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Self-Maps of Projective Bundles on Projective Spaces

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Introduction

As shown in [3; 7; 9], the basic dynamical properties of the self-maps of projective spaces can be summarized as follows. Any holomorphic self-map $\mathbb{P}^r \xrightarrow{f} \mathbb{P}^r$ lifts through the canonical map $\mathbb{C}^{r+1} \setminus 0 \xrightarrow{q} \mathbb{P}^r$ to a self-map $\mathbb{C}^{r+1} \xrightarrow{F} \mathbb{C}^{r+1}$, $qF = fq$, whose components are homogeneous polynomials of degree d, the algebraic degree of f. When $d > 1$, the origin is a super-attracting fixed point for F with bounded and complete circular basin of attraction. The Green function associated to this basin is given by the formula $G = \lim_j \log\|F^j\|/d^j$, so it is plurisubharmonic. If $\mathcal{H}$ denotes the open set where G is pluriharmonic, then the Fatou set $\mathcal{F}$ of f equals $q(\mathcal{H})$. It follows that $\mathcal{F}$ is Stein and Kobayashi hyperbolic and that, when $r \geq 2$, the complement $\mathcal{J}$ of $\mathcal{F}$ is connected.

In this paper we extend these results to the context of projective bundles on projective manifolds. In the first part, we discuss the structure of the self-maps of a projective bundle $\mathbb{P}E \xrightarrow{p} B$ (fiber-degree, algebraic degree, completely invariant sub-bundles, dimension of the space of self-maps, lifting to E'). In the second part, we introduce Green functions and use them in the analysis of the basic dynamical features of a self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ (pseudoconvexity and hyperbolicity of the Fatou set, connectedness of the Julia set). We prove the following theorem.

Theorem 1. *Let $\mathbb{P}E \to \mathbb{P}^n$ be a projective bundle with nonzero discriminant, and let $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ be a self-map with topological degree at least* 2. *Then*:

(1) *f has well-defined algebraic degree*;
(2) *the Fatou components of f are Stein and Kobayashi hyperbolic*;
(3) *when* $\operatorname{rank}(E) \geq 3$, *the Julia set of f is connected.*

Note that the first two statements fail in the trivial case $\mathbb{P}E = \mathbb{P}^n \times \mathbb{P}^n$.

Preliminaries. Fix a vector bundle $E \xrightarrow{\pi'} B$, $\operatorname{rank}(E) = r + 1$, over a projective manifold B, $\dim(B) = n$, and let $E' \xrightarrow{\pi} B$ denote its dual. The homogeneous lines in E' form the projective manifold $\mathbb{P}E$, which is endowed with the projection $\mathbb{P}E \xrightarrow{p} B$. The pull-back p^*E' admits a canonical line sub-bundle,

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$0 \to \mathcal{T} \to p^*E'$, whose fiber over a line $l \in \mathbb{P}E$ is $l \subset E'_{\pi(l)}$. Dualizing this yields the line bundle $\mathcal{O}_{\mathbb{P}E}(1) := \mathcal{T}'$ as a quotient of p^*E. The Chow ring $\mathcal{A}(\mathbb{P}E)$ is a free $\mathcal{A}(B)$-module generated by H^j, $0 \le j \le r$, where H is the class of $\mathcal{O}_{\mathbb{P}E}(1)$. We have $\sum_0^{r+1}(-1)^j C_j H^{r+1-j} = 0$, where $C_j \in \mathcal{A}(B)$ are the Chern classes of E. Put $C(T) := \sum_0^{r+1}(-1)^j C_j T^{r+1-j} \in \mathcal{A}(B)[T]$. The Picard group of $\mathbb{P}E$ is $\operatorname{Pic}(\mathbb{P}E) = p^* \operatorname{Pic}(B) \oplus \mathbb{Z}H$. Let $\simeq$ denote the equality in Pic. The canonical class of $\mathbb{P}E$ is $K_{\mathbb{P}E} \simeq -(r+1)H + p^* \det(E) + p^*K_B$. When $n \ge 2$, let $\Delta := C_1^2 - \frac{2(r+1)}{r} C_2 \in \mathcal{A}^2(B) \otimes \mathbb{Q}$; it does not change when E is tensored by a line bundle on B. We call Δ the *discriminant* of $\mathbb{P}E$.

When $B = \mathbb{P}^n$, let $L := p^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then $p^*C_j = c_j L^j$ for some $c_j \in \mathbb{Z}$, with $c_0 = 1$ and $c_j = 0$ for $j > n$. Let $C(T,S) := \sum_0^{r+1}(-1)^j c_j T^{r+1-j} S^j \in \mathbb{Z}[T,S]$. Then $\operatorname{Pic}(\mathbb{P}E) = \mathbb{Z}L \oplus \mathbb{Z}H$ and $\mathcal{A}(\mathbb{P}E) = \mathbb{Z}[L,H]/L^{n+1}$, where $L^{n+1} = C(H,L) = 0$. If $n \ge 2$ then $\delta := c_1^2 - \frac{2(r+1)}{r} c_2 \in \mathbb{Q}$.

A vector bundle E is totally decomposable if and only if it is a direct sum of line bundles. Every vector bundle on $\mathbb{P}^1$ is totally decomposable. Given $m \in \mathbb{Z}^{r+1}$ with $m_0 \ge \cdots \ge m_r$, put $E_m := \bigoplus_0^r \mathcal{O}_{\mathbb{P}^n}(m_j)$ and $\delta_m := \frac{1}{r} \sum_{j<k}(m_j - m_k)^2$. When $n \ge 2$ we have $\delta_{E_m} = \delta_m$, so δ_E is coherently defined for every vector bundle E on $\mathbb{P}^n$.

Given a map $X \xrightarrow{r} B$ and a line bundle $\chi \in \operatorname{Pic}(X)$, the set of maps $X \xrightarrow{f} \mathbb{P}E$ that satisfy $pf = r$ and $f^*\mathcal{O}_{\mathbb{P}E}(1) \simeq \chi$ can be identified with the projectivized set of nonvanishing global sections in $r^*E' \otimes \chi$.

Let ω denote the zero-section in E', and let $E' \setminus \omega \xrightarrow{q} \mathbb{P}E$ be the map that associates to $e' \ne 0$ the line passing through e'. Note that $pq = \pi$ and that $q^*\mathcal{O}_{\mathbb{P}E}(1)$ is trivial.

1. Self-Maps

1.1. Fiber-Degree and Algebraic Degree

DEFINITION 1.1. A finite self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ is *over B* if and only if there exists $B \xrightarrow{g} B$ with $pf = gp$. In this case, we say that f is over g.

THEOREM 1.2. *Let E be a vector bundle on $\mathbb{P}^n$, and let $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ be a finite self-map. Then some iterate of f is over $\mathbb{P}^n$.*

Proof. We have to show that some iterate f^i maps fibers of $\mathbb{P}E$ to fibers. Write $f^*H \simeq aH + bL$ and $f^*L \simeq cH + dL$. Note that, by the projection formula, we can take $i = 1$ when $c = 0$. Let σ denote the topological degree of f. There are three cases to discuss: $r > n$, $r = n$, and $r < n$.

When $r > n$, Bézout's theorem implies that the restriction of pf to any fiber of $\mathbb{P}E$ is constant. In this case, we can take $i = 1$. Consider the case $r = n$.

When $r = n = 1$, we may assume $E = E_{(0,\varepsilon)}$ with $\varepsilon \le 0$. If $\varepsilon = 0$, then f is a finite self-map of $\mathbb{P}^1 \times \mathbb{P}^1$ and we can take $i = 2$. If $\varepsilon < 0$, then $\mathbb{P}E$ contains a unique negative curve and it follows that $f^*H = aH = f_*H$, where $a > 0$, $\sigma = a^2$, and (with the projection formula) $c\varepsilon = a - d$. We show that $c = 0$, so that we can take $i = 1$ in this case. Otherwise, since $\deg(L^2) = 0$, we get $c\varepsilon = -2d$ and hence $d = -a < 0$. If s is a section in $\mathbb{P}E$ with $\deg(H \cdot s) = 0$, then $0 \leq \deg(f^*L \cdot s) = d$. This is a contradiction.

When $r = n > 1$ and $c \neq 0$ we have $(cT + dS)^{n+1} = c^{n+1}C(T,S) + d^{n+1}S^{n+1}$, so that $C(T,S) = (T - mS)^{n+1} - (-1)^{m+1}m^{n+1}S^{n+1}$ in $\mathbb{Q}[T,S]$, where $m = -d/c$. Since $(n+1)m^n = c_n$, we get $m \in \mathbb{Z}$. Replacing E by $E \otimes \mathcal{O}_{\mathbb{P}^n}(-m)$, we may assume that $m = 0$ and hence $C(T,S) = T^{n+1}$. But then $(aT + bS)^{n+1} = a^{n+1}T^{n+1} + b^{n+1}S^{n+1}$ in $\mathbb{Q}[T,S]$; that is, $ab = 0$. Since f is finite, it follows that $b \neq 0$ and hence $a = 0$. Therefore, we can take $i = 2$ in this case.

When $1 \leq r < n$ and $c \neq 0$, we have $C(aT + bS, cT + dS) = a^{r+1}C(T,S)$ and $(cT + dS)^{n+1} - d_0 S^{n+1} = c_{n-r}(T,S)C(T,S)$, with $c_{n-r} \in \mathbb{Z}[T,S]$ and $d_0 \in \mathbb{Z}$. Let $m_j \in \mathbb{C}$ be the roots of $C(T,1)$. We have $(cm_j + d)^{n+1} = d_0$ for all $0 \leq j \leq r$. Not all m_j are equal. Indeed, otherwise $m_j = m \in \mathbb{Z}$ for all $0 \leq j \leq r$ and we could assume that $C(T,S) = T^{r+1}$. It follows immediately that $b = 0$ and $d \neq 0$, but then $c = 0$. Consequently, $d_0 \neq 0$, the polynomial $(cT + d)^{n+1} - d_0$ has no multiple roots, and all m_j are distinct. The Möbius transformation $\mu(z) = \frac{az+b}{cz+d}$ leaves invariant the set $\{m_j, 0 \leq j \leq r\}$, so $\mu^{(r+1)!}$ fixes all of the points m_j. Put

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \bar{M} = M^{(r+1)!} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

Then $\binom{m_j}{1}$ is an eigenvector of $\bar{M}$ with eigenvalue $\bar{c}m_j + \bar{d}$. Since $m_0 \neq m_1$ and $(\bar{c}m_j + \bar{d})^{n+1} = \bar{d}_0$, we have $B^{n+1} = d_0 I$ and can take $i = (r+1)!\,(n+1)$. □

REMARK 1.3. Let E be a vector bundle of rank $r+1$ on a projective manifold B of dimension n, $B \neq \mathbb{P}^n$. If $r \geq n$, then any finite self-map of $\mathbb{P}E$ is over B.

Proof. Fix a fiber F of $\mathbb{P}E$ and consider the restriction $F \xrightarrow{\phi} B$ of pf. Let $G = \phi(F)$ and $\gamma = \dim(G)$. Since B is smooth, Lazarsfeld's result implies $\gamma < r$. Choose a finite map $G \xrightarrow{\nu} \mathbb{P}^\gamma$. The composition $F \xrightarrow{\nu\phi} \mathbb{P}^\gamma$ is surjective, hence $\gamma = 0$. □

REMARK 1.4. In the totally split case of $E = E_m$ with $\delta_m \neq 0 = m_0$, any finite self-map of $\mathbb{P}E_m$ is over $\mathbb{P}^n$.

Proof. We keep the notation from the proof of Theorem 1.2. Lemma 1.8 (to follow) implies that a, b, c, and d are nonnegative. We need to show that $c = 0$. This is clear when $r \geq n$. In the case $1 \leq r < n$, assume $c \neq 0$. Then all m_j are distinct and $(cm_j + d)^{n+1} = d^{n+1}$ for all j. It follows that n is odd, $r = 1$, $d > 0$, and $m_1 = -2d/c$. Moreover, $\{\mu(0), \mu(m_1)\} = \{0, m_1\}$. Since $\mu(0) = b/d \geq 0$, we obtain $\mu(0) = 0 = b$ and $\mu(m_1) = m_1$. But $\mu(m_1) = 2a \geq 0$, which is a contradiction. □

DEFINITION 1.5. A finite self-map $X \xrightarrow{f} X$ has well-defined *algebraic degree d* if and only if $\mathrm{Pic}(X) \xrightarrow{f^*} \mathrm{Pic}(X)$ is given by multiplication with d.

Clearly, self-maps of $\mathbb{P}E$ with well-defined algebraic degree must be over B.

DEFINITION 1.6. Assume that $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ is over $B \xrightarrow{g} B$. If the induced maps of fibers $F_b \xrightarrow{f} F_{g(b)}$ have algebraic degree d independent of $b \in B$, we say that f has *fiber-degree d*.

PROPOSITION 1.7. *Any finite self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ over $B \xrightarrow{g} B$ has well-defined fiber-degree $d(f)$. If $\beta(E, f) \in \mathrm{Pic}(B)$ is the line bundle that satisfies $f^*H \simeq d(f)H - p^*\beta(E, f)$, then*:

(1) $\beta(E \otimes \alpha, f) \simeq \beta(E, f) - g^*\alpha + d(f)\alpha$ *for every line bundle* $\alpha \in \mathrm{Pic}(B)$;
(2) $\beta(E, \bar{f}f) \simeq d(\bar{f})\beta(E, f) + g^*\beta(E, \bar{f})$ *for every self-map* $\mathbb{P}E \xrightarrow{\bar{f}} \mathbb{P}E$ *over* B;
(3) $\beta(E, f) = \frac{dC_1(E) - g^*C_1(E)}{r+1}$ *in* $\mathcal{A}(B) \otimes \mathbb{Q}$;
(4) *when* $n \geq 2$, $g^*\Delta_E = d^2\Delta_E$ *in* $\mathcal{A}(B) \otimes \mathbb{Q}$.

Proof. Write $f^*H \simeq dH - p^*\beta$ with $d \in \mathbb{Z}$ and $\beta \in \mathrm{Pic}(B)$. Then $f^*H \cdot F_b = dH \cdot F_b$ for all $b \in B$. Let $d(b) > 0$ be the algebraic degree of $F_b \xrightarrow{f} F_{g(b)}$. Then $f_*F_b = d(b)^r F_{g(b)}$ and $f_*H_b = d(b)^{r-1}H_{g(b)}$, where $H_b := \mathcal{O}_{F_b}(1)$. Since $H|_{F_b} \simeq H_b$ in $\mathrm{Pic}(F_b)$, the projection formula implies

$$\begin{aligned} dd(b)^{r-1}H_{g(b)} &= df_*(H_b) = df_*(H \cdot F_b) = f_*(f^*H \cdot F_b) \\ &= H \cdot f_*(F_b) = d(b)^r H \cdot F_{g(b)} = d(b)^r H_{g(b)} \end{aligned}$$

and so $(d - d(b))d(b)^{r-1}H_{g(b)} = 0$ in $\mathcal{A}(F_{g(b)})$. This implies that $d(b) = d$ for all $b \in B$.

Parts (1) and (2) of the proposition are straightforward calculations. Let $P(T) := \sum_0^{r+1}(-1)^j g^*C_j(dT - \beta)^{r+1-j} - d^{r+1}C(T) \in \mathcal{A}(B)[T]$. Since $\deg(P) \leq r$, it follows that $P(T) = 0$. Looking at the coefficient of T^r, we get part (3). Looking at the coefficient of T^{r-1}, we get $g^*C_2/d^2 + r\beta \cdot g^*C_1/d + r(r+1)\beta^2/2 = C_2$, and part (4) follows from (3). □

The canonical projection $E_m \to \bigoplus_{k \neq j} \mathcal{O}_{\mathbb{P}^n}(m_k) \to 0$ defines a hypersurface H_j of $\mathbb{P}E_m$, with $H_j + m_jL \simeq H$, for all $0 \leq j \leq r$.

LEMMA 1.8. *If a and b are integers such that $h^0(\mathbb{P}E_m, aH - bL) > 0$, then $a \geq 0$ and $b \leq m_0 a$.*

Proof. Let D be an effective divisor on $\mathbb{P}E_m$, $D \simeq aH - bL$. We may assume that H_0 is not an irreducible component of D. Indeed, if H_0 appears in D with multiplicity $c > 0$ and if $D' = D - cH_0 \simeq a'H - b'L$, then $a = a' + c$ and $b = b' + m_0c$. If the statement is true for D', it follows immediately that it is also true for D.

We proceed by induction over r. If $r = 1$ then $a = \deg(D \cdot L^n)$ and $m_1a - b = \deg(D \cdot H_0 \cdot L^{n-1})$; hence, since L^n and L^{n-1} are nef, we deduce that $a \geq 0$ and $m_0a - b \geq m_1a - b \geq 0$. If $r \geq 2$ then the induction hypothesis applied to $D|_{H_0}$ implies that $a \geq 0$ and $b \leq m_1a \leq m_0a$, and the induction is complete. □

THEOREM 1.9. *Let E be a vector bundle on $\mathbb{P}^n$, and let $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ be a finite self-map over $\mathbb{P}^n$. If $\delta_E \neq 0$, then f has well-defined algebraic degree.*

Proof. Let $\mathbb{P}^n \xrightarrow{g} \mathbb{P}^n$ be the self-map induced by f, and let $d > 0$ be the fiber-degree of f. In $\mathrm{Pic}(\mathbb{P}E)$ we have $f^*L \simeq \gamma L$ and $f^*H \simeq dH - bL$, where $\gamma > 0$ is the algebraic degree of g and $b \in \mathbb{Z}$ is the degree of $\beta(E, f) \in \mathrm{Pic}(\mathbb{P}^n)$. By Proposition 1.7, $b = (d-\gamma)c_1/(r+1)$. Also note that $f_*H = \gamma^{n-1}d^{r-1}(\gamma H + bL)$.

Assume first that $n \geq 2$. By Proposition 1.7, $(\gamma^2 - d^2)\delta = 0$. Since $\delta \neq 0$, we get $\gamma = d$ and then $b = 0$. In the remaining case ($n = 1$), $E = E_m$ for some $m \in \mathbb{Z}^{r+1}$ and we may assume that $m_0 = 0$, so that $H \simeq H_0$ is effective. Since $\delta \neq 0$, we also get $c_1 < 0$. Lemma 1.8 applied to f^*H and f_*H implies that $b \leq 0$ and $b \geq 0$; that is, $b = 0$ and so $d = \gamma$. □

1.2. *Lifting to E'*

A *section* of p is a map $B \xrightarrow{s} \mathbb{P}E$ with $ps = 1_B$. Its image $S = s(B)$ is a section of $\mathbb{P}E$.

DEFINITION 1.10. A section $S \subset \mathbb{P}E$ with $\mathcal{O}_{\mathbb{P}E}(1)|_S \simeq 0$ is an *affine* section.

The set of affine sections in $\mathbb{P}E$ is the projectivized set of nonvanishing global sections in E'. We need the following well-known Bertini-type result.

LEMMA 1.11. *If* $\mathrm{rank}(E) > \dim(B)$ *and if E' is globally generated, then the generic section of $\mathbb{P}E$ is affine and is not contained in any proper subvariety of $\mathbb{P}E$.*

Proof. We need to show first that the generic global section of E' is nonvanishing. Let $V := \Gamma(B, E')$ and let $K := \{(v, b) \in V \times B : v(b) = 0\}$, a subvariety of $V \times B$. For $b \in B$, the evaluation map $V \xrightarrow{v_b} \mathbb{C}^{r+1}$ is surjective and so its kernel V_b has dimension $\dim(V) - r - 1$. Since $r \geq n$, it follows that $\dim(K) < \dim(V)$. The projection $V \times B \xrightarrow{P} V$ is proper, so $P(K)$ is a proper subvariety of V.

Let Z be a proper subvariety of $\mathbb{P}E$. We may assume that $p(Z) = B$, since otherwise Z does not contain any section in $\mathbb{P}E$. Given $b \in B$, the set $Z_b := \{0\} \cup q^{-1}(Z \cap F_b)$ is analytic in $\mathbb{C}^{r+1} = \pi^{-1}(b)$. Since $\lambda Z_b \subset Z_b$ for all $\lambda \in \mathbb{C}$, it follows that Z_b is algebraic in $\mathbb{C}^{r+1}$ and hence $U := \bigcap_{b \in B}\{s \in V : s(b) \in Z_b\}$ is algebraic in V. Moreover, $U \neq V$. Indeed, pick $b \in B$ such that $Z_b \neq \mathbb{C}^{r+1}$, and choose $t \in \mathbb{C}^{r+1} \setminus Z_b$. Because E' is globally generated, there exists a $v \in V$ with $v(b) = t$ and hence $v \notin U$. It follows that the generic section of $\mathbb{P}E$ is not contained in Z. □

DEFINITION 1.12. The self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ *lifts to E'* if and only if there exists a self-map $E' \setminus \omega \xrightarrow{F} E' \setminus \omega$ such that $qF = fq$.

LEMMA 1.13. *Let $U \xrightarrow{g} \bar{U}$ be holomorphic from the unit ball $U \subset \mathbb{C}^n$ to some analytic space $\bar{U}$. If $U \times \mathbb{P}^r \xrightarrow{f} \bar{U} \times \mathbb{P}^r$ is a holomorphic map over g with positive fiber-degree, then f admits a holomorphic lifting $U \times (\mathbb{C}^{r+1} \setminus 0) \xrightarrow{F} \bar{U} \times (\mathbb{C}^{r+1} \setminus 0)$.*

Proof. Let $\omega := U \times 0$ and $\bar{\omega} := \bar{U} \times 0$, and denote by

$$U \times \mathbb{P}^r \xrightarrow{p} U,$$

$$U \times \mathbb{C}^{r+1} \setminus \omega \xrightarrow{q} U \times \mathbb{P}^r,$$

$$U \times \mathbb{C}^{r+1} \xrightarrow{\pi} U$$

the canonical projections, and similarly for $\bar{p}, \bar{q}, \bar{\pi}$. Let t_j $(0 \le j \le r)$ denote the coordinates in $\mathbb{C}^{r+1}$. For $1 \le j \le r$, the functions t_j/t_0 are meromorphic on $\bar{U} \times \mathbb{P}^r$ and holomorphic outside the hypersurface $T_0 = \{t_0 = 0\} \subset \bar{U} \times \mathbb{P}^r$. Because f has fiber-degree $d > 0$, its image is not contained in T_0 and so the compositions $\phi_j := (t_j/t_0) f q$ are meromorphic on $U \times \mathbb{C}^{r+1} \setminus \omega$ and holomorphic outside the hypersurface $q^{-1}(f^{-1}(T_0))$. Levi's extension theorem implies that ϕ_j is meromorphic on $U \times \mathbb{C}^{r+1}$ for all $1 \le j \le r$. Note that $\phi_j(u, \lambda t) = \phi_j(u, t)$ for all $(u, t) \in U \times \mathbb{C}^{r+1}$ and all $\lambda \in \mathbb{C}^*$. The extension theorem of Thullen, Remmert, and Stein implies that the point set closure H_0 of $q^{-1}(f^{-1}(T_0)) \subset U \times \mathbb{C}^{r+1}$ is an analytic hypersurface in $U \times \mathbb{C}^{r+1}$. Since $\lambda H_0 \subset H_0$ for all $\lambda \in \mathbb{C}^*$, we deduce that $\omega \subset H_0$. Note that H_0 contains no fibers of π, since $d > 0$.

Fix $b \in U$ and let $h_0 = 0$ be a local defining equation of H_0 near $(b, 0)$. For a large enough integer m, the functions $\psi_j = h_0^m \phi_j$ are holomorphic near $(b, 0)$ for all $1 \le j \le r$. Using the Taylor expansion near $(b, 0)$, write $h_0^m = \sum_{k=0}^{\infty} \psi_{0,k}$ with $\psi_{0,k}$ holomorphic in u and homogeneous of degree k in t. Similarly, $\psi_j = \sum_{k=0}^{\infty} \psi_{j,k}$. Because the ϕ_j are homogeneous in t, we deduce that $\phi_j \psi_{0,k} = \psi_{j,k}$ for all $1 \le j \le r$ and all $k \ge 0$. Since $\pi^{-1}(b) \not\subset H_0$, we can find $l > 0$ such that $\psi_{0,l}(b, \cdot) \not\equiv 0$. Let $\gamma = \gcd(\psi_{0,l}, \dots, \psi_{r,l})$ in $\mathcal{O}_b[t]$, which is a UFD. Clearly, $\gamma(u, t)$ is a homogeneous polynomial in t. Define $P_j := \psi_{j,l}/\gamma$ for $0 \le j \le r$, and let e be their common algebraic degree in t. Then $\phi_j = P_j/P_0$ as meromorphic functions on $V \times \mathbb{C}^{r+1}$, where $b \in V \subset U$ is a sufficiently small ball. Shrinking V, we may assume that $P_0(u, \cdot) \not\equiv 0$ for all $u \in V$. If $Z \subset V \times \mathbb{C}^{r+1}$ denotes the set of common zeros of the polynomials $P_0, \dots, P_r$, then the map

$$V \times \mathbb{C}^{r+1} \setminus Z \xrightarrow{F_V} \bar{U} \times \mathbb{C}^{r+1} \setminus \bar{\omega}, \quad F_V = (P_0, \dots, P_r),$$

is a lifting of f. Note that Z contains no fiber of π, and $\lambda Z \subset Z$ for all $\lambda \in \mathbb{C}^*$; hence $\omega \cap \pi^{-1}(V) \subset Z$.

We show that $d = e$. Otherwise $d < e$ and, for all $u \in V$, the polynomials $P_0(u, t), \dots, P_r(b, t)$ have nontrivial common factors. Since p is proper, there exists an irreducible component C of $q(Z)$ with $p(C) = V$. For $u \in V$, $C \cap p^{-1}(u)$ is a nonempty sum of irreducible components of $q(Z) \cap p^{-1}(b)$. Since $q(Z) \cap p^{-1}(b)$ has pure dimension $r - 1$, it follows that C is a hypersurface in $V \times \mathbb{P}^r$ and hence $q^{-1}(C)$ is a hypersurface in $V \times \mathbb{C}^{n+1} \setminus \omega$. As before, the point set closure of $q^{-1}(C)$ is a hypersurface Y in $V \times \mathbb{C}^{n+1}$, and $\omega \cap \pi^{-1}(V) \subset Y \subset Z$. If $y = 0$ is a local equation of $Y \subset V \times \mathbb{C}^{r+1}$ near $(b, 0)$, then there exist holomorphic germs $Q_0, \dots, Q_r \in \mathcal{O}_{(b,0)}$ with $P_j = y Q_j$ for all $0 \le j \le r$. The order o of y in t is positive because $\pi^{-1}(b) \not\subset Y$. Therefore, the homogeneous term y_o of y is a nontrivial common divisor of $P_0, \dots, P_r$. This contradiction shows that $d = e$. Since the map $p^{-1}(u) \xrightarrow{f} \bar{p}^{-1}(g(u))$ is regular for all $u \in V$, we obtain $Z = \omega \cap \pi^{-1}(V)$.

Therefore, for all $b \in U$ there exist a ball $b \in V \subset U$ and a lifting of f,

$$V \times \mathbb{C}^{r+1} \setminus \omega \xrightarrow{F_V} \bar{U} \times \mathbb{C}^{r+1} \setminus \bar{\omega}.$$

Given two such liftings F_V and F_W, we have $c_{VW} := F_V/F_W \in \mathcal{O}^*(V \cap W)$. Therefore, the system (V, c_{VW}) defines a 1-cocycle of the sheaf $\mathcal{O}_U^*$. Since $H^1(U, \mathcal{O}_U^*) = 0$, we obtain $c_{VW} = c_V/c_W$ with $c_V \in \mathcal{O}^*(V)$ for all V. Gluing together the liftings F_V/c_V, we get the lifting F. □

PROPOSITION 1.14. *Any finite self-map* $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ *over* $B \xrightarrow{g} B$ *determines a line bundle* $D(E, f) \in \mathrm{Pic}(B)$, *so that* $D(E, f) \simeq 0$ *if and only if* f *lifts to* E'. *Moreover*:

(1) $D(\bar{E}, f|_{\mathbb{P}\bar{E}}) \simeq D(E, f)$ *when* $\mathbb{P}\bar{E} \subset \mathbb{P}E$ *is an* f*-invariant sub-bundle*;
(2) $D(E, f) \simeq dL_2 - g^*L_1$ *when the* $\mathbb{P}L_i$ *are sections in* $\mathbb{P}E$ *with* $f(\mathbb{P}L_1) \subset \mathbb{P}L_2$;
(3) $D(E \otimes \alpha, f) \simeq D(E, f) - g^*\alpha + d(f)\alpha$ *for all* $\alpha \in \mathrm{Pic}(B)$;
(4) $D(E, \bar{f}f) \simeq d(\bar{f})D(E, f) + g^*D(E, \bar{f})$ *for all* $\mathbb{P}E \xrightarrow{\bar{f}} \mathbb{P}E$ *over* B;
(5) *if* $r \geq n$ *then* $(r+1)D(E, f) \simeq (r+1)\beta(E, f)$.

Proof. Since g is finite, it is also open. Take an $\mathcal{O}^*$-acyclic covering $\mathcal{U} = \{U_k\}$ of B such that $E|_{U_k}$ and $E|_{V_k}$ are trivial for all k, where $\mathcal{V} := g(\mathcal{U})$. Let $(\mathcal{U}, e_{kl})$ and $(\mathcal{V}, \phi_{kl})$ be cocycles defining E'. By Lemma 1.13, there exist local liftings to E', $(E' \setminus \omega)|_{U_k} \xrightarrow{F_k} (E' \setminus \omega)|_{V_k}$. Write $F_k(b, t) = (g(b), P_k(b, t))$, where $U_k \times \mathbb{C}^{r+1} \xrightarrow{P_k} \mathbb{C}^{r+1}$ is a polynomial of degree $d = d(f)$ in t. Over $U_k \cap U_l$, both F_k and F_l are liftings of f; hence there exists $c_{kl} \in \mathcal{O}^*(U_k \cap U_l)$ with $\phi_{kl}(g(b))P_l(b, t) = c_{kl}(b)P_k(b, e_{kl}(b)t)$. We can write, simply, $\phi_{kl}(g)P_l = c_{kl}P_k(e_{kl})$. The cocycle $(\mathcal{U}, \phi_{kl}(g))$ defines g^*E'. Clearly, $(\mathcal{U}, c_{kl})$ is a cocycle whose class D is independent of the choice of local liftings F_k. Therefore, if f lifts to E' then $D \simeq 0$. Conversely, if $D \simeq 0$ then $c_{kl} = c_k/c_l$, with $c_k \in \mathcal{O}^*(U_k)$. Gluing together the local liftings $\bar{F}_k(b, t) = (g(b), P_k(b, t)/c_k(b))$, we obtain a global lifting of f to E'.

Let $0 \to \hat{E} \to E \to \bar{E} \to 0$ define an f-invariant sub-bundle $\mathbb{P}\bar{E}$. We may assume that $\bar{E}'$ and $\hat{E}'$ are defined over $\mathcal{U}, \mathcal{V}$ by cocycles $\bar{e}_{kl}, \bar{\phi}_{kl}$ and $\hat{e}_{kl}, \hat{\phi}_{kl}$, respectively. Then E' is defined over $\mathcal{U}$ and $\mathcal{V}$ by

$$e_{kl} = \begin{pmatrix} \hat{e}_{kl} & 0 \\ u_{kl} & \bar{e}_{kl} \end{pmatrix} \quad \text{and} \quad \phi_{kl} = \begin{pmatrix} \hat{\phi}_{kl} & 0 \\ v_{kl} & \bar{\phi}_{kl} \end{pmatrix}.$$

Moreover,

$$P_k = \begin{pmatrix} \bar{P}_k \\ \hat{P}_k \end{pmatrix} \quad \text{with} \quad \bar{P}_k|_{\bar{E}'} = 0,$$

while $\mathbb{P}\bar{E} \xrightarrow{f} \mathbb{P}\bar{E}$ is given over U_k by $(g, \hat{P}_k|_{\bar{E}'})$. Restricting to $\bar{E}'$ the relation $\phi_{kl}(g)P_l = c_{kl}P_k(e_{kl})$, we get part (1) of the proposition.

Take quotients $E \to L_i \to 0$ that define sections with $f(\mathbb{P}L_1) \subset \mathbb{P}L_2$. We may assume that L_1' is defined by $(\mathcal{U}, p_{kl})$ and L_2' by $(\mathcal{V}, \lambda_{kl})$. Reasoning as in (1), we get $\lambda_{kl}(g(b))\hat{P}_l(b, \bar{0}, \hat{t}) = c_{kl}(b)\hat{P}_k(b, \bar{0}, p_{kl}(b)\hat{t}))$ for all k, l and all $b \in U_k \cap U_l$,

$\hat{t} \in \mathbb{C}$. But $\hat{P}_k(b, \bar{0}, \hat{t}) = p_k(b)\hat{t}^d$ with $p_k \in \mathcal{O}^*(U_k)$. We then have $\lambda_{kl}(g)p_l = c_{kl}p_k p_{kl}^d$, so $g^*L_2' \simeq D + dL_1'$ and thus part (2) is proved.

Let $\tilde{E} = E \otimes \alpha$. We may assume that α is defined by $(\mathcal{U}, a_{kl})$ and $(\mathcal{V}, b_{kl})$; then $\tilde{E}'$ is defined by $(\mathcal{U}, e_{kl}/a_{kl})$ and $(\mathcal{V}, \phi_{kl}/b_{kl})$. If $\tilde{c}_{kl}$ is the cocycle that defines $D(\tilde{E}, f)$, then $(\phi_{kl}(g)/b_{kl}(g))P_l = \tilde{c}_{kl}P_k(e_{kl}/a_{kl})$. We get $b_{kl}(g)\tilde{c}_{kl}/a_{kl}^d = c_{kl}$, and (3) follows.

To prove part (4), let $B \xrightarrow{\bar{g}} B$ be the map induced by $\bar{f}$. We may assume that E' is defined by $(\bar{g}(\mathcal{V}), \gamma_{kl})$ and that $\bar{f}$ admits local liftings $\bar{F}_k$ over each V_k, $\bar{F}_k(b,t) = (\bar{g}(b), \bar{P}_k(b,t))$ for $b \in V_k$ and $t \in \mathbb{C}^{r+1}$. We have $\gamma_{kl}(\bar{g})\bar{P}_l = \bar{c}_{kl}\bar{P}_k(\phi_{kl})$, where $(\mathcal{V}, \bar{c}_{kl})$ defines $D(E, \bar{f})$. Then $D(E, \bar{f}f)$ is defined by a cocycle x_{kl} satisfying $\gamma_{kl}(\bar{g}g)\bar{P}_l(g, P_l) = x_{kl}\bar{P}_k(g, P_k(e_{kl}))$. But

$$\begin{aligned}\gamma_{kl}(\bar{g}g)\bar{P}_l(g, P_l) &= \bar{c}_{kl}(g)\bar{P}_k(g, \phi_{kl}(g)P_l) = \bar{c}_{kl}(g)\bar{P}_k(g, c_{kl}P_k(e_{kl}))\\ &= \bar{c}_{kl}(g)c_{kl}^{d(\bar{f})}\bar{P}_k(g, P_k(e_{kl})),\end{aligned}$$

hence $x_{kl} = \bar{c}_{kl}(g)c_{kl}^{d(\bar{f})}$.

Finally, we prove part (5). Since $D(E, f)$ and $\beta(E, f)$ behave identically when E is tensored by $\alpha \in \mathrm{Pic}(B)$, we may assume (by Serre's theorem) that E' is globally generated. In the equality $\phi_{kl}(g)P_l(t) = c_{kl}P_k(e_{kl}t)$, we take Jacobian determinants with respect to t and obtain $\det(\phi_{kl}(g))j_l(t) = c_{kl}^{r+1}j_k(e_{kl}t)\det(e_{kl})$. Therefore, if $M := -(r+1)D + \det(E) - g^*\det(E)$, then the functions j_k define a B-map $E' \xrightarrow{j} M$. Using Lemma 1.11, choose a nonvanishing global section $\mathcal{O} \xrightarrow{v} E'$ such that the image of $s := qv$ is not contained in the support of the divisor J that is locally defined by j_k. Both s^*J and M are given by the vanishing of $j(v)$, so $s^*J \simeq M$. Note that $J = R_f - p^*R_g$, where R_f and R_g are the ramification divisors of f and g, respectively. By Riemann–Hurwitz,

$$\begin{aligned}J &\simeq K_{\mathbb{P}E} - f^*K_{\mathbb{P}E} - p^*K_B + p^*g^*K_B\\ &\simeq (d-1)(r+1)H + p^*\det(E) - p^*g^*\det(E) - (r+1)p^*\beta.\end{aligned}$$

Hence $M \simeq J_S \simeq -(r+1)\beta + \det(E) - g^*\det(E)$, and we get $(r+1)\beta \simeq (r+1)D$. □

LEMMA 1.15. *If $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ has well-defined algebraic degree, then there exists an $\alpha \in \mathrm{Pic}(B)$ such that f extends to a self-map of $\mathbb{P}(E \oplus \alpha)$.*

Proof. Let $B \xrightarrow{g} B$ be the self-map induced by f, let d be the algebraic degree of f, and let $s \in H^0(\mathbb{P}E, p^*g^*E' \otimes (\mathcal{O}_{\mathbb{P}E}(1))^d)$ be a nonvanishing section that defines f. Put $G := E \oplus \alpha$ with projection $\mathbb{P}G \xrightarrow{r} B$. The adjunction formula implies that $\mathcal{O}_{\mathbb{P}G}(\mathbb{P}E) \simeq \mathcal{O}_{\mathbb{P}G}(1) \otimes r^*\alpha'$. If $\mathbb{P}G \xrightarrow{f_+} \mathbb{P}G$ extends f, then f_+ has the algebraic degree of f. We need a nonvanishing section $s_+ \in H^0(\mathbb{P}G, r^*g^*G' \otimes (\mathcal{O}_{\mathbb{P}G}(1))^d)$ that extends s. Note that

$$r^*g^*G' \otimes (\mathcal{O}_{\mathbb{P}G}(1))^d) \simeq (r^*g^*E' \otimes (\mathcal{O}_{\mathbb{P}G}(1))^d) \oplus (\mathcal{O}_{\mathbb{P}G}(\mathbb{P}E))^d.$$

We show first that, for $\alpha \gg 0$, s extends to $t \in H^0(\mathbb{P}G, r^*g^*G' \otimes (\mathcal{O}_{\mathbb{P}G}(1))^d)$. Indeed, Leray's spectral sequence together with the projection formula give the exact sequence

$$H^1(B, g^*E' \otimes \alpha \otimes r_*((\mathcal{O}_{\mathbb{P}G}(1))^{d-1})) \to H^1(\mathbb{P}G, r^*g^*E' \otimes r^*\alpha \otimes (\mathcal{O}_{\mathbb{P}G}(1))^{d-1}) \\ \to H^0(B, g^*E' \otimes \alpha \otimes R^1 r_*((\mathcal{O}_{\mathbb{P}G}(1))^{d-1})).$$

But $R^1 r_*((\mathcal{O}_{\mathbb{P}G}(1))^{d-1}) = 0$, and Serre's asymptotic vanishing theorem implies that $H^1(B, g^*E' \otimes \alpha \otimes r_*((\mathcal{O}_{\mathbb{P}G}(1))^{d-1})) = 0$ for $\alpha \gg 0$. It follows that $H^1(\mathbb{P}G, r^*g^*E' \otimes r^*\alpha \otimes (\mathcal{O}_{\mathbb{P}G}(1))^{d-1}) = 0$ for $\alpha \gg 0$. The short exact sequence $0 \to \mathcal{O}_{\mathbb{P}G}(-\mathbb{P}E) \to \mathcal{O}_{\mathbb{P}G} \to \mathcal{O}_{\mathbb{P}E} \to 0$ implies that the restriction map $H^0(\mathbb{P}G, r^*g^*G' \otimes (\mathcal{O}_{\mathbb{P}G}(1))^d) \to H^0(\mathbb{P}E, p^*g^*E' \otimes (\mathcal{O}_{\mathbb{P}E}(1))^d)$ is surjective; hence s extends to t, a section that does not vanish on $\mathbb{P}E$. $\square$

Let $s_0 \in H^0(\mathbb{P}G, \mathcal{O}_{\mathbb{P}G}(\mathbb{P}E))$ be a section that vanishes precisely on $\mathbb{P}E$, and define $s_+ := (t, s_0^d)$.

THEOREM 1.16. *Any finite self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ with well-defined algebraic degree lifts to all $E' \otimes \alpha$, $\alpha \in \mathrm{Pic}(B)$.*

Proof. We need to show that $D(E, f) \simeq 0$. Lemma 1.15 and parts (1) and (5) of Proposition 1.14 imply that $(r+n)D(E, f) \simeq 0$ and $(r+n+1)D(E, f) \simeq 0$. $\square$

REMARK 1.17. Let $\mathbb{P}E \xrightarrow{p} B$ be the ruled surface with invariant 1 over an elliptic curve. If $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ induces the identity self-map of B and has fiber-degree greater than 1, then f does not lift to any $E' \otimes \alpha$, $\alpha \in \mathrm{Pic}(B)$.

Proof. Otherwise, $d-1$ would divide

$$\deg(D(E, f)) = \deg(\beta(E, f)) = (d-1)/2. \qquad \square$$

1.3. Completely Invariant Sub-bundles

Given $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$, a subset $A \subset \mathbb{P}E$ is *completely invariant* if and only if $f^{-1}(A) = A$.

LEMMA 1.18. *For every $0 \le j \le r$, fix a section $s_j \in H^0(\mathbb{P}E_m, \mathcal{O}_{\mathbb{P}E_m}(H_j))$ that vanishes precisely on H_j. Assume $\delta_m \ne 0 = m_0$ and let $0 \le k < r$ be determined by $0 = m_k > m_{k+1}$. Then, for all integers $d \ge 0$, $H^0(\mathbb{P}E_m, dH)$ is the set of degree-d homogeneous polynomials in $s_0, \dots, s_k$.*

Proof. The statement is clearly true when $d = 0$. It is also true when $k = 0$. Indeed, otherwise let $a > 0$ be minimal with $h^0(\mathbb{P}E_m, aH_0) > 1$, and choose an effective divisor $aH_0 \ne D \simeq aH_0$ so that H_0 is not an irreducible component of D. Lemma 1.8 applied to $D|_{H_0}$ implies that $am_0 \le am_1$, which contradicts $m_0 > m_1$. To prove the general case, we use induction over $k + d$.

Let $P(T_0, \dots, T_k)$ be a homogeneous polynomial with $P(s_0, \dots, s_k) = 0$. Restricting to H_0, we get $p(s_1, \dots, s_k) = 0$, where $p(T_1, \dots, T_k) = P(0, T_1, \dots, T_k)$. The induction hypothesis implies $p = 0$ and so $P = T_0 Q$, where $\deg(Q) = \deg(P) - 1$ and $Q(s_0, \dots, s_k) = 0$. By induction, $Q = 0$, hence $P = 0$.

The exact sequence $0 \to \mathcal{O}_{\mathbb{P}E_m}((d-1)H) \to \mathcal{O}_{\mathbb{P}E_m}(dH) \to \mathcal{O}_{H_0}(dH) \to 0$ implies that $h^0(\mathbb{P}E_m, dH) \le h^0(\mathbb{P}E_m, (d-1)H) + h^0(H_0, dH|_{H_0})$. By induction, $h^0(\mathbb{P}E_m, dH) \le \binom{d+k}{k}$. This finishes the proof. $\square$

For $0 \leq k < r$, define $\mathcal{P}_k := \bigcap_0^k H_j \subset \mathbb{P}E_m$.

THEOREM 1.19. *If $m_k > m_{k+1}$, then $\mathcal{P}_k$ is completely invariant for all finite self-maps of $\mathbb{P}E_m$.*

Proof. By an inductive argument, we may assume that $0 = m_0 = m_k$. Let d be the algebraic degree of f. With the notation of Lemma 1.18, for every $0 \leq j \leq k$ we have that f^*H_j is the zero-locus of $P_j(s_0, \dots, s_k)$, where P_j is a homogeneous polynomial of degree d. It is enough to show that the polynomials $P_0, \dots, P_k$ have no common zeros in $\mathbb{P}^k$. Assume that $p \in \mathbb{P}^k$ is a common zero. Since f is finite, $\operatorname{codim}_{\mathbb{P}E_m}(f^{-1}(\mathcal{P}_k)) = \operatorname{codim}_{\mathbb{P}E_m}(\mathcal{P}_k) = k + 1$. The set $Z := \{z \in \mathbb{P}E_m : s_i(z)p_j = s_j(z)p_i \ \forall 0 \leq i, j \leq k\}$ is defined by k equations, so $\operatorname{codim}_{\mathbb{P}E_m}(Z) \leq k$. But $Z \subset f^{-1}(\mathcal{P}_k)$, hence $\operatorname{codim}_{\mathbb{P}E_m}(Z) \geq k + 1$. This contradiction finishes the proof. $\square$

COROLLARY 1.20. *Assume that $B = \mathbb{P}^1$ and $m_0 > m_1 > \cdots > m_r$. Fix a rational function $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$ of algebraic degree d. The space of self-maps over g of $\mathbb{P}E_m$ is the complement of $r + 1$ hyperplanes in general position in $\mathbb{P}^N$, where*

$$N = -1 + \binom{d+r+1}{r} + \sum_0^r m_j \left(\binom{d+r+1}{r+1} - (d+1)\binom{d+j}{j} \right).$$

Proof. We start with $B = \mathbb{P}^n$, and we assume $n = 1$ only when calculating N. By Theorem 1.9, any self-map $\mathbb{P}E_m \xrightarrow{f} \mathbb{P}E_m$ over g has algebraic degree d. By Theorem 1.16, any such f comes from a self-map $E'_m \xrightarrow{F} E'_m$ over g. Up to a multiplicative constant, F is uniquely determined by f.

Fix such F. Let $b = [b_0, \dots, b_n]$ be the homogeneous coordinates in $\mathbb{P}^n$, and define $A_j = \{b \in \mathbb{P}^n : b_j \neq 0\}$ and $A_j^k = A_j \cap g^{-1}(A_k)$. In E'_m, we have $(b, t)_j = (b, (b_k/b_j)^m t)_k$. For all j and k, $A_j^k \times \mathbb{C}^{r+1} \xrightarrow{F} A_k \times \mathbb{C}^{r+1}$ is given by the formula $F((b,t)_j) = (g(b), F_j^k(b,t))_k$ with $F_j^k \in \mathcal{O}^{r+1}(A_j^k)[t]$. For an arbitrarily large integer $p \gg 0$ we have $F_j^k = (b_j g_k)^{-p} P_j^k$, where $P_j^k \in \mathbb{C}^{r+1}[b,t]$, $\deg_b(P_j^k) = p(1+d)$, and $\deg_t(P_j^k) = d$. Set $P_0^0 = P = \sum_J P_J t^J$, where $P_J \in \mathbb{C}^{r+1}[b]$ and where the sum ranges over the nonnegative multi-indices $J \in \mathbb{Z}^{r+1}$ with length $|J| = d$. The gluing condition becomes $P_j^k(b,t) = (b_j/b_0)^p (g_k/g_0)^{p+m} P(b, (b_0/b_j)^m t)$ for all j, k, b, and t. This means that $(b_j/b_0)^{p - J\cdot m}(g_k/g_0)^{p+m} P_J \in \mathbb{C}^{r+1}[b]$ for all j, k, and J, where $J \cdot m = \sum_0^r J_i m_i$. We may assume that $p \geq m_0 d$. Since $g_0, \dots, g_r$ have no common factor, it follows that $P_J = b_0^{p - J\cdot m} g_0^{p+m} Q_J$ with $Q_J \in \mathbb{C}^{r+1}[b]$ and $\deg(Q_J) = J \cdot m - dm$, where $dm = (dm_0, \dots, dm_r)$. We get $F_j^k = \sum_J b_j^{-J\cdot m} g_k^m Q_J t^J$ on $A_j^k \times \mathbb{C}^{r+1}$.

Write $Q_J = (Q_{J0}, \dots, Q_{Jr})$, with $Q_{Jl} \in \mathbb{C}[b]$ and $\deg(Q_{Jl}) = J \cdot m - dm_l$ for all J and l. By Theorem 1.19, $Q_{Jl} = 0$ whenever there exists an $i > l$ such that $J_i \neq 0$. Consequently, the condition $F^{-1}(\omega) = \omega$ means that $Q_{J^l l} \neq 0$ for all $0 \leq l \leq r$, where $J_i^l = d\delta_i^l$. Here, δ_i^l denotes the Kronecker symbol.

In conclusion, we can identify F with the map it induces,

$$A_0^0 \times \mathbb{C}^{r+1} \ni (b,t)_0 \xrightarrow{F} \left(g(b), g_0^m(b) \sum_J \frac{Q_J(b)}{b_0^{J\cdot m}} t^J \right)_0 \in A_0 \times \mathbb{C}^{r+1},$$

where $Q_{Jl} \in \mathbb{C}[b]$ are indexed by $0 \le l \le r$ and $0 \le J \in \mathbb{Z}^{l+1}$ with $|J| = d$ and also satisfy $\deg(Q_{Jl}) = -dm_l + \sum_0^l J_i m_i$ and $Q_{J'l} \neq 0$.

Therefore, if $l(J) = l$ denotes the dimension of $J \in \mathbb{Z}^{l+1}$, we have

$$N(n,m,d) = \sum_J \binom{n + J \cdot m - dm_{l(J)}}{n},$$

where the sum ranges over all multi-indices J with $l(J) \le r$ and $|J| = d$. The formula for $N(1,m,d)$ follows by a lengthy but straightforward calculation that uses the identity $\sum_0^a \binom{b+i}{i} = \binom{a+b+1}{a}$. □

EXAMPLE 1.21. Assume that $B = \mathbb{P}^1$, let $F_n = \mathbb{P}E_{(0,-n)}$ be the Hirzebruch surface with invariant $n > 0$, and fix a rational function $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$ of degree $d > 0$, $g[x,y] = [u,v]$. Any self-map $F_n \xrightarrow{f} F_n$ over g determines a polynomial $w(x,y,t)$ that satisfies $w(0,0,1) \neq 0$ and is homogeneous of degree d with respect to the weights $(1/n, 1/n, 1)$, so that, as a rational self-map of $\mathbb{P}^2$, f is given by the formula $f[x,y,z] = [u^n, u^{n-1}v, w(x,y,zx^{n-1})]$.

Proof. Indeed, the proof of Corollary 1.20 shows that f is given by a rational self-map F of $\mathbb{P}^1 \times \mathbb{C}^2$: $F(x,y;t,z) = \left(u,v;u^n x^{-nd} t^d, \sum_0^d w_j x^{-jn} t^j z^{d-j}\right)$, with $w_j \in \mathbb{C}[x,y]$, $\deg(w_j) = nj$, and $w_0 \neq 0$. As a rational self-map of $\mathbb{P}^2$, $f[x,y,z] = \left[u^n, u^{n-1}v, \sum_0^d w_j z^{d-j} x^{(n-1)(d-j)}\right]$. We put $w(x,y,t) := \sum_0^d w_j(x,y) t^{d-j}$. □

EXAMPLE 1.22. Let $X \xrightarrow{f} X$ be a finite self-map, where $X \xrightarrow{p} \mathbb{P}^n$ is the blow-up at a point with exceptional divisor e. Then e is completely f-invariant, and f induces (through p) a regular self-map $\mathbb{P}^n \xrightarrow{g} \mathbb{P}^n$. Moreover, f can be identified with a regular self-map of $\mathbb{P}^{n+1}$ of the form $[b, b_{n+1}] \to [g(b), g_{n+1}(b, b_{n+1})]$, where $g_{n+1} \in \mathbb{C}[b, b_{n+1}]$ is homogeneous and $g_{n+1}(0,1) \neq 0$.

Proof. Since $X = \mathbb{P}E_{(0,-1)}$, Theorem 1.19 implies that e is completeletely invariant. Let d be the algebraic degree of f. As in the proof of Corollary 1.20, f is given by a rational self-map F of $\mathbb{P}^n \times \mathbb{C}^2$:

$$F(b;t_0,t_1) = \left(g(b); t_0^d, \frac{1}{g_0(b)} \sum_0^d q_j(b) t_1^j b_0^{-j}\right),$$

with $q_j \in \mathbb{C}[b]$, $\deg(q_j) = d - j$, and $q_d \neq 0$. We put $g_{n+1} := \sum_0^d q_j(b) b_{n+1}^j$. □

2. Dynamics

We work from now on in the following context: $E \xrightarrow{\pi'} B$ is a vector bundle, with $\operatorname{rank}(E) = r+1$ and $\dim(B) = n$, whose dual $E' \xrightarrow{\pi} B$ is endowed with a Hermitian metric $\|\cdot\|$; $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ is a finite self-map over $B \xrightarrow{g} B$, of fiber-degree $d \ge 2$, that is assumed to admit a lifting to E', $E' \setminus \omega \xrightarrow{F} E' \setminus \omega$. We study the basic dynamical propeties of f, adapting to this context the methods of [3; 7; 9].

2.1. Green Function

For $r > 0$, we have $B_r := \{x \in E' : \|x\| < r\}$. Put $U := \partial B_1$. A set $B \subset E'$ is

(a) *bounded* iff there exists $r > 0$ such that $B \subset B_r$;
(b) a neighborhood of ω iff there exists $r > 0$ such that $B \supset B_r$; and
(c) *complete circular* iff $\lambda x \in B$ for all $x \in B$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

Put $E' \setminus \omega \xrightarrow{\alpha} \mathbb{R}$ with $\alpha := \log\|F\| - d\log\|\cdot\|$ and $E' \setminus \omega \xrightarrow{\beta} \mathbb{R}$ with $\beta := \log\|F\| - \log\|\cdot\|$. Note that α is continuous and *homogeneous of degree* 0, meaning that α is constant on the homogeneous lines of E' and hence α is bounded on $E' \setminus \omega$.

LEMMA 2.1. *There exists an $r > 0$ such that $\beta \leq -1$ on B_r and $\beta \geq 1$ outside $B_{1/r}$.*

Proof. We can choose $r := e^{-(m+1)/(d-1)}$, where $m := \max_U |\alpha|$. $\square$

DEFINITION 2.2. $\mathcal{A} := \{x \in E' : \lim_{j\to\infty}\|F^j(x)\| = 0\}$ is the *basin of attraction* of ω, and $\mathcal{A}_\infty := \{x \in E' : \lim_{j\to\infty}\|F^j(x)\| = \infty\}$ is the basin of attraction of ∞.

The sets $\mathcal{A}$ and $\mathcal{A}_\infty$ are disjoint and are completely F-invariant domains of E'; also, $\mathcal{A} = \bigcup_j F^{-j}B_r$ with r given by Lemma 2.1. Note that $\mathcal{A}$ is a bounded and complete circular neighborhood of ω.

PROPOSITION 2.3. *If $\mathcal{A} \subset E'$ is a bounded and complete circular neighborhood of ω, then there exists a unique function $E' \setminus \omega \xrightarrow{G} \mathbb{R}$ with the following properties*:

(1) $G - \log\|\cdot\|$ *is homogeneous of degree* 0; *and*
(2) $G|_{\partial\mathcal{A}} \equiv 0$.

Moreover, $G - \log\|\cdot\|$ is upper semi-continuous and upper bounded on $E' \setminus \omega$.

Proof. To prove the uniqueness of G, assume that G_1 and G_2 both satisfy these two properties. Then $h := G_1 - G_2$ is homogeneous of degree 0. Note that, for every homogeneous line l in E', $l \cap \mathcal{A}$ is a nondegenerate disk centered at the origin. Since $h|_{\partial\mathcal{A}} \equiv 0$, we deduce that $h|_l \equiv 0$ for every l and hence $h \equiv 0$.

In order to construct G, first define $E' \setminus \omega \xrightarrow{r} (0, \infty)$ and $r(x) = \sup\{\lambda > 0 : \lambda x \in \mathcal{A}\}$. It is clear that $r(\lambda x) = r(x)/|\lambda|$ for all $(\lambda, x) \in \mathbb{C}^* \times (E' \setminus \omega)$ and that $r|_{\partial\mathcal{A}} \equiv 1$. Consequently, $G := -\log r$ satisfies properties (1) and (2) of the proposition.

Observe that r is lower semi-continuous. Indeed, given $a > 0$, it follows that $r(x) > a$ if and only if $ax \in \mathcal{A}$. Given $x \in E' \setminus \omega$ with $r(x) > a$, there exists a neighborhood V of ax with $V \subset \mathcal{A}$. Then V/a is a neighborhood of x and $a(V/a) \subset A$, so that $r > a$ on V/a. Therefore, G is upper semi-continuous. If $C := \max_U G$, then $G(x) - \log\|x\| = G(x/\|x\|) \leq C$ for all $x \in E' \setminus \omega$. $\square$

We call G the *Green function* of $\mathcal{A}$. Clearly, $\mathcal{A} \setminus \omega = \{x \in E' \setminus \omega : G(x) < 0\}$ and $\partial\mathcal{A} = \{x \in E' \setminus \omega : G(x) = 0\}$.

Let $\mathcal{H}$ be the set of points $x \in E' \setminus \omega$ with the property that G is pluriharmonic in a neighborhood of x, and put $\Omega := q(\mathcal{H})$. Then $\mathcal{H} = q^{-1}(\Omega)$ by Proposition 2.3.

PROPOSITION 2.4. *Ω contains no fiber of $\mathbb{P}E$.*

Proof. Assume that there exists a fiber $F_b \subset \Omega$ such that $\mathbb{C}^{r+1} \setminus 0 = q^{-1}(F_b) \subset \mathcal{H}$. The restriction $G|_{\mathbb{C}^{r+1}\setminus 0}$ is pluriharmonic and hence extends to a pluriharmonic function on $\mathbb{C}^{r+1}$. This is not possible, since $G|_{\mathbb{C}^{r+1}\setminus 0}(0) = -\infty$. □

PROPOSITION 2.5. *Ω equals the set of points l with the property that $E' \setminus \omega \xrightarrow{q} \mathbb{P}E$ admits a section defined near l, with image contained in $\partial\mathcal{A}$.*

Proof. Given $l^0 \in \mathbb{P}E$ with $b^0 := p(l^0) \in B$, let $U \subset B$ be a neighborhood of b^0 on which E' is trivial, $E'|_U = U \times \mathbb{C}^{r+1}$, so that $l^0 = (b^0, [T^0])$ with $[T^0] \in \mathbb{P}^r$. Permuting the coordinates in $\mathbb{C}^{r+1}$, we may assume $T_0^0 \neq 0$ so that $[T^0] = [1, t^0] \in \mathbb{P}^r$. If $A_0 := \{[T] \in \mathbb{P}^r : T_0 \neq 0\} = \mathbb{C}^r$, then $E' \cap q^{-1}(U \times A_0) = U \times \mathbb{C}^* \times \mathbb{C}^r$, and $E' \cap q^{-1}(U \times A_0) \xrightarrow{q} U \times A_0$ is given by the formula $q(b, \lambda, x) = (b, x/\lambda)$. In these coordinates, $l^0 = (b^0, t^0) \in U \times A_0$ and $G(b, \lambda, x) = \log|\lambda| + G(b, 1, x/\lambda)$. Let $U \times \mathbb{C}^* \times \mathbb{C}^r \xrightarrow{\Lambda} \mathbb{C}^*$ be the projection on $\mathbb{C}^*$, and let $U \times A_0 \xrightarrow{s_0} q^{-1}(U \times A_0)$ be the local section of q given by $s_0(b, t) = (b, 1, t)$. Define $U \times \mathbb{C}^r \xrightarrow{\gamma} \mathbb{R}$ with $\gamma(b, t) = G(b, 1, t)$. Then $\gamma = Gs_0$ on $U \times A_0$ and $G = \log|\Lambda| + \gamma q$ on $q^{-1}(U \times A_0)$. Since $\mathcal{H} = q^{-1}(\Omega)$, it follows that $l^0 \in \Omega$ if and only if γ is pluriharmonic near $l^0 = (b^0, t^0)$.

Assume that $l^0 \in \Omega$, choose $\phi \in \mathcal{O}_{l^0}$ with $\Re\phi = \gamma$, and define near l^0 the map $s(b, t) := (b, e^{-\phi(b,t)}, te^{-\phi(b,t)})$. Then

$$qs(b, t) = (b, t) \quad \text{and} \quad Gs(b, t) = \log|e^{-\phi(b,t)}| + \gamma(b, t) = 0,$$

meaning that s is a local section of q whose image is contained in $\partial\mathcal{A}$.

Conversely, let s be a local section of q near l^0, let $s(b, t) = (b, \sigma(b, t), t\sigma(b, t))$ with $\sigma \in \mathcal{O}_{l^0}^*$, and assume that $Gs \equiv 0$. Since $Gs = \log|\sigma| + \gamma$, it follows that $\gamma = -\log|\sigma|$ is pluriharmonic near l^0. □

REMARK 2.6. If s_0 and s_1 are two germs at $l^0 \in \Omega$ of local sections of q with image in $\partial\mathcal{A}$, then $s_0 = cs_1$ for some $c \in \mathbb{C}^*$ with $|c| = 1$.

Proof. Indeed, in local coordinates near l^0 we have $s_i(b, t) = (b, \sigma_i(b, t), t\sigma_i(b, t))$, with $\sigma_i \in \mathcal{O}_{l^0}^*$ satisfying $\gamma = -\log|\sigma_i|$ for $i \in \{0, 1\}$. It follows that $|\sigma_1/\sigma_0| \equiv 1$, which implies the existence of $c \in \mathbb{C}^*$ with $|c| = 1$ and so $\sigma_1 = c\sigma_0$; that is, $s_1 = cs_0$. □

We recall that, when $\mathcal{A}$ is the basin of attraction of ω under the action of F, the corresponding G is called the Green function of F.

PROPOSITION 2.7. *The Green function G of a self-map $E' \setminus \omega \xrightarrow{F} E' \setminus \omega$ of fiber-degree $d \geq 2$ satisfies the following properties*:

(1) $G(F) = dG$;
(2) *for any* $E' \setminus \omega \xrightarrow{\nu} \mathbb{R}$ *such that* $\nu - \log\|\cdot\|$ *is bounded,* $G = \lim_{j\to\infty} \nu(F^j)/d^j$ *uniformly on* $E' \setminus \omega$.

Consequently, $G - \log\|\cdot\|$ *is continuous and bounded on* $E' \setminus \omega$.

Proof. If $\mu := \nu(F) - d\nu$, then Lemma 2.1 implies that μ is bounded on $E' \setminus \omega$. Since $\nu(F^j)/d^j = \sum_0^{j-1} \mu(F^k)/d^k$, there exists an $E' \setminus \omega \xrightarrow{G_\nu} \mathbb{R}$ such that $G_\nu = \lim_{j\to\infty} \nu(F^j)/d^j$ uniformly in $E' \setminus \omega$. Since $\nu - \log\|\cdot\|$ is bounded, $G_\nu = G_0$ with $G_0 = \lim_{j\to\infty} \log\|F^j\|/d^j$. It follows immediately from the definition of G_0 that $G_0(F) = dG_0$. As a uniform limit of continuous functions, G_0 is continuous. For $(\lambda, x) \in \mathbb{C}^* \times (E' \setminus \omega)$ we have $\log\|F^j(\lambda x)\|/d^j = \log|\lambda| + \log\|F^j(x)\|/d^j$, so $G_0(\lambda x) = \log|\lambda| + G_0(x)$; that is, $G_0 - \log\|\cdot\|$ is constant on the homogeneous lines of E'. By continuity, this implies that $G_0 - \log\|\cdot\|$ is bounded.

By Proposition 2.3, it remains to show that $G_0|_{\partial\mathcal{A}} \equiv 0$, since this will imply that $G = G_0$. Fix $x \in E' \setminus \omega$. If $G_0(x) > 0$, then $\|F^j(x)\| > \exp[G_0(x)d^j/2]$ for all $j \gg 0$ and hence $x \in \mathcal{A}_\infty$. If $G_0(x) < 0$, then $\|F^j(x)\| < \exp[G_0(x)d^j/2]$ for all $j \gg 0$ and hence $x \in \mathcal{A}$. Since $\partial\mathcal{A} \cap \mathcal{A} = \emptyset$ and $\partial\mathcal{A} \cap \mathcal{A}_\infty = \emptyset$, it follows that $G_0|_{\partial\mathcal{A}} \equiv 0$. □

2.2. *Fatou Set*

The Fatou set $\mathcal{F}$ of $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ is the set of points $l \in \mathbb{P}E$ that have a neighborhood $V \subset \mathbb{P}E$ on which the sequence of iterates $V \xrightarrow{f^j} \mathbb{P}E$ is a normal family. By definition, $\mathcal{F}$ is open in $\mathbb{P}E$. It is easy to see that the Fatou set does not change when the self-map is replaced by an iterate. The complement $\mathcal{J} = \mathbb{P}E \setminus \mathcal{F}$ is the *Julia set* of f. Let $\mathcal{F}_g$ denote the Fatou set of $B \xrightarrow{g} B$.

PROPOSITION 2.8. $\mathcal{F} \subset p^{-1}(\mathcal{F}_g)$.

Proof. Fix $l \in \mathcal{F}$ with a neighborhood V on which $(f^j)_j$ is a normal family. If $U \subset B$ is a sufficiently small neighborhood of $b := p(l)$, then there exists a local section of p, $U \xrightarrow{s} \mathbb{P}E$, with $s(U) \subset V$. Then $(g^j|_U)_j = (pf^js)_j$ is normal on U. □

PROPOSITION 2.9. *If E is globally generated, then $\mathcal{F} \subset \Omega$.*

Proof. Fix $l \in \Omega$ and let $(f^{j_k})_k$ be a subsequence of iterates of f that converges uniformly in a compact neighborhood $\bar{V}$ of l to a map $\bar{V} \xrightarrow{\phi} \mathbb{P}E$. Let $l^0 := \phi(l)$ and fix $x^0 \in l^0 \setminus 0 \subset E' \setminus \omega$. Because E is globally generated, we can find a $w \in \Gamma(B, E)$ with $x^0(w) \neq 0$ and put $\varepsilon := |x^0(w)|/\|x^0\| > 0$. Define $\bar{W} := \{x \in E' \setminus \omega : |x(w)|/\|x\| \leq \varepsilon/2\}$. Clearly, $\bar{W} = q^{-1}(q(\bar{W}))$ and $l^0 \notin q(\bar{W})$. Shrinking $\bar{V}$, we may assume that $\phi(\bar{V}) \cap q(\bar{W}) = \emptyset$. Thus we may also assume that $f^{j_k}(\bar{V}) \cap q(\bar{W}) = \emptyset$ for all k.

Define $E' \setminus \omega \xrightarrow{\nu} \mathbb{R}$ with $\nu(x) := \log\max(\|x\|, 2|x(w)|/\varepsilon)$, so that $\nu(x) = \log\|x\|$ on W and $\nu(x) = \log(2|x(w)|/\varepsilon)$ outside $\bar{W}$. The function $\nu - \log\|\cdot\|$ is continuous and homogeneous of degree 0, so it is bounded on $E' \setminus \omega$. By Proposition 2.7, $\lim_k \nu(F^{j_k})/d^{j_k} = G$ uniformly on $E' \setminus \omega$. If $x \in q^{-1}(\bar{V})$ then $qF^{j_k}(x) = f^{j_k}(q(x)) \in f^{j_k}(\bar{V})$, so that $F^{j_k}(x) \notin \bar{W}$. Therefore, uniformly on $q^{-1}(\bar{V})$, $G = \lim_k \log(2|F^{j_k}(w)|/\varepsilon)/d^{j_k}$. We conclude that G is pluriharmonic on $q^{-1}(\bar{V})$ and that $l \in \Omega$. □

PROPOSITION 2.10. $p^{-1}(\mathcal{F}_g) \cap \Omega \subset \mathcal{F}$.

Proof. Fix $l \in \Omega$ with $b := p(b) \in \mathcal{F}_g$ and fix an arbitrary subsequence $(f^{j_k})_k$ of iterates of f. We may assume that $(g^{j_k})_k$ converges uniformly in a compact neighborhood $\bar{U} \subset B$ of b to a map $\bar{U} \xrightarrow{\gamma} B$. Put $b' := \gamma(b)$, fix local coordinates near $0 := b'$, and let $D \subset B$ be a neighborhood of b', biholomorphic to the unit ball in C^n, such that $E|_D$ is trivial. Shrinking $\bar{U}$, we may assume that $\gamma(\bar{U}) \subset D/2$, and then we may assume that $g^{j_k}(\bar{U}) \subset D$ for all k.

By Proposition 2.5, there exists a neighborhood $V \subset \mathbb{P}E$ of l and a local section of q, $V \xrightarrow{s} \partial\mathcal{A} \subset E' \setminus \omega$. Shrinking V, we may assume that $p(V) \subset \bar{U}$.

We see that $F^{j_k}s(V) \subset F^{j_k}s(p^{-1}(\bar{U})) \subset \pi^{-1}(g^{j_k}(\bar{U})) \subset \pi^{-1}(D)$ and $F^{j_k}s(V) \subset F^{j_k}(\partial\mathcal{A}) \subset \partial\mathcal{A}$, so that $F^{j_k}s(V) \subset \pi^{-1}(D) \cap \partial\mathcal{A}$. Since $\pi^{-1}(D) \cap \partial\mathcal{A}$ is a bounded set in $\mathbb{C}^{n+r}$, Montel's theorem implies that $(F^{j_k}s)_k$ is a normal family. Since $f^{j_k}|_V = qF^{j_k}s$, there exists a sub-subsequence $(f^{j_{k_h}})_h$ that converges on V, and we conclude that $l \in \mathcal{F}$. □

THEOREM 2.11. *If E is globally generated, then $\mathcal{F} = \Omega \cap p^{-1}(\mathcal{F}_g)$.*

Proof. This collects the results of the previous three propositions. □

EXAMPLE 2.12. For $n > 0$ and $d > 1$, consider the rational self-map s of $\mathbb{P}^2$, $[x_0, x_1, z] \xrightarrow{s} [x_0^{nd}, x_0^{(n-1)d} x_1^d, z^d]$. As in Example 1.21, s can be viewed as a regular self-map $F_n \xrightarrow{f} F_n$, $f((x_0, x_1; t, z)_j) = (x_0^d, x_1^d; t^d, z^d)_j$, for $j = 0, 1$. The Green function of f is $G((x_0, x_1; t, z)_j) = \log\max(|t|, |x_0/x_j|^n|z|, |x_1/x_j|^n|z|)$. The mapping $F_n \xrightarrow{f} F_n$ has four Fatou components, all basins of attraction, that are biholomorphic to the 2-disk.

From now on, we assume that E is globally generated.

LEMMA 2.13. *There exists a Hermitian metric $\|\cdot\|$ on E' such that $\log\|\cdot\|$ is plurisubharmonic on E'.*

Proof. Because E is the quotient of a trivial bundle $\mathcal{O}_B^N$, it follows that E' is a sub-bundle of $\mathcal{O}_B^N$. We choose $\|\cdot\|$ to be the restriction to E' of the trivial metric on $\mathcal{O}_B^N$. □

COROLLARY 2.14. *The Green function G of F is plurisubharmonic on E'.*

Proof. By Proposition 2.7, G is the uniform limit on $E' \setminus \omega$ of the sequence $\log\|F^j\|/d^j$, $j \geq 0$. Consequently, G is plurisubharmonic on $E' \setminus \omega$. Since

$G - \log\|\cdot\|$ is continuous and bounded on $E' \setminus \omega$, the function $E' \xrightarrow{G} [-\infty, \infty)$ is continuous; $G^{-1}(-\infty) = \omega$, hence G is plurisubharmonic on E'. □

The following result is well known.

LEMMA (Cegrell). *Given a plurisubharmonic function $M \xrightarrow{h} [-\infty, \infty)$ on a complex manifold M, let $\mathcal{H}$ be the set of points $m \in M$ that possess a neighborhood $V \subset M$ such that $h|_V$ is pluriharmonic. If $\mathcal{H} \neq \emptyset$, then $\mathcal{H}$ is pseudoconvex in M.*

COROLLARY 2.15. *Ω is pseudoconvex in $\mathbb{P}E$.*

Proof. Fix $l \in \partial\Omega$, and let $V \xrightarrow{s} E' \setminus \omega$ be a local section of q defined in a neighborhood V of l. Then $\mathcal{H} \cap s(V)$ is pseudoconvex in $s(V)$, so $\Omega \cap V$ is pseudoconvex in V. □

LEMMA 2.16. *If $\mathcal{F}_g$ is pseudoconvex in B, then $p^{-1}(\mathcal{F}_g)$ is pseudoconvex in $\mathbb{P}E$.*

Proof. Fix $l \in p^{-1}(\mathcal{F}_g)$, and choose a neighborhood $U \subset B$ of $b := p(l)$ on which E is trivial. Since $\mathcal{F}_g \cap U$ is pseudoconvex in U, we deduce that $p^{-1}(\mathcal{F}_g) \cap p^{-1}(U) = (\mathcal{F}_g \cap U) \times \mathbb{P}^r$ is pseudoconvex in $p^{-1}(U) = U \times \mathbb{P}^r$. □

THEOREM 2.17. *If $\mathcal{F}_g$ is pseudoconvex in B, then $\mathcal{F}$ is pseudoconvex in $\mathbb{P}E$. If $\mathcal{F}_g$ is Stein, then $\mathcal{F}$ is Stein.*

Proof. According to Theorem 2.11, Corollary 2.15, and Lemma 2.16, $\mathcal{F}$ is an intersection of two pseudoconvex sets and hence is pseudoconvex.

Assume now that $\mathcal{F}_g$ is Stein. Since $\mathcal{F}$ is pseudoconvex in $\mathbb{P}E|_{\mathcal{F}_g}$ and since, by Proposition 2.4, $\mathcal{F}$ contains no fiber of $\mathbb{P}E$, Brun's result on the Levi problem in projective bundles with a Stein basis can be applied to conclude that $\mathcal{F}$ is Stein. □

LEMMA 2.18. *Let U_1 and U_2 be open and proper subsets of $\mathbb{P}^r$, with $\mathbb{P}^r = U_1 \cup U_2$. If $U_1 \cap U_2$ is pseudoconvex, then $r = 1$.*

Proof. Note first that U_1 and U_2 are Stein. Indeed, if $x \in \partial U_1$ then $x \in U_2$, so that U_1 is pseudoconvex. Fujita's result on the Levi problem in $\mathbb{P}^r$ implies that U_1 is Stein. Leray's lemma implies that the cohomological dimension of $\mathbb{P}^r$ is at most 1; that is, $r = 1$. □

THEOREM 2.19. *Assume $r \geq 2$. If $\mathcal{F}_g$ is pseudoconvex in B, then $\mathcal{J}$ is connected.*

Proof. Assume that $\mathcal{J}$ is a disjoint union of proper subsets $\mathcal{J}_1$ and $\mathcal{J}_2$. By Proposition 2.4, $\mathcal{J}$ intersects every fiber of $\mathbb{P}E$ and hence $p(\mathcal{J}_1) \cup p(\mathcal{J}_2) = B$. Since p is a closed map and B is connected, there exists a $b \in p(\mathcal{J}_1) \cap p(\mathcal{J}_2)$. Put $U_i = F_b \setminus \mathcal{J}_i$ for $i \in \{1, 2\}$. Then U_1 and U_2 are open and proper subsets of F_b, and $U_1 \cap U_2 = F_b \setminus \mathcal{J} = F_b \cap \mathcal{F}$. Theorem 2.17 implies that $U_1 \cap U_2$ is pseudoconvex in $F_b = \mathbb{P}^r$. By Lemma 2.18, this is not possible when $r \geq 2$. □

THEOREM 2.20. *If $\mathcal{F}_g$ is Kobayashi hyperbolic, then $\mathcal{F}$ is Kobayashi hyperbolic.*

Proof. By Theorem 2.11, $\mathcal{F} = \Omega \cap p^{-1}(\mathcal{F}_g)$. Let $\mathcal{F}_0$ be a connected component of $\mathcal{F}$ and fix $l \in \mathcal{F}_0$. Let $\mathcal{G}$ denote the abstract complex manifold of germs of local sections of q with domain in $\mathcal{F}_0$ and image in $\partial\mathcal{A}$, endowed with the local biholomorphism $\mathcal{G} \xrightarrow{Q} \mathcal{F}_0$ and the evaluation map $\mathcal{G} \xrightarrow{e} \partial\mathcal{A} \cap q^{-1}(\mathcal{F}_0) \subset E' \setminus \omega$. Proposition 2.5 implies that Q is surjective, and Remark 2.6 implies that e is injective. The map $\mathcal{G}$ may be viewed as a covering of $\mathcal{F}_0$ with infinitely many sheets and indexed by the unit circle.

Let $\mathcal{V} \xrightarrow{\nu} \mathcal{F}_0$ be the universal covering of $\mathcal{F}_0$, and fix $s_0 \in G_l := Q^{-1}(l)$. Since Q is a local biholomorphism, any piecewise-smooth path $[0,1] \xrightarrow{\gamma} \mathcal{F}_0$ with $\gamma(0) = l$ determines a local section of Q, $V_\gamma \xrightarrow{s_\gamma} \mathcal{G}$, defined near γ with $Q(s_\gamma(l)) = s_0$. Since $s_\gamma(\gamma(1))$ depends only on the homotopy class of γ, we obtain an $\mathcal{F}_0$-map $\mathcal{V} \xrightarrow{s} \mathcal{G}$ with $s[\gamma] := s_\gamma(\gamma(1))$.

Let $\mathcal{K}$ be the subset of the fundamental group $\pi(\mathcal{F}_0, l)$ formed by the classes of loops $[\gamma]$ with the property that $s[\gamma] = s_0$. In other words, $[\gamma] \in \mathcal{K}$ if and only if $s_\gamma\gamma$ is a loop in $\mathcal{G}$. Clearly, $\mathcal{K}$ is a subgroup of $\pi(\mathcal{F}_0, l)$ and hence defines unramified coverings $\mathcal{V} \xrightarrow{\rho} \mathcal{M}$ and $\mathcal{M} \xrightarrow{\mu} \mathcal{F}_0$ with $\mu\rho = \nu$. By definition of $\mathcal{K}$ and $\mathcal{M}$, there exists an injective map $\mathcal{M} \xrightarrow{t} \mathcal{G}$ with $t\rho = s$ and $Qt = \mu$.

Since E is globally generated, E' is a sub-bundle of a trivial vector bundle, $E' \subset B \times \mathbb{C}^N$. By Lemma 2.1 we have $\partial\mathcal{A} \subset B \times \mathcal{B}$, where $\mathcal{B} := \{z \in \mathbb{C}^N : \|z\| \leq 1/r\}$. Therefore, $\partial\mathcal{A} \cap q^{-1}(\mathcal{F}_0) \subset \partial\mathcal{A} \cap \pi^{-1}(\mathcal{F}_g) \subset \mathcal{F}_g \times \mathcal{B}$. Let $\partial\mathcal{A} \cap q^{-1}(\mathcal{F}_0) \xrightarrow{i} \mathcal{F}_g \times \mathcal{B}$ denote this inclusion map. As a product of hyperbolic manifolds, $\mathcal{F}_g \times \mathcal{B}$ is hyperbolic. Since the map $\mathcal{M} \xrightarrow{iet} \mathcal{F}_g \times \mathcal{B}$ is injective, $\mathcal{M}$ is hyperbolic. Since $\mathcal{M} \xrightarrow{\mu} \mathcal{F}_0$ is a covering, $\mathcal{F}_0$ is hyperbolic. □

COROLLARY 2.21. *Let $\mathbb{P}E \to \mathbb{P}^n$ be a projective bundle with nonzero discriminant, and let $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ be a self-map with topological degree at least* 2. *Then the Fatou components of f are Stein and Kobayashi hyperbolic. When* $\operatorname{rank}(E) \geq 3$, *the Julia set of f is connected.*

Proof. By Theorem 1.9, such a mapping $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ has well-defined algebraic degree $d > 1$. Let $\mathbb{P}^n \xrightarrow{g} \mathbb{P}^n$ be the self-map induced by f. By Theorem 1.16, we may assume that E is globally generated and that f lifts to E'. Theorem 2.17 implies that $\mathcal{F}_g$ is Stein, and then $\mathcal{F}$ is Stein. By Theorem 2.19, $\mathcal{J}$ is connected when $\operatorname{rank}(E) \geq 3$. By Theorem 2.20, $\mathcal{F}$ is hyperbolic. □

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