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Convergence in Capacity of the Perron–Bremermann Envelope

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1. Introduction

In [CK1], Cegrell and Kołodziej constructed a sequence of measures in a ball μ_j converging to μ in the weak* topology such that the solutions of the Dirichlet problems

$$(dd^c u_j)^n = d\mu_j, \quad u_j = 0 \text{ on the boundary}$$

are uniformly bounded yet u_j does not converge to u, the solution of the Dirichlet problem

$$(dd^c u)^n = d\mu, \quad u = 0 \text{ on the boundary.}$$

In [CK2] the authors gave conditions on the Monge–Ampère mass of the solutions u_j, with fixed continuous boundary values φ, that guarantee the stability of the complex Monge–Ampère operator. They introduced the set $\mathcal{A}(\mu)$ of all solutions u of the Dirichlet problem $u \in \mathcal{F}(\varphi)$, $(dd^c u)^n = g d\mu$, where μ is a positive finite measure that does not put mass on pluripolar sets and where g varies over all μ-measurable functions satisfying $0 \le g \le 1$. Cegrell and Kołodziej proved that, in $\mathcal{A}(\mu)$, weak* convergence is equivalent to convergence in capacity.

Our main goal is to generalize this statement by admitting a large variation of the boundary data. Let Ω be a bounded domain in $\mathbb{C}^n$, let f be a bounded function on $\partial\Omega$, and let μ be a positive Borel measure on Ω. Following [BT1], we define

$$\mathrm{PB}(f,\mu) = \{u \in \mathcal{F}(g) : (dd^c u)^n \ge \mu,\ g \le f, \text{ and } g \text{ is upper semicontinuous on } \partial\Omega\}.$$

We shall refer to the following function as the *Perron–Bremermann envelope*:

$$U(f,\mu) = \sup\{v : v \in \mathrm{PB}(f,\mu)\}.$$

For a fixed positive finite measure μ that does not put mass on pluripolar sets and for a fixed positive constant k, we shall consider the family $\mathcal{D}(\mu,k)$ of plurisubharmonic functions $U(f, g d\mu)$, where g is μ-measurable function such that $0 \le g \le 1$ and where f varies over all upper semicontinuous functions on the boundary such that $|f| \le k$. We shall prove that, in $\mathcal{D}(\mu,k)$, pointwise convergence is equivalent to convergence in capacity.

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This paper is organized as follows. In Section 2 we recall some basic facts about the harmonic measures that will be useful in the subsequent sections.

Section 3 is devoted to the Cegrell classes. We recall the definitions of the Cegrell classes with continuous boundary data and give the analogous definition for upper semicontinuous boundary data. We also prove that, for f a bounded and upper semicontinuous function and for μ a positive finite measure that does not put mass on pluripolar sets, the function $U(f,\mu)$ belongs to the Cegrell class $\mathcal{F}(f)$ and $\limsup_{z\to w} U(f,\mu)(z) = f(w)$ whenever the function f is continuous at $w \in \partial\Omega$.

In Section 4 we give the proof of the main theorem. We also show that, if uniformly bounded upper semicontinuous functions f_j tend pointwise (as $j \to \infty$) to a bounded upper semicontinuous function f, then $U(f_j,\mu) \to U(f,\mu)$ in capacity as $j \to \infty$. Then, by a theorem of Xing [X],

$$(dd^c(U(f_j,\mu))^n \to (dd^c U(f,\mu))^n$$

weakly as $j \to \infty$.

2. Preliminaries

We begin with two easy propositions, which will be useful later. For the proofs, see [Ru].

PROPOSITION 2.1. *Let X be a complete metric space and f a function that is upper semicontinuous in X. Then there exists a family V_q, $q \in \mathbb{Q}$, of open dense sets in X such that f is continuous in Y, where*

$$Y = \bigcap_{q\in\mathbb{Q}} V_q$$

is a dense G_δ-subset in X.

PROPOSITION 2.2. *Let X be a complete metric space, $x_0 \in X$, and let $f\colon X \to \mathbb{R}$ be a bounded function that is continuous at x_0. Then there exists a continuous function $g\colon X \to \mathbb{R}$ such that $g \le f$ and $g(x_0) = f(x_0)$.*

Now we recall some basic facts about harmonic measures. All definitions and theorems concerning harmonic functions and harmonic measures can be found in [ArG].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then for any $x \in \Omega$ there exists a positive measure ω_x such that $\operatorname{supp}\omega_x \subset \partial\Omega$, $\omega_x(\partial\Omega) = 1$, and such that

$$h(x) = \int_{\partial\Omega} h\, d\omega_x$$

for all $h \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. For any Borel set B in $\partial\Omega$, we write

$$\omega(x,B) = \int_B d\omega_x.$$

Observe that $\omega(\cdot, B)$ is a positive harmonic function. Recall that the harmonic measure in the ball $B(z,r)$ is equal to

$$d\omega_w(\xi) = \frac{1}{\sigma(\partial B(z,r))} \frac{r^2 - |w-z|^2}{|\xi - w|^{2n}} d\sigma \tag{2.1}$$

for $w \in B(z,r)$, where $d\sigma$ is the Lebesgue measure on $\partial B(z,r)$.

Let f be a bounded function on $\partial\Omega$. We will denote the Perron envelope for subharmonic functions by

$$H_f = \sup\{u \in \mathrm{SH}(\Omega) : \limsup_{z \to \partial\Omega} u \le f\}.$$

It it a well-known fact that H_f is harmonic in the regular, bounded domain $\Omega \subset \mathbb{R}^n$ and

$$H_f(x) = \int_{\partial\Omega} f \, d\omega_x.$$

Let us finally recall the definition of the relative capacity and of convergence in capacity.

Definition 2.3. The *relative capacity* of the Borel set $E \subset \Omega \subset \mathbb{C}^n$ with respect to Ω is defined in [BT2] as

$$\mathrm{cap}(E, \Omega) = \sup\left\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega), \ -1 \le u \le 0 \right\}.$$

Definition 2.4. Let $u_j, u \in \mathrm{PSH}(\Omega)$. We say that a sequence u_j *converges in capacity* to u if, for any $\varepsilon > 0$ and $K \subset\subset \Omega$,

$$\lim_{j\to\infty} \mathrm{cap}(K \cap \{|u_j - u| > \varepsilon\}) = 0.$$

We will need the following well-known proposition (see [K]).

Proposition 2.5. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let a sequence $u_j \in \mathrm{PSH}(\Omega)$ be uniformly bounded. If u_j increases almost everywhere (with respect to the Lebesgue measure) to some $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ as $j \to \infty$, then $u_j \to u$ in capacity as $j \to \infty$.*

Proof. Fix $\varepsilon > 0$ and $K \subset\subset \Omega$. Because the functions u_j and u are uniformly bounded, we can assume that $u_j < -1$ and $u < -1$ in Ω. Let ψ be the exhaustion function for Ω. Then there exists a constant $A > 0$ such that $A\psi < u_j$ on K for all $j \ge 1$. Now define

$$v_j = \max(u_j, A\psi),$$
$$v = \max(u, A\psi).$$

Observe that v_j increases a.e. to v as $j \to \infty$, so $(dd^c v_j)^n \to (dd^c v)^n$ weakly as $j \to \infty$ (see [BT2]). We also know that $v_j = v = A\psi$ in some neighborhood of $\partial\Omega$ and that $v_j = u_j$ and $v = u$ on K. By [X] we get $v_j \to v$ in capacity as $j \to \infty$, so

$$\lim_{j\to\infty} \operatorname{cap}(K \cap \{|u_j - u| > \varepsilon\}) = \lim_{j\to\infty} \operatorname{cap}(K \cap \{|v_j - v| > \varepsilon\}) = 0,$$

completing the proof. □

3. The Cegrell Classes with Upper Semicontinuous Boundary Data

The Cegrell classes were first introduced in [C1]. More information about Cegrell classes can be found in [C2; C3; A; ACz].

Recall that a bounded domain Ω in $\mathbb{C}^n$ is called *hyperconvex* (see [Kl]) if every boundary point of Ω admits a weak plurisubharmonic barrier or (equivalently) if there exists a smooth, strictly plurisubharmonic function ψ in Ω such that $\lim_{z\to\partial\Omega} \psi(z) = 0$.

DEFINITION 3.1. Let Ω be bounded hyperconvex domain in $\mathbb{C}^n$. We say that $u \in \mathcal{E}_0$ if u is a bounded plurisubharmonic function in Ω, $\lim_{z\to\partial\Omega} u(z) = 0$, and

$$\int_\Omega (dd^c u)^n < +\infty.$$

For $p \geq 1$, define $\mathcal{E}_p$ to be the class of all plurisubharmonic functions u in Ω for which there exists a decreasing sequence $u_j \in \mathcal{E}_0$ such that $u_j \searrow u$ as $j \to \infty$ and

$$\sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty. \tag{3.1}$$

If the sequence u_j can be chosen so that it satisfies also the condition

$$\sup_j \int_\Omega (dd^c u_j)^n < +\infty, \tag{3.2}$$

then we say that u belongs to the class $\mathcal{F}_p$.

We say that $u \in \mathcal{E}$ if u is a plurisubharmonic function in Ω and if, for each $z \in \Omega$, there exist a neighborhood ω of z and a decreasing sequence $u_j \in \mathcal{E}_0$ satisfying (3.2) such that $u_j \searrow u$ on ω as $j \to \infty$. If the sequence can be chosen so that u_j converges pointwise to u on all of Ω, then we say that u belongs to the class $\mathcal{F}$.

Now we recall the definition of the Cegrell class with continuous boundary data.

DEFINITION 3.2. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{E}_p, \mathcal{F}_p, \mathcal{F}, \mathcal{E}\}$, let Ω be bounded hyperconvex domain in $\mathbb{C}^n$, and let $f \in \mathcal{C}(\partial\Omega)$ be such that $\lim_{z\to w} U(f,0)(z) = f(w)$ for all $w \in \partial\Omega$. A plurisubharmonic function u on Ω belongs to the class $\mathcal{K}(f)$ if there exists a $v \in \mathcal{K}$ such that

$$U(f,0) \geq u \geq v + U(f,0). \tag{3.3}$$

A bounded domain Ω in $\mathbb{C}^n$ is called *B-regular* (see [S]) if every boundary point of Ω admits a strong plurisubharmonic barrier or (equivalently) if there exists a smooth, strictly plurisubharmonic function φ in Ω such that $\lim_{z\to\partial\Omega} \varphi(z) = 0$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k \geq |\alpha|^2 \quad \forall \alpha \in \mathbb{C}^n.$$

Sibony [S] showed that these conditions are equivalent to the condition that every continuous function on $\partial\Omega$ is extendable to a continuous plurisubharmonic function on Ω.

As in the continuous case, we can define the Cegrell classes with upper semicontinuous boundary data.

Definition 3.3. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{E}_p, \mathcal{F}_p, \mathcal{F}, \mathcal{E}\}$, let Ω be bounded B-regular domain in $\mathbb{C}^n$, and let f be an upper semicontinuous function on $\partial\Omega$. A plurisubharmonic function u on Ω belongs to the class $\mathcal{K}(f)$ if there exists a $v \in \mathcal{K}$ such that

$$U(f,0) \geq u \geq v + U(f,0).$$

Later we shall give an example (see Example 3.11) showing that the Cegrell classes with upper semicontinuous boundary values are different from the Cegrell classes with continuous boundary values.

Definition 3.4. Define the class $\mathcal{F}^a$ (resp. $\mathcal{F}^a(f)$) to be the set of all $u \in \mathcal{F}$ (resp. $u \in \mathcal{F}(f)$) such that $(dd^c u)^n$ vanishes on every pluripolar set; that is,

$$\int_E (dd^c u)^n = 0$$

for any pluripolar set E.

Cegrell proved in [C1] that the comparison principle holds in the class $\mathcal{F}^a$. Later, Åhag [A] and Cegrell [C2] separately proved that the comparison principle holds also in the class $\mathcal{F}^a(f)$. Following the method used in [C2] and [A], one can prove the following theorem.

Theorem 3.5. *Let Ω be bounded B-regular domain in $\mathbb{C}^n$, let f and g be bounded upper semicontinuous functions on $\partial\Omega$ such that $f \geq g$, and let $u \in \mathcal{F}^a(f)$ and $v \in \mathcal{F}(g)$. If $(dd^c u)^n \leq (dd^c v)^n$, then $v \leq u$ on Ω.*

Now we define the following class of measures.

Definition 3.6. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Define $\mathcal{MF}^a = \mathcal{MF}^a(\Omega)$ to be the set of all positive, finite measures μ on Ω such that μ vanishes on all pluripolar sets in Ω.

Definition 3.7. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Define $\mathcal{MF}_p = \mathcal{MF}_p(\Omega)$ $(p \geq 1)$ to be the set of all positive, finite measures μ on Ω for which there exists a constant $A > 0$ such that, for every $\varphi \in \mathcal{E}_0$,

$$\int_\Omega (-\varphi)^p \, d\mu \leq A \left(\int_\Omega (-\varphi)^p (dd^c \varphi)^n \right)^{p/(n+p)}.$$

Cegrell [C1] proved that, for every $p \geq 1$, $\mathcal{MF}_p \subset \mathcal{MF}^a$.

The first part of the following theorem was proved in [C1], and the second part was proved in [A].

THEOREM 3.8. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. For every $\mu \in \mathcal{MF}^a$ (resp. $\mu \in \mathcal{MF}_p$, $p \geq 1$) there exists a unique $u \in \mathcal{F}$ (resp. $u \in \mathcal{F}_p$) such that $(dd^c u)^n = \mu$. Moreover, if $f \in \mathcal{C}(\partial\Omega)$ and if $\lim_{z\to w} U(f,0)(z) = f(w)$ for all $w \in \partial\Omega$, then for every $\mu \in \mathcal{MF}^a$ (resp. $\mu \in \mathcal{MF}_p$, $p \geq 1$) there exists a unique $u \in \mathcal{F}(f)$ (resp. $u \in \mathcal{F}_p(f)$) such that $(dd^c u)^n = \mu$ and $\limsup_{z\to w} u(z) = f(w)$ for all $w \in \partial\Omega$.*

Next we prove the following lemma.

LEMMA 3.9. *Let Ω be a bounded, B-regular domain in $\mathbb{C}^n$, let $\mu \in \mathcal{MF}^a$ (resp. $\mu \in \mathcal{MF}_p$, $p \geq 1$), and let f be a bounded, upper semicontinuous function on $\partial\Omega$. Then $U(f,\mu) \in \mathcal{F}(f)$ (resp. $U(f,\mu) \in \mathcal{F}_p(f)$), $(dd^c U(f,\mu))^n = \mu$, and $\limsup_{z\to w} U(f,\mu)(z) \leq f(w)$ for all $w \in \partial\Omega$. Moreover, if f is continuous at $w_0 \in \partial\Omega$ then $\limsup_{z\to w_0} U(f,\mu)(z) = f(w_0)$.*

Proof. Assume that $\mu \in \mathcal{MF}^a$; the proof is analogous in the case $\mu \in \mathcal{MF}_p$. First observe that $\mathrm{PB}(f,\mu) \neq \emptyset$ because, by Theorem 3.8, $U(\inf_{\partial\Omega} f, \mu) \in \mathrm{PB}(f,\mu)$. Since f is an upper semicontinuous function, there exists a sequence of continuous functions f_j decreasing to f. By Theorem 3.8 we know that there exists a sequence $u_j \in \mathcal{F}(f_j)$ such that $(dd^c u_j)^n = \mu$ and $\limsup_{z\to w} u_j(z) = f_j(w)$ for all $w \in \partial\Omega$. By Theorem 3.5, u_j is decreasing and so there exists a $v \in \mathrm{PSH}(\Omega)$ such that $u_j \searrow v$ as $j \to \infty$ and $(dd^c v)^n = \mu$. By Theorem 3.8 there exists a $\varphi \in \mathcal{F}$ such that $(dd^c\varphi)^n = \mu$. Hence from Theorem 3.5 we obtain

$$U(0, f_j) \geq u_j \geq \varphi + U(0, f_j),$$

and if $j \to \infty$ then we have

$$U(0, f) \geq v \geq \varphi + U(0, f).$$

So $v \in \mathcal{F}(f)$, which implies that $v \leq U(f,\mu)$.

On the other hand, if $u \in \mathrm{PB}(f,\mu)$ and $u \in \mathcal{F}(g)$, where g is an upper semicontinuous function on $\partial\Omega$ and $g \leq f$, then

$$\limsup_{z\to w} u(z) \leq g(w) \leq f(w) \leq f_j(w)$$

for every $w \in \partial\Omega$; thus, by Theorem 3.5, $u \leq u_j$ in Ω. Taking supremum over all u yields $U(f,\mu) \leq v$, so $U(f,\mu) = v$. This implies that $U(f,\mu) \in \mathcal{F}(f)$, since $U(0,\mu) \in \mathcal{F}$ and

$$U(f,0) \geq U(f,\mu) \geq U(0,\mu) + U(f,0).$$

Now suppose that f is continuous at $w_0 \in \partial\Omega$. By Proposition 2.2 there exists a $g \in \mathcal{C}(\partial\Omega)$ such that $g \leq f$ and $g(w_0) = f(w_0)$. Hence there exists a $u \in \mathcal{F}(g)$ such that $(dd^c u)^n = \mu$ and $\limsup_{z\to w} u(z) = g(w)$ for all $w \in \partial\Omega$. Now observe that $u \in \mathrm{PB}(f,\mu)$, so $u \leq U(f,\mu)$ on Ω. We also have

$$f(w_0) = g(w_0) = \limsup_{z\to w_0} u(z) \leq \limsup_{z\to w_0} U(f,\mu)(z) \leq f(w_0).$$

This implies that $\limsup_{z\to w_0} U(f,\mu)(z) = f(w_0)$. □

REMARK. If we assume in Lemma 3.9 that the measure $\mu = (dd^c u)^n$ where $u \in \mathcal{E}_0$, then $U(f,\mu)$ is a bounded plurisubharmonic function. Moreover, if f is continuous at $w_0 \in \partial\Omega$ then $\lim_{z\to w_0} U(f,\mu)(z) = f(w_0)$.

The following example shows that it is not possible to obtain that, for every $w \in \partial\Omega$, $\limsup_{z\to w} U(f,\mu)(z) = f(w)$.

EXAMPLE 3.10. Define the upper semicontinuous function f on the boundary of the unit ball in $\mathbb{C}^2$ by

$$f(z,w) = \begin{cases} 1 & \text{if } |z| = 1, \\ 0 & \text{if } |z| < 1. \end{cases}$$

Note that the sequence $f_j(z,w) = |z|^j$ is decreasing to f on $\partial B(0,1)$ as $j \to \infty$. Moreover, the function $u_j(z,w) = |z|^j$ satisfies the conditions $(dd^c u_j)^2 = 0$ and $u_j = f_j$ on $\partial B(0,1)$. So by Lemma 3.9 we have $U(f,0) = \lim_{j\to\infty} u_j = 0$, which implies that $\limsup_{z\to\partial B(0,1)} U(f,0) \neq f$.

The next example shows that Cegrell classes with upper semicontinuous boundary values are nontrivial generalizations of Cegrell classes with continuous boundary values. We show that there exist both an upper semicontinuous function f on the boundary of the unit ball $\partial B(0,1)$ in $\mathbb{C}^2$ and a bounded plurisubharmonic function u on $B(0,1)$ such that

$$u \in \mathcal{E}_0(f) \Big\backslash \Big(\bigcup_{g\in\mathcal{C}(\partial B)} \mathcal{E}_0(g) \Big).$$

EXAMPLE 3.11. Let $B(0,1)$ be the unit ball in $\mathbb{C}^2$. We define on $\partial B(0,1)$ the function

$$f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1,$$

where $r_1^2 + r_2^2 = 1$ and $\theta_1, \theta_2 \in (0, 2\pi]$. Observe that f is an upper semicontinuous function but not a continuous function. By Lemma 3.9, $\lim_{z\to w} U(f,0)(z) = f(w)$ for all $w \in \partial B(0,1)$ for which $f(w) \neq 2\pi$. By definition we have $U(f,0) \in \mathcal{E}_0(f)$, but there does not exist a $g \in \mathcal{C}(\partial B(0,1))$ such that $U(f,0) \in \mathcal{E}_0(g)$. To prove this, suppose by contradiction that there exists a $g \in \mathcal{C}(\partial B(0,1))$ such that $U(f,0) \in \mathcal{E}_0(g)$. Then $\lim_{z\to w} U(f,0)(z) = g(w)$ for all $|w| = 1$; but this implies that $g(w) = f(w)$ for all $w \in \partial B(0,1)$, which is impossible because f has no continuous extension on the whole boundary.

From Lemma 3.9 we have the following corollary.

COROLLARY 3.12. *Let Ω be a bounded, B-regular domain in $\mathbb{C}^n$ and let f be a bounded, upper semicontinuous function on $\partial\Omega$. Then, for every $\mu \in \mathcal{MF}^a$ (resp.*

$\mu \in \mathcal{MF}_p$, $p \geq 1$), *there exists a unique* $u \in \mathcal{F}(f)$ (*resp.* $u \in \mathcal{F}_p(f)$) *such that* $(dd^c u)^n = \mu$.

4. Main Results

The main theorem of this paper is as follows.

THEOREM 4.1. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded B-regular domain, let* $\mu \in \mathcal{MF}^a$, *let* f_j *be a uniformly bounded sequence of upper semicontinuous functions on* $\partial\Omega$, *and let* f *be a bounded upper semicontinuous function on* $\partial\Omega$. *If* $f_j \to f$ *pointwise as* $j \to \infty$, *then* $U(f_j, \mu) \to U(f, \mu)$ *in capacity as* $j \to \infty$.

REMARK. Theorem 4.1 does not hold if we replace pointwise convergence $f_j \to f$ by convergence in $L^p(d\sigma)$, where $d\sigma$ is the surface measure and $p \geq 1$; this was pointed out to the author by Professor Urban Cegrell. See [R] for the example of a sequence of continuous functions f_j defined on the boundary of the unit ball such that, as $j \to \infty$, $f_j \to 0$ in $L^p(d\sigma)$ (for all $p \geq 1$) but $U(f_j, 0) \nrightarrow 0$ in capacity.

In order to prove Theorem 4.1, we need the following lemma.

LEMMA 4.2. *Let* Ω *be a bounded domain in* $\mathbb{C}^n$, *and let the sequence* $f_j : \partial\Omega \to \mathbb{R}$ *be uniformly bounded. If* $f_j \geq 0$ *and* $f_j \to 0$ *in* $L^1(\omega_{z_0})$, *as* $j \to \infty$, *for some* $z_0 \in \Omega$, *then* $U(f_j, 0) \to 0$ *locally uniformly as* $j \to \infty$.

Proof. Note that

$$0 \leq U(f_j, 0)(z) \leq H_{f_j}(z) = \int_{\partial\Omega} f_j \, d\omega_z.$$

Fix a compact set $K \subset \Omega$. By Harnack's inequality there exists a constant $C > 0$ such that

$$\|U(f_j, 0)\|_K \leq C \int_{\partial\Omega} f_j \, d\omega_{z_0}. \tag{4.1}$$

By our assumptions we know that $f_j \to 0$ in $L^1(\omega_{z_0})$ as $j \to \infty$, so we have proved that $U(f_j, 0) \to 0$ locally uniformly as $j \to \infty$. □

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. First we prove the theorem for $f = 0$ and $\mu = 0$. Let us define the upper semicontinuous functions $h_k = \min(f_k, 0)$ and $g_k = \max(f_k, 0)$. Observe that

$$h_k \leq f_k \leq g_k$$

and

$$U(h_k, 0) \leq U(f_k, 0) \leq U(g_k, 0). \tag{4.2}$$

We want to prove that $U(f_k, 0) \to 0$ in capacity as $k \to \infty$. By (4.2) it is enough to show that $U(g_k, 0) \to 0$ and $U(h_k, 0) \to 0$ in capacity as $k \to \infty$. From Lemma

4.2 we get $U(g_k, 0) \to 0$ locally uniformly as $k \to \infty$, so also $U(g_k, 0) \to 0$ in capacity as $k \to \infty$.

It remains to check that $U(h_k, 0) \to 0$ in capacity as $k \to \infty$. We define the upper semicontinuous function $l_k = \inf_{j \geq k} h_j$. Then

$$l_k \leq h_k \leq 0,$$

and $l_k \to 0$ as $k \to \infty$ since $h_k \to 0$ as $k \to \infty$.

Now observe that $U(l_k, 0)$ is increasing, so there exists a ψ such that $U(l_k, 0) \nearrow \psi$; hence, by Proposition 2.5, $\psi^* \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$, $(dd^c\psi^*)^n = 0$, and $U(l_k, 0) \to \psi^*$ in capacity as $k \to \infty$. We must show that $\psi^* = 0$ in Ω. By Proposition 2.1 there exists a dense G_δ-set $G_k \subset \partial\Omega$ such that l_k is continuous on G_k. Let

$$G = \bigcap_k G_k.$$

By Baire's theorem, G is a dense subset of $\partial\Omega$ and so, by Lemma 3.9,

$$0 = \lim_{k\to\infty} l_k(w) \leq \psi^*(w) \leq 0$$

for $w \in G$. We have $\psi^* = 0$ on G and therefore $\psi^* = 0$ on $\partial\Omega$.

Now fix $z \in \Omega$. For all $k \in \mathbb{N}$ there exists a decreasing sequence of nonpositive functions $l_k^j \in \mathcal{C}(\partial\Omega)$ such that $l_k^j \to l_k$ as $j \to \infty$. Moreover, there exists an increasing sequence $j(k)$ such that

$$U(l_k, 0)(z) \geq U(l_k^{j(k)}, 0)(z) - \frac{1}{k}.$$

Let $L_k = l_k^{j(k)}$. Observe that $L_k \to 0$ as $k \to \infty$, since $l_k \to 0$ as $k \to \infty$. Define

$$\varphi = \sup_{k\in\mathbb{N}} U(L_k, 0).$$

Then $\varphi^* \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$, and we will prove that $\lim_{z\to\partial\Omega} \varphi^*(z) = 0$. Fix $w \in \partial\Omega$. Since $\varphi^* \geq U(L_k, 0)$ for all k, we obtain

$$0 \geq \limsup_{z\to w} \varphi^*(z) \geq \liminf_{z\to w} \varphi^*(z) \geq \liminf_{z\to w} U(L_k, 0)(z) = L_k(w).$$

But $\lim_{k\to\infty} L_k(w) = 0$, so $\lim_{z\to w} \varphi^*(z) = 0$.

Observe also that $\liminf_{z\to\partial\Omega}(\varphi^*(z) - \psi^*(z)) \geq 0$ and $\varphi^* \geq \psi^*$ in Ω, so by the comparison principle we have

$$\int_\Omega (dd^c\varphi^*)^n \leq \int_\Omega (dd^c\psi^*)^n = 0.$$

This implies that $(dd^c\varphi^*)^n = 0$ and therefore $\varphi^* = 0$ in Ω. But we also have

$$\psi^*(z) = \Big(\lim_{k\to\infty} U(l_k, 0)(z)\Big)^* \geq \left(\limsup_{k\to\infty}\left(U(L_k, 0)(z) - \frac{1}{k}\right)\right)^* = \varphi^*(z) = 0,$$

which means that $\psi^*(z) = 0$. So we have proved that $\psi^* = 0$ in Ω.

For the general case we define the functions

$$g_j = \max(f_j, f) \quad \text{and} \quad h_j = \min(f_j, f).$$

Then g_j, h_j are upper semicontinuous functions and, as $j \to \infty$, we have $g_j \to f$ and $h_j \to f$. From the definition of g_j and h_j we obtain

$$h_j \le f_j \le g_j,$$

so

$$U(h_j, \mu) - U(f, \mu) \le U(f_j, \mu) - U(f, \mu) \le U(g_j, \mu) - U(f, \mu). \tag{4.3}$$

If $v \in \mathrm{PB}(h_j - f, 0)$ then, for all $w \in \partial\Omega$,

$$\limsup_{z\to w}(v + U(f, \mu))(z) \le \limsup_{z\to w} v(z) + \limsup_{z\to w} U(f, \mu)(z) \le h_j(w).$$

Moreover, $(dd^c(v + U(f, \mu)))^n \ge (dd^c U(f, \mu))^n = \mu$ and so $v + U(f, \mu) \in \mathrm{PB}(h_j, \mu)$. This implies that $v + U(f, \mu) \le U(h_j, \mu)$ and then

$$U(h_j - f, 0) + U(f, \mu) \le U(h_j, \mu). \tag{4.4}$$

In exactly the same way we can prove that

$$U(g_j, 0) - U(f, \mu) \le -U(f - g_j, \mu). \tag{4.5}$$

From (4.3), (4.4), and (4.5) it follows that

$$|U(f_j, \mu) - U(f, \mu)| \le \max(|U((h_j - f), 0)|, |U((f - g_j), 0)|).$$

We know (see [BT2]) that $U((h_j - f), 0) = U((h_j - f), 0)^*$ and $U((f - g_j), 0) = U((f - g_j), 0)^*$ outside a pluripolar set. To finish the proof it is enough to show that $U((f - g_j), 0)^*$ and $U((h_j - f), 0)^*$ approach 0 in capacity as $j \to \infty$.

First we prove that $U((h_j - f), 0)^* \to 0$ in capacity as $j \to \infty$. There exists a decreasing sequence of continuous functions $F_j \searrow f$ as $j \to \infty$. Define the function $H_j = h_j - F_j$. Then H_j is an upper semicontinuous function such that $H_j \to 0$ pointwise as $j \to \infty$. Moreover, $H_j \le h_j - f \le 0$ and so

$$U(H_j, 0) \le U(h_j - f, 0) \le U(h_j - f, 0)^* \le 0.$$

By the first part of the proof we know that $U(H_j, 0) \to 0$ in capacity as $j \to \infty$, so also $U(h_j - f, 0)^* \to 0$ in capacity as $j \to \infty$.

In exactly the same way we can prove that $U((f - g_j), 0)^* \to 0$ in capacity as $j \to \infty$. There exist sequences of continuous functions $G_k^j \searrow g_j$, as $k \to \infty$, for all $j \ge 0$. Let us define $L_j = f - G_j^j$. Then L_j is an upper semicontinuous function such that $L_j \to 0$ pointwise as $j \to \infty$ and

$$L_j \le f - g_j \le 0.$$

Therefore,

$$U(L_j, 0) \le U(f - g_j, 0) \le U(f - g_j, 0)^* \le 0.$$

By the first part of the proof, $U(L_j, 0) \to 0$ in capacity as $j \to \infty$, so also $U(f - g_j, 0)^* \to 0$ in capacity as $j \to \infty$. This finishes the proof of Theorem 4.1. $\square$

The main purpose of this paper is to generalize the following theorem, which was proved by Cegrell and Kołodziej in [CK2].

THEOREM 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain, and let $\mu \in \mathcal{MF}^a$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Let the functions $u_j, u \in \mathcal{F}(\varphi)$ solve the Dirichlet problems*

$$(dd^c u_j)^n = g_j d\mu \quad \text{and} \quad (dd^c u)^n = g d\mu$$

for g_j, g μ-measurable functions with values in $[0,1]$ and satisfying $g_j d\mu \to g d\mu$ as $j \to \infty$. Then $u_j \to u$ in capacity as $j \to \infty$. Moreover, if $\mathcal{A}(\mu)$ denotes the set of all solutions of the Dirichlet problem

$$u \in \mathcal{F}(\varphi), \quad (dd^c u)^n = g d\mu,$$

where g varies over all μ-measurable functions with $0 \le g \le 1$, then in $\mathcal{A}(\mu)$ it follows that convergence in $L^1_{\mathrm{loc}}(\Omega)$ is equivalent to convergence in capacity.

Theorem 4.1 gives us a condition that guarantees the stability of the solutions of the complex Monge–Ampère operator. In this condition we fix the Monge–Ampère mass of the solutions while the boundary values of the solutions remain uniformly bounded. Thus we may prove the following theorem.

THEOREM 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded B-regular domain, and let $\mu \in \mathcal{MF}^a$ and $k > 0$. Denote by $\mathcal{B}(\mu, k)$ the set of all plurisubharmonic functions $U(f, \mu)$, where $f : \partial\Omega \to \mathbb{R}$ is an upper semicontinuous function such that $|f| \le k$. Then, in $\mathcal{B}(\mu, k)$, pointwise convergence is equivalent to convergence in capacity.*

Proof. Let $u_j, u \in \mathcal{B}(\mu, k)$ and let $u_j \to u$ pointwise as $j \to \infty$. Suppose that u_j does not converge in capacity to u as $j \to \infty$. Then, for some $\varepsilon > 0$ and $K \subset\subset \Omega$, there exist a subsequence u_{j_l} and constants $c > 0$ and $N > 0$ such that, for $j_l \ge N$,

$$\operatorname{cap}(K \cap \{|u_{j_l} - u| > \varepsilon\}) \ge c. \tag{4.6}$$

There exist balls $B(z_1, r), \dots, B(z_m, r)$ such that $K \subset \bigcup_{p=1}^m B(z_p, r)$. There exists a subsequence (denoted also by u_{j_l}) such that, for any $B(z_p, r)$ with $1 \le p \le m$, u_{j_l} restricted to $\partial B(z_p, r)$ tend pointwise to u restricted to $\partial B(z_p, r)$ as $l \to \infty$. By Theorem 4.1, $u_{j_l} \to u$ in capacity on $B(z_p, r)$ as $l \to \infty$, so $u_{j_l} \to u$ in capacity on $\bigcup_{p=1}^m B(z_p, r)$ as $l \to \infty$; this contradicts (4.6), concluding the proof. □

Combining Theorem 4.3 and Theorem 4.4, we can obtain a new condition that gives stability of the solutions of the complex Monge–Ampère operator.

THEOREM 4.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly pseudoconvex domain, and let $\mu \in \mathcal{MF}^a$. Let f_j, f be uniformly bounded upper semicontinuous functions on the boundary and let $f_j \to f$ pointwise as $j \to \infty$. Let the functions g_j and g be μ-measurable with values in $[0,1]$ and satisfying $g_j d\mu \to g d\mu$ as $j \to \infty$. Then $U(f_j, g_j d\mu) \to U(f, g d\mu)$ in capacity as $j \to \infty$.*

Proof. Let f_j, f be uniformly bounded upper semicontinuous functions on $\partial\Omega$ such that $f_j \to f$ as $j \to \infty$, and let g_j, g be μ-measurable functions such that

$0 \le g_j, g \le 1$, and $g_j d\mu \to g d\mu$ as $j \to \infty$. Then, by Lemma 3.9, $U(f_j, g_j d\mu) \in \mathcal{F}(f_j)$ and $U(f, gd\mu) \in \mathcal{F}(f)$. From the proof of Theorem 4.1 we can obtain

$$U(\min(f_j - f, 0), |g_j - g| d\mu) \le U(f_j, g_j d\mu) - U(f, g d\mu) \\ \le -U(\min(f - f_j, 0), |g_j - g| d\mu).$$

Hence it is sufficient to show that $U(\min(f_j - f, 0), |g_j - g| d\mu)^* \to 0$ and $U(\min(f - f_j, 0), |g_j - g| d\mu)^* \to 0$ in capacity as $j \to \infty$. We shall prove this for the function $U(\min(f - f_j, 0), |g_j - g| d\mu)^*$; the proof for the second function is analogous.

We have the inequality

$$U(0, |g_j - g| d\mu) + U(\min(f - f_j, 0), 0) \le U(\min(f - f_j, 0), |g_j - g| d\mu)^* \le 0.$$

By Theorem 4.3 we know that $U(0, |g_j - g| d\mu) \to 0$ in capacity as $j \to \infty$. Repeating again the argument from the proof of Theorem 4.1, one can show that $U(\min(f - f_j, 0), 0)^* \to 0$ in capacity as $j \to \infty$. □

Theorem 4.5 yields the following corollary.

COROLLARY 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly pseudoconvex domain, and let $\mu \in \mathcal{MF}^a$ and $k > 0$. Denote by $\mathcal{D}(\mu, k)$ the set of all plurisubharmonic functions $U(f, gd\mu)$, where $f : \partial\Omega \to \mathbb{R}$ is an upper semicontinuous function such that $|f| \le k$ and g is a μ-measurable function such that $0 \le g \le 1$. Then, in $\mathcal{D}(\mu, k)$, pointwise convergence is equivalent to convergence in capacity.*

Proof. The proof follows from Theorem 4.4 and the proof of Theorem 4.5. □

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References

[A] P. Åhag, *The complex Monge–Ampère operator on bounded hyperconvex domains,* Ph.D. thesis, Umeå University, 2002.
[ACz] P. Åhag and R. Czyż, *On the Cegrell classes,* IMUJ Preprint 11 (2004).
[ArG] D. H. Armitage and S. J. Gardiner, *Classical potential theory,* Springer Monogr. Math., Springer-Verlag, London, 2001.
[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge–Ampère equation,* Invent. Math. 37 (1976), 1–44.
[BT2] ———, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.
[C1] U. Cegrell, *Pluricomplex energy,* Acta Math. 180 (1998), 187–217.
[C2] ———, *The comparison principle for the complex Monge–Ampère equation,* Mid Sweden University Research Reports, 10, 2003.
[C3] ———, *The general definition of the complex Monge–Ampère operator,* Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[CK1] U. Cegrell and S. Kołodziej, *The Dirichlet problem for the complex Monge–Ampère operator: Perron classes and rotation-invariant measures,* Michigan Math. J. 41 (1994), 563–569.

[CK2] ———, *The equation of complex Monge–Ampère type and stability of solutions,* IMUJ Preprint 23, 2003.

[Kl] M. Klimek, *Pluripotential theory,* London Math. Soc. Monogr. (N.S.), Oxford Univ. Press, New York, 1991.

[K] S. Kołodziej, *The complex Monge–Ampère equation and methods of pluripotential theory,* Cubo 6 (2004), 259–279.

[R] A. Rashkovskii, *A few examples on boundary values of maximal plurisubharmonic functions,* Mid Sweden University Research Reports, 22, 2000.

[Ru] W. Rudin, *Real and complex analysis,* McGraw-Hill, New York, 1987.

[S] N. Sibony, *Une classe de domaines pseudoconvexes,* Duke Math. J. 55 (1987), 299–319.

[X] Y. Xing, *Continuity of the complex Monge–Ampère operator,* Proc. Amer. Math. Soc. 124 (1996), 457–467.

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