. Revision received August 17, 2005.
This work was supported by the grant Proj. No. R14-2002-003-01001-0 from Korea Research Foundation, Korea.

Here Dim, Codim, and Scalar denote (respectively) the complex dimension, the full complex codimension, and the scalar curvature of M_i, $i = 1, \dots, 7$.

Using a holomorphic pinching condition such that $H \geq \frac{1}{2}$, Ros [8] asserted the result of Theorem B after proving that the second fundamental form is parallel. Moreover, in [4] it can be easily seen that, except for M_3 in Theorem B, all the foregoing results are Einstein (see also Besse [1]).

Now let us consider the totally real bisectional curvature and the Ricci curvature of the third imbedded submanifold $M_3 = \mathbb{C}P^{n-s}(1) \times \mathbb{C}P^s(1)$ in $\mathbb{C}P^n(1)$. This curvature can be calculated as follows (see also Ki and Suh [2]). First, the totally real bisectional curvature of M_3 is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{1}{2} & \text{if } A = a \text{ and } B = b, \\ 0 & \text{if } A = a \text{ and } B = u, \\ R_{\bar{u}uv\bar{v}} = \frac{1}{2} & \text{if } A = u \text{ and } B = v, \end{cases}$$

where $A \neq B$ and the indices $A, B, \dots; 1, \dots, n-s, n-s+1, \dots, n$ and $a, b, \dots :$ $1, \dots, n-s, u, v, \dots : n-s+1, \dots, n$.

Next, the Ricci tensor can be given by

$$\begin{aligned} S_{A\bar{B}} &= \sum_C R_{\bar{B}AC\bar{C}} = \sum_a R_{\bar{B}Aa\bar{a}} + \sum_r R_{\bar{B}Ar\bar{r}} \\ &= \begin{cases} \frac{n-s+1}{2}\delta_{bc} & \text{if } A = b \text{ and } B = c, \\ 0 & \text{if } A = b \text{ and } B = u, \\ \frac{s+1}{2}\delta_{uv} & \text{if } A = u \text{ and } B = v. \end{cases} \end{aligned}$$

As usual, if $n - s + 1 \neq s + 1$ then $M_3 = \mathbb{C}P^{n-s}(1) \times \mathbb{C}P^s(1)$ is not Einstein. But we need more restrictions for the dimension of M_3 whose Ricci curvature is not less than $n/2$. In Section 3 we will show that, among the manifolds of type M_3, it should be $\mathbb{C}P^1(1) \times \mathbb{C}P^1(1)$ and Einstein.

The purpose of this paper is to give a complete classification of complete Kähler submanifolds immersed in $\mathbb{C}P^{n+p}(1)$ whose Ricci curvature is greater than or equal to $n/2$. In order to do this we will show that complete Kähler submanifolds satisfying such a condition have parallel second fundamental form. Then, using Theorem B, we will prove the following.

MAIN THEOREM. *Let M be an immersed Kähler submanifold of dimension n in the complex projective space $\mathbb{C}P^{n+p}(1)$ of constant holomorphic sectional curvature* 1. *If the Ricci curvatures are greater than or equal to $n/2$, then M is congruent to the standard imbedding of one of the following Kähler–Einstein submanifolds*:

$$\mathbb{C}P^n(1), \quad \mathbb{C}P^1(1) \times \mathbb{C}P^1(1), \quad Q^n,\ n \geq 3.$$

Of course, all manifolds mentioned in this theorem are known to be submanifolds in $\mathbb{C}P^{n+p}(1)$ with its codimension 0, 1, and 1 (respectively).

In Section 2 we recall some basic formulas that will be used in the proof of our Main Theorem. Also we introduce the second fundamental form and the Ricci tensor defined on a Kähler submanifold in a complex projective space $\mathbb{C}P^{n+p}(1)$, and we give the Bochner type identity for the complex Laplacian of the second fundamental form.

In Section 3, by estimating the Laplacian of the squared norm of the second fundamental tensor and under our assumption concerning the Ricci curvature, we prove that the second fundamental form is parallel. It follows that its scalar curvature is also constant.

Finally, in Section 4 we introduce another proof of our Main Theorem by using Ros's theorem for the holomorphic pinching condition $H \geq \frac{1}{2}$.

2. Kähler Submanifolds

In this section we recall some basic facts about complex submanifolds of a Kähler manifold. First of all, the basic formulas for the theory of complex submanifolds are prepared.

Let M' be an $(n + p)$-dimensional Kähler manifold with Kähler structure (g', J'). Let M be an n-dimensional complex submanifold of M' and let g be the induced Kähler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, when restricted to M, the frames $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices will be used unless otherwise stated:

$$\begin{aligned} A, B, \ldots &= 1, \ldots, n, n+1, \ldots, n+p; \\ i, j, \ldots &= 1, \ldots, n; \\ x, y, \ldots &= n+1, \ldots, n+p. \end{aligned}$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_j, \omega_y\}$ denote its dual frame fields. Then the canonical forms ω_A and a connection form ω_{AB} of the ambient space M' satisfy the following structure equations:

$$\begin{gathered} d\omega_A + \sum \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}; \\ \Omega'_{AB} = \sum_{C,D} K'_{ABCD} \omega_C \wedge \bar{\omega}_D. \end{gathered} \tag{2.1}$$

Here Ω'_{AB} (resp., K'_{ABCD}) denotes the curvature form (resp., the components of the Riemannian curvature tensor K') of M'; see [2; 3].

Restricting these forms to the submanifold M yields

$$\omega_x = 0, \tag{2.2}$$

and the induced Kähler metric tensor g of M is given by $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex-valued 1-forms of type (1, 0) on M. Moreover, $\{\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n\}$ are linearly independent, and $\{\omega_j\}$ is the set of canonical forms on M. It follows from (2.2) and Cartan's lemma that the exterior derivative of (2.2) gives rise to

$$\omega_{xi} = \sum_j h^x_{ij}\omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{2.3}$$

The quadratic form $h = \sum_{i,j,x} h^x_{ij}\omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M.

The structure equations for the Kähler submanifold M are similarly given by

$$\begin{gathered} d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}; \\ \Omega_{ij} = \sum_{k,l} K_{\bar{i}jk\bar{l}}\,\omega_k \wedge \bar{\omega}_l. \end{gathered} \tag{2.4}$$

From (2.1), (2.3), and (2.4), it follows that the Gauss equation between the Riemannian curvature tensors K and K' of M and M' (respectively) is given by

$$K_{\bar{i}jk\bar{l}} = K'_{\bar{i}jk\bar{l}} - \sum_x h^x_{jk}\bar{h}^x_{il}. \tag{2.5}$$

Now let the ambient space $M' = \mathbb{C}P^{n+p}(c)$ be an $(n+p)$-dimensional complex projective space of constant holomorphic sectional curvature $c > 0$. Then from (2.5) we have that the curvature tensor K, the Ricci tensor S, and the scalar curvature r of a Kähler submanifold M in $\mathbb{C}P^{n+p}(c)$ are given (respectively) by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ji}\delta_{kl} + \delta_{ki}\delta_{jl}) - \sum_x h^x_{jk}\bar{h}^x_{il}, \tag{2.6}$$

$$S_{i\bar{j}} = \frac{c}{2}(n+1)\delta_{ij} - h^2_{i\bar{j}}, \tag{2.7}$$

and

$$r = n(n+1)c - 2h_2, \tag{2.8}$$

where the tensor's components $S_{i\bar{j}}$ and $h^2_{i\bar{j}}$, the function h_2, and the scalar curvature r are defined by $S_{i\bar{j}} = \sum_k K_{\bar{j}ik\bar{k}}$, $h^2_{i\bar{j}} = \sum_{k,x} h^x_{ik}\bar{h}^x_{kj}$, $h_2 = \sum_i h^2_{i\bar{i}}$, and $r = 2\sum_i S_{i\bar{i}}$. Moreover, we know that

$$\begin{aligned} h^x_{ijk\bar{l}} = {} & \frac{c}{2}(h^x_{ij}\delta_{kl} + h^x_{jk}\delta_{il} + h^x_{ki}\delta_{jl}) \\ & - \sum_{r,y}(h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}. \end{aligned} \tag{2.9}$$

Then the Laplacian of the function h_2 can be given by

$$\begin{aligned}\Delta h_2 &= 2\sum_k (h_2)_{k\bar{k}} \\ &= (n+2)ch_2 - 4h_4 - 2\operatorname{Tr} A^2 + 2\sum_{i,j,k,x} h^x_{ijk}\bar{h}^x_{ijk},\end{aligned} \tag{2.10}$$

where $h_4 = \sum_{i,j} h^2_{i\bar{j}} h^2_{j\bar{i}}$, $\operatorname{Tr} A^2 = \sum_{x,y} A^x_y A^y_x$, and $A^x_y = \sum_{i,j} h^x_{ij}\bar{h}^y_{ij}$.

3. Proof of Main Theorem

Before giving a complete proof of our Main Theorem, we prove the following lemma.

LEMMA 3.1. *Let M be an n-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex projective space $\mathbb{C}P^{n+p}(1)$ of constant holomorphic sectional curvature 1. If the Ricci curvatures are greater than or equal to $n/2$, then the second fundamental tensor is parallel and the scalar curvature is constant.*

Proof. Now let us denote by S_j the Ricci curvature of M defined in such a way that

$$S_j = S_{j\bar{j}} = \frac{n+1}{2} - \lambda_j,$$

where we have used (2.7) and $h^2_{i\bar{j}} = \lambda_i \delta_{ij}$. By the assumption that $S_j \geq n/2$ we then know the eigenvalue

$$\lambda_j = h^2_{j\bar{j}} \leq \frac{1}{2}, \tag{3.1}$$

which means that

$$h_2 \leq \frac{n}{2}.$$

From (3.1) we assert that the matrix $\left\{\frac{1}{2}I - (h^2_{i\bar{j}})\right\}$ is a positive semidefinite Hermitian matrix. Moreover, we know that the matrix $(h^2_{i\bar{j}})$ is a positive semidefinite Hermitian matrix. Because such matrices can be transformed simultaneously by a unitary matrix into a diagonal matrix, it follows that the matrix

$$(h^2_{i\bar{j}})\left\{\tfrac{1}{2}I - (h^2_{i\bar{j}})\right\} = \left(\tfrac{1}{2}h^2_{i\bar{j}} - h^4_{i\bar{j}}\right)$$

can be transformed into another positive semidefinite Hermitian matrix. From this, contracting $i = j$ and summing up yields

$$\tfrac{1}{2}h_2 - h_4 \geq 0.$$

Then, by this formula and (2.10), the Laplacian of the function $h_2 = \sum_{i,j,x} h^x_{ij}\bar{h}^x_{ij}$ can be estimated in such a way that

$$\begin{aligned}\Delta h_2 = 2\sum_k (h_2)_{k\bar{k}} &= 2\|\nabla h\|^2 + (n+2)h_2 - 4h_4 - 2\operatorname{Tr} A^2 \\ &\geq (n+2)h_2 - 2h_2 - 2h_2^2 \\ &= h_2\{n - 2h_2\} \geq 0,\end{aligned} \tag{3.2}$$

where A denotes the matrix $A = (A^x_y)$, which is a positive definite Hermitian matrix given by $A^x_y = \sum_{j,k} h^x_{jk}\bar{h}^y_{jk}$ and where A's eigenvalues $A^x_y = \mu_x \delta^x_y$ are nonnegative functions on M. We have used the fact that $h_2^2 \geq \operatorname{Tr} A^2$ (see [2]) in the third inequality.

The Ricci curvature S_j for any j is bounded from below (by assumption) and so, by a theorem of Myers (see Kobayashi and Nomizu [3]), we assert that the complete Kähler manifold M is compact. Then we are able to apply Hopf's maximum principle to such a situation, so we know that the function h_2 is constant on M. Hence, by (3.2) either $h_2 = 0$ or $h_2 = n/2$. This means that $\Delta h_2 = 0$ and, given the second equality in (3.2), that the second fundamental form is parallel. Of course, we know that the scalar curvature is constant. □

By Lemma 4.1 and Theorem B, we assert that a complete Kähler submanifold M in $\mathbb{C}P^{n+p}(1)$ is an imbedded submanifold congruent to one of the seven types M_i, $i = 1, \dots, 7$, mentioned in Theorem B.

On the other hand, from the condition in our Main Theorem its scalar curvature is bounded from

$$r = 2\sum_j S_{j\bar{j}} \geq n^2, \tag{3.3}$$

where we have put $c = 1$. Then, among all seven types, only some of the three types M_1, M_3, and M_4 are candidates that could satisfy condition (3.3). As mentioned in Section 1, the Ricci curvatures of M_3 are given by $S_{A\bar{A}} = (n-s+1)/2$ or $(s+1)/2$. Then the condition $S_{A\bar{A}} \geq n/2$ implies $n = 2$ and $s = 1$. Hence $M_3 = \mathbb{C}P^1(1) \times \mathbb{C}P^1(1)$ and should be Einstein; the others, $M_1 = \mathbb{C}P^n(1)$ and $M_4 = Q^n$, are also Einstein. From this we complete the proof of our Main Theorem.

4. Another Proof of Main Theorem

In a communication with some differential geometers we have learned of some works by Zheng [9; 10; 11] that are concerned with Kähler–Einstein manifolds and their bisectional curvatures. Using Zheng's notation, it is not difficult to show the implication from the condition "the Ricci curvature is greater than or equal to n" mentioned in our Main Theorem to the holomorphic pinching that we shall describe. That is, the holomorphic sectional curvature H of M in $\mathbb{C}P^{n+p}(c)$ can be given by $c/2 \leq H \leq c$. Here, if we assume that $c = 2$ (resp. $c = 1$), then the holomorphic pinching of H is given by $1 \leq H \leq 2$ (resp. $\frac{1}{2} \leq H \leq 1$) as follows.

Let M be a complex n-dimensional compact Kähler manifold in a complex projective space $\mathbb{C}P^{n+p}(2)$ with constant holomorphic sectional curvature $c = 2$. Let TM be the holomorphic tangent bundle and let $h\colon TM \times TM \to TM^{\perp}$ be the second fundamental form of M in $\mathbb{C}P^{n+p}(2)$. At any point $p \in M$, let X and Y be two complex vectors in T_pM and let $\{e_1, \dots, e_n\}$ be any unitary basis of T_pM. Then, by the Gauss equation, the totally real bisectional curvature of M in $\mathbb{C}P^{n+p}(2)$ is given by

$$R_{\bar{X}XY\bar{Y}} = \|X\|^2\|Y\|^2 + g(X,Y)^2 - \|h(X,Y)\|^2,$$

where g denotes the Kähler metric on M induced from $\mathbb{C}P^{n+p}(2)$. Then the Ricci curvature of M in the direction of X is given by

$$\mathrm{Ric}(X) = n + 1 - \sum_{i=1}^{n} \|h(X, e_i)\|^2 \geq n,$$

where we have put $\|X\| = 1$. Thus the condition that "the Ricci curvature is greater than or equal to n" means

$$\sum_{i=1}^{n} \|h(X, e_i)\|^2 \leq 1.$$

In particular, if we pick $\{e_i\}$ so that $e_1 = X$ then $\|h(X, X)\|^2 = 1$. Therefore, the holomorphic sectional curvature in the direction of X is

$$H(X) = R_{\bar{X}XX\bar{X}} = 2 - \|h(X, X)\|^2 \in [1, 2].$$

Within this scenario we may recall the following theorem of Ros [8] with regard to the holomorphic pinching.

THEOREM C. *Let M be an n-dimensional compact Kähler submanifold of an $(n + p)$-dimensional complex projective space $\mathbb{C}P^{n+p}(1)$ with the holomorphic sectional curvature $H \geq \frac{1}{2}$. Then M is an imbedded submanifold congruent to the same type as in Theorem B.*

Using this theorem, (3.3), and the same method given in Section 3, we can show that $\mathbb{C}P^n(1)$, $\mathbb{C}P^1(1) \times \mathbb{C}P^1(1)$, and Q^n are the only Kähler–Einstein submanifolds satisfying the conditions of our Main Theorem.

ACKNOWLEDGMENT. The authors would like to express their sincere gratitude to the referee for a careful reading of our manuscript and for valuable suggestions that improved the exposition.

References

[1] A. L. Besse, *Einstein manifolds,* Springer-Verlag, Berlin, 1986.

[2] U.-H. Ki and Y. J. Suh, *On semi-Kaehler manifolds whose totally real bisectional curvature is bounded from below,* J. Korean Math. Soc. 33 (1996), 1009–1038.

[3] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, I and II,* Wiley, New York, 1963 and 1969.

[4] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space,* J. Math. Soc. Japan 28 (1976), 638–667.

[5] K. Ogiue, *Positively curved complex submanifolds immersed in a complex projective space,* J. Differential Geom. 7 (1972), 603–606.

[6] ———, *Positively curved complex submanifolds immersed in a complex projective space II,* Hokkaido Math. J. 1 (1972), 16–20.

[7] ———, *Differential geometry of Kaehler submanifolds,* Adv. Math. 13 (1974), 73–114.

[8] A. Ros, *A characterization of seven compact Kaehler submanifolds by holomorphic pinching,* Ann. of Math. (2) 121 (1985), 377–382.
[9] F. Zheng, *Non-positively curved Kähler metrics on product manifolds,* Ann. of Math. (2) 137 (1993), 671–673.
[10] ———, *Kähler–Einstein surfaces with nonpositive bisectional curvature,* Proc. Amer. Math. Soc. 121 (1994), 1217–1220.
[11] ———, *Complex differential geometry,* AMS/IP Stud. Adv. Math., 18 (S.-T. Yau, ed.), Amer. Math. Soc., Providence, RI, 2000.

Department of Mathematics
Kyungpook National University
Taegu, 702-701
Korea

yjsuh@mail.knu.ac.kr