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# On Orientability and Degree of Fredholm Maps

Shuguang Wang

## 1. Introduction

The degree of a map between two manifolds has played important roles in various mathematical areas. Certain orientability is always required in order to make sense of the concept of degree. In the case of finite-dimensional nonorientable manifolds, this goes back to Hopf, Olum, and Steenrod, after Brouwer's pioneering work on orientable manifolds (cf. [9] and references therein). Elworthy and Tromba [4] took the first study in the case of infinite-dimensional Banach manifolds, where they introduced the degree on orientable Fredholm manifolds. This orientability restriction on manifolds is, however, often too severe and unnatural. It was Fitzpatrick, Pejsachowicz, and Rabier [6] who pointed out explicitly that the only requirement was the orientability of *maps* involved rather than that of *manifolds.* (The finite-dimensional version was in Olum's work.) Their approach is based on the concept of parity of paths, which makes it particularly useful in problems dealing with crossing singular strata. Indeed this is often the only practical way to check the orientability of a map. More recently, Benevieri and Furi [1] took another approach to orienting Fredholm maps that is conceptually more clear and seems more natural, since it comes directly from pointwise orientations of all Fredholm operators.

The approach taken in this paper has a more geometric flavor and also provides an instance where geometry and analysis interact nicely. The use of a determinant line bundle that arises from geometry links conveniently the notions of Benevieri–Furi and Fitzpatrick–Pejsachowicz–Rabier. In fact, many properties in [1], [2], and [6] become much easier to understand through our new approach. Conversely, the geometric approach allows us to apply functional analysis tools to some problems in gauge theory involving a real structure, where the relevant manifolds are often nonorientable or with no natural orientation, hence making it necessary to orient relevant maps instead. More details will appear in [10].

## 2. Fredholm Operator Families

We first consider orientability for families of Fredholm operators. To motivate the definition, we start with the case of a finite-dimensional manifold. Here orientability of the manifold can be characterized as the triviality of the orientation

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line bundle, namely the determinant of the tangent bundle of the manifold. To any smooth map, one can also associate the determinant bundle. Indeed this can be carried out for any Fredholm map between two Banach manifolds, which we now review.

Let $\Lambda$ be a topological space and let $E, F$ be Banach spaces. We use $\Phi_n(E, F)$ to denote the set of index-$n$ Fredholm operators with the usual norm topology. Consider a continuous family of operators parameterized by $\Lambda$—namely, a continuous map $h\colon \Lambda \to \Phi_n(E, F)$.

The dimensions $\dim\ker h(\lambda)$ and $\operatorname{coker} h(\lambda)$ can jump at points in $\Lambda$; hence $\ker h$ and $\operatorname{coker} h$ in general do not form vector bundles over $\Lambda$, although $\operatorname{ind} h = \ker h - \operatorname{coker} h$ can be viewed as a virtual bundle in the $K$-theory $KO(\Lambda)$. However, using some elementary algebra involving exact sequences, one can show that the determinant

$$\det\operatorname{ind} h = \wedge^{\max}\ker h \otimes (\wedge^{\max}\operatorname{coker} h)^*$$

is a continuous line bundle on $\Lambda$, where the maximum wedge product $\wedge^{\max}\ker h = \wedge^{\dim\ker h}\ker h$ and where the $^*$ signifies the dual space. Since the construction will be used afterwards, let us sketch the argument; the interested reader can check [3, Chap. 5] for more details.

It suffices to show that $\det\operatorname{ind} h$ is a continuous line bundle locally. At any point $\lambda_0 \in \Lambda$, since $\dim\operatorname{coker} h(\lambda_0) < \infty$ and since surjective operators form an open set, it is possible to find a neighborhood $U$ of $\lambda_0$, a vector space $V$ of a finite dimension $N \geq \dim\operatorname{coker} h(\lambda)$, and a linear map $\varphi\colon V \to F$ such that $\varphi$ stabilizes $h$ on $U$; namely, $h \oplus \varphi\colon E \oplus V \to F$ is surjective on $U$. Thus $\ker(h \oplus \varphi) \to U$ is a vector bundle of rank $\operatorname{ind} h + N$, and there exists a canonical isomorphism

$$\mu : \det\operatorname{ind} h \approx \wedge^{\max}\ker(h \oplus \varphi) \otimes (\wedge^{\max} V)^* \tag{1}$$

on $U$ that induces a continuous line bundle structure on the left side over $U$. As an elementary algebraic result, the isomorphism (1) in turn follows from the canonical isomorphism

$$\wedge^{\max}\ker h \otimes \wedge^{\max} V \approx \wedge^{\max}\ker(h \oplus \varphi) \otimes \wedge^{\max}\operatorname{coker} h,$$

which is associated with the exact sequence

$$0 \to \ker h \to \ker(h \oplus \varphi) \to V \xrightarrow{\varphi} \operatorname{coker} h \to 0. \tag{2}$$

(That is, collect even and odd terms together and then take the tensor product for each group.)

REMARK. In order to glue together the line bundles on two different open sets $U, U'$, one must choose consistently the parity of the dimensions $N, N'$, an overlooked requirement that was recently pointed out by Froyshov [7]. (Of course, one can always increase the value of $N$ by any integer.) Precisely, let $\varphi'\colon V' \to F$ be a second map satisfying a similar condition as $\varphi$. Then, as shown in [7], the transition function $\mu' \circ \mu^{-1}$ on $U \cap U'$ is $(-1)^{(N+N')\dim\ker h}$ up to a continuous factor. Since $\dim\ker h(\lambda) \bmod 2$ is not a local constant in general, one needs to impose $N + N' \equiv 0 \bmod 2$ to guarantee the continuity of $\mu' \circ \mu^{-1}$. Using either even- or odd-dimensional vector spaces $V$ throughout the stabilizing, one obtains two continuous determinant bundles, which are then naturally isomorphic via the homeomorphism of fiberwise multiplication by $(-1)^{\dim\ker h(x)}$. For certainty, we will work with even parity in this paper. (According to the referee, there is another approach to the topology of $\det\operatorname{ind} h$ that is included in a forthcoming book by P. Kronheimer and T. Mrowka.)

Note that when $h$ is an isomorphism (i.e., when $\ker h = \operatorname{coker} h = \mathbf{R}^0 = \{0\}$), one should apply the convention that $\wedge^{\max}\mathbf{R}^0$ equals $\mathbf{R}$ canonically, as is required in the foregoing argument.

We now concentrate on the case of index-0 Fredholm operators for the consideration of degree. Given that the determinant line bundle characterizes the orientability of a finite-dimensional manifold, it seems natural to make the following definition.

Definition. Let $h\colon \Lambda \to \Phi_0(E, F)$ be a continuous family of index-0 Fredholm operators. We say $h$ is *$*$-orientable* (for lack of better terminology) if the determinant line bundle $\det\operatorname{ind} h$ is orientable. If orientable, a *$*$-orientation* of $h$ is that of $\det\operatorname{ind} h$.

In other words, $h$ is $*$-orientable if and only if $\det\operatorname{ind} h$ is trivial or, equivalently, iff the bundle has a nowhere vanishing section (i.e., a trivilization). A $*$-orientation is then an equivalence class of trivilizations in which any two differ by a factor of a positive function.

Note that our definition is not redundant, since the entire family $\Phi_0(E, F)$ is not orientable in general—for example, when $E, F$ are infinite-dimensional separable Hilbert spaces (by a well-known result of Kuiper).

It turns out that our formulation is closely related to that of Benevieri and Furi [1]. The definition of their orientation is recalled here for the reader's convenience. Consider a Fredholm operator $L \in \Phi_0(E, F)$. A *corrector* $A\colon E \to F$ of $L$ is by definition a finite-rank operator; that is, $\dim\operatorname{Im} A < \infty$ such that $L + A\colon E \to F$ is an isomorphic operator (equivalently $L + A$ is surjective, since its index is 0). Denote the set of all correctors by $\mathcal{C}(L)$, and let $A' \in \mathcal{C}(L)$ be another corrector. Consider the following automorphism on $F$:

$$\begin{aligned} T &:= (L + A)(L + A')^{-1} \\ &= (L + A' + A - A')(L + A')^{-1} \\ &= I + (A - A')(L + A')^{-1}. \end{aligned}$$

Clearly $S := (A - A')(L + A')^{-1}$ has a finite rank, which implies that $\det T$ is well-defined. (Indeed, $\det T = \det(T|_{\operatorname{Im} S}\colon \operatorname{Im} S \to \operatorname{Im} S)$.) Then an equivalence relation can be defined on $\mathcal{C}(L)$ as $A \sim A'$ if $\det T = \det(L + A)(L + A')^{-1} > 0$. A *Benevieri–Furi orientation* $\alpha(L)$ of $L$ is then just one of the two equivalence classes in $\mathcal{C}(L)/{\sim}$, and each corrector in the chosen class is called a *positive* corrector of the Benevieri–Furi orientation. In particular, if $L$ is already an isomorphic operator then it carries a canonical Benevieri–Furi orientation represented by the trivial corrector $A = 0$.

Given a continuous family $h\colon \Lambda \to \Phi_0(E, F)$, we call $h$ Benevieri–Furi orientable if $h$ carries a Benevieri–Furi orientation—namely, a continuous choice of orientations $\alpha(\lambda)$ of $h(\lambda)$ for $\lambda \in \Lambda$. Continuous choice means that $\alpha$ can be represented by the same corrector locally; equivalently, any positive corrector of $\alpha$ at a point is a positive corrector of $\alpha$ in a neighborhood of the point. Again, as surjective operators form an open set, $h$ is always locally Benevieri–Furi orientable.

The proof of the next theorem will require a slight extension of the previous constructions to the bundle versions. A continuous family $h\colon \Lambda \to \Phi_n(E, F)$ can be viewed as a continuous homomorphism $h\colon \underline{E} \to \underline{F}$ between the trivial vector bundles. In general, we can consider a continuous Fredholm homomorphism $h\colon \tilde{E} \to \tilde{F}$ between two bundles of Banach fibers over $\Lambda$. Then the determinant bundle $\det\operatorname{ind} h$ can be topologized locally using a bundle homomorphism $\varphi\colon \tilde{V} \to \tilde{F}$ over $U$, where $\tilde{V}$ is a vector bundle of a finite (even) rank such that $h \oplus \varphi\colon \tilde{E} \oplus \tilde{V} \to \tilde{F}$ is surjective fiberwise on $U$. Then $\det\operatorname{ind} h$ inherits a topology using a canonical isomorphism similar to (1):

$$\mu\colon \det\operatorname{ind} h \approx \wedge^{\max} \ker(h \oplus \varphi) \otimes (\wedge^{\max}\tilde{V})^*.$$

Analogously, the bundle homomorphism $h$ with index $n = 0$ is said to have a Benevieri–Furi orientation $\alpha(\lambda)$ if locally there is a *continuous* bundle homomorphism $A\colon \tilde{E} \to \tilde{F}$ on $U$ with a fiberwise finite rank such that $h + A\colon \tilde{E} \to \tilde{F}$ is an isomorphism on $U$ and $A(\lambda) \in \alpha(\lambda)$ for all $\lambda \in U$.

THEOREM 1. *Suppose $\Lambda$ is a locally connected topological space. Then a continuous family $h\colon \Lambda \to \Phi_0(E, F)$ is $*$-orientable if and only if it is orientable in the sense of Benevieri–Furi. Moreover, if $h$ is orientable then there is a canonical correspondence between the two orientation sets.*

*Proof.* We first establish the canonical (algebraic) correspondence between the pointwise orientations in the two setups. Set $L = h(\lambda_0)$ at a point $\lambda_0 \in \Lambda$ and let $\alpha \in \mathcal{C}(L)/\!\sim$ be a Benevieri–Furi orientation class of $L$. We intend to assign a unique orientation class $\tilde{\alpha}$ of the fiber $\det\operatorname{ind} h(\lambda_0) = \wedge^{\max} \ker L \otimes (\wedge^{\max} \operatorname{coker} L)^*$ of the determinant bundle. The idea is to use a finite-dimensional reduction in the Benevieri–Furi theory that parallels the finite-dimensional stabilizing spaces in the definition of determinant bundles.

Choose an even-dimensional vector space $F_1$ so that $F = \operatorname{Im} L + F_1$. Let $E_1 = L^{-1}(F_1)$. Then $L$ restricts to an index-0 operator $L_1\colon E_1 \to F_1$ and, in particular, $\dim E_1 = \dim F_1$ is finite. Choose any vector space $E_0$ so that $E = E_0 \oplus E_1$, and let $F_0 = L(E_0)$. Then $L$ restricts to an isomorphism $L_0\colon E_0 \to F_0$ and $F = F_0 \oplus F_1$. Moreover $L = L_0 \oplus L_1$, $\ker L = \ker L_1$, and $\operatorname{coker} L = \operatorname{coker} L_1$ naturally (and independent of the choice of $E_0$). Therefore we have a natural isomorphism

$$\det\operatorname{ind} L = \det\operatorname{ind} L_1. \tag{3}$$

Any corrector $A_1\colon E_1 \to F_1$ of $L_1$ yields a corrector $A = 0 \oplus A_1\colon E \to F$ of $L$. Further, two correctors $A_1, A_1'$ are equivalent if and only if $A, A'$ are equivalent. Hence there is a canonical one-to-one correspondence between the Benevieri–Furi orientation classes of $L_1$ and $L$.

Choose a corrector $A_1$ of $L_1$ that is compatible with $\alpha$, namely, such that $A = 0 \oplus A_1 \in \alpha$. (So $A_1$ represents the orientation class $\alpha_1$ of $L_1$ that corresponds to $\alpha$.) Now the isomorphism $K := L_1 + A_1 \colon E_1 \to F_1$ gives an isomorphism $\wedge K \colon \wedge^{\max} E_1 \to \wedge^{\max} F_1$ and hence a nonzero vector in $(\wedge^{\max} E_1)^* \otimes \wedge^{\max} F_1$, the last being a 1-dimensional space. We can therefore take the dual vector $s$ of $\wedge K$ in $\wedge^{\max} E_1 \otimes (\wedge^{\max} F_1)^*$. The exact sequence

$$0 \to \ker L_1 \to E_1 \xrightarrow{L_1} F_1 \to \operatorname{coker} L_1 \to 0 \tag{4}$$

yields a canonical isomorphism $\det \operatorname{ind} L_1 = \wedge^{\max} E_1 \otimes (\wedge^{\max} F_1)^*$. Combining this with (3) gives $\det \operatorname{ind} L = \wedge^{\max} E_1 \otimes (\wedge^{\max} F_1)^*$. Hence we have a well-defined nonzero vector $s \in \det \operatorname{ind} L$. Define $\tilde{\alpha} = [s]$ to be the orientation class of the fiber $\det \operatorname{ind} L$ associated to the Benevieri–Furi orientation class $\alpha$.

We need to check that $\tilde{\alpha}$ is independent of all choices made in the process. Independence of $A_1$: If $A_1' \sim A_1$ is another corrector, then $(\wedge K)^{-1} \circ (\wedge K') \colon \wedge^{\max} E_1 \to \wedge^{\max} E_1$ is equal to $\det[(L_1 + A_1)^{-1}(L_1 + A_1')]$, which is positive. Hence $[s] = [s']$. Independence of $E_0$: the choice of $E_0$ only affects $A$ up to equivalence, hence not the class $[s]$. Independence of $F_1$: Suppose $F_1'$ is another even-dimensional vector space satisfying $F = \operatorname{Im} L + F_1'$. We can assume $F_1' \supset F_1$ without loss of generality. Define $E_1' = L^{-1}(F_1')$ and $L_1' \colon E_1' \to F_1'$, similar to $E_1$ and $L_1$. Since $\operatorname{Im} L + F_1' = \operatorname{Im} L + F_1$ it is easy to see that $F_1' = \operatorname{Im} L_1' + F_1$. Replace $F$ by $F_1'$ and $L$ by $L_1'$ and repeat the foregoing construction. Then a corrector $A_1$ of $L_1$ yields a corrector $A_1'$ of $L_1'$, and $A_1$ is compatible with $\alpha$ if and only if $A_1'$ is compatible with $\alpha$. Furthermore, the nonzero vector $s' \in \wedge^{\max} E_1' \otimes (\wedge^{\max} F_1')^*$ associated to $A_1'$ is the vector $s \in \wedge^{\max} E_1 \otimes (\wedge^{\max} F_1)^*$, after both vector spaces are naturally identified with $\det \operatorname{ind} L$. (Recall that the corrector 0 should correspond to the vector 1 in the determinant fiber of the isomorphism $L_0'$.)

One sees that $-\alpha$ corresponds to $-\tilde{\alpha}$ by reversing $A_1$. Thus we have established a one-to-one correspondence between the orientation classes of $L$ and $\det \operatorname{ind} L$.

Next we consider the topological part. Suppose $h$ is Benevieri–Furi orientable with continuous orientation $\alpha(\lambda)$, $\lambda \in \Lambda$. We intend to show $\det \operatorname{ind} h$ is orientable by showing that $\tilde{\alpha}(\lambda)$ is continuous (i.e., locally represented by continuous sections). We continue with the preceding construction. In a neighborhood $U$ of $\lambda_0$, $F = \operatorname{Im} h(\lambda) + F_1$ continues to hold for a fixed $F_1$. So we have the similar decompositions $E = E_0(\lambda) \oplus E_1(\lambda)$, $F = F_0(\lambda) \oplus F_1$, and $h(\lambda) = h_0(\lambda) \oplus h_1(\lambda)$. By continuity of $\alpha(\lambda)$ it follows that $A = 0 \oplus A_1 \colon E \to F$ is in $\alpha(\lambda)$ for all $\lambda \in U$. Note that $\dim E_1(\lambda) = \dim F_1$ is constant (and even), so $\tilde{E}_1 = E_1(\lambda) \to U$ gives a subbundle of $\underline{E}$ on $U$. Moreover, the natural bundle isomorphism

$$\wedge \tilde{E}_1 \otimes (\wedge F_1)^* \to \det \operatorname{ind} h_1 = \det \operatorname{ind} h \tag{5}$$

arising from (3) and (4) is continuous on $U$, since $\dim F_1$ is even and since $\det \operatorname{ind} h$ has been given the even-parity topology throughout this paper.

Consider the bundle homomorphism $h_1(\lambda) \colon \tilde{E}_1 \to (U \times F_1)$ once more. Here we need to use the bundle version of the Benevieri–Furi theory outlined previously. Extend $A_1(\lambda_0) = A_1$ to a continuous bundle homomorphism $A_1(\lambda) \colon \tilde{E}_1 \to (U \times F_1)$ over $U$. Then $A(\lambda) = 0 \oplus A_1(\lambda) \colon E \to F$ is a continuous family of correctors of $h(\lambda)$ on $U$. By definition, $A(\lambda)$ represents $\alpha(\lambda)$ at $\lambda_0$ and hence (by continuity of $\alpha$) should represent $\alpha$ on $U$. Thus $A_1(\lambda)$ represents the orientation $\alpha_1(\lambda)$ on $U$. The bundle isomorphism $h_1(\lambda)+A_1(\lambda)\colon \tilde{E}_1 \to (U \times F_1)$ yields a continuous section $s = s(\lambda)$ of $\det \operatorname{ind} h$ on $U$ via the isomorphism (5). Since $A_1(\lambda) \in \alpha_1(\lambda)$, $s(\lambda)$ represents $\tilde{\alpha}(\lambda)$ on $U$ by definition of $\tilde{\alpha}$. Thus $\tilde{\alpha}$ is represented locally by a continuous section at each point $\lambda_0$ and, as a consequence, the determinant bundle $\det \operatorname{ind} h$ is orientable on $\Lambda$.

Conversely, suppose $\det \operatorname{ind} h$ is orientable and we are given a family of fiberwise orientations $\tilde{\alpha}$ that is locally represented by continuous sections of $\det \operatorname{ind} h$. We need to show that the corresponding pointwise-defined Benevieri–Furi orientation $\alpha$ is continuous. Take any point $\lambda_0$ and a neighborhood $U$. One can assume $U$ is connected since $\Lambda$ is locally connected. Since $h$ is locally orientable in Benevieri–Furi and $*$ senses both, we have exactly two continuous orientations: $\beta', \beta''$ for Benevieri–Furi and $\tilde{\beta}', \tilde{\beta}''$ for $*$, both on $U$. By the argument in the preceding paragraph, one must match the two pairs entirely on $U$ under the algebraic correspondence introduced before: say, $\beta'$ matches $\tilde{\beta}'$ and $\beta''$ matches $\tilde{\beta}''$. Now $\tilde{\alpha}$ becomes one of $\tilde{\beta}', \tilde{\beta}''$ entirely on $U$ by continuity of $\tilde{\alpha}$. Hence $\alpha$ must be one of $\beta', \beta''$ also on $U$, since it corresponds to $\tilde{\alpha}$. Therefore $\alpha$ is continuous on $U$ (i.e., locally at each point $\lambda_0$) and consequently on $\Lambda$ as well. □

(The topological part of the proof is essentially to compare the orientability/orientations of the determinant bundle and the associated principal $\mathbf{Z}_2$-bundle that is defined using the pointwise Benevieri–Furi orientations. See the remark at the end of this section.)

Let $R_h \subset \Lambda$ be the set of regular points of $h$, in other words, those points where $\operatorname{coker} h = 0$. Combining this with Benevieri–Furi's result [2] then yields the equivalence of all three notions of orientability, but under some conditions.

COROLLARY 2. *Suppose that the family $h\colon \Lambda \to \Phi_0(E, F)$ is nondegenerate (namely, $R_h$ is nonempty) and that $\Lambda$ is connected and locally path connected. Then the following are equivalent*:

(i) *$h$ is orientable in the Fitzpatrick–Pejsachowicz–Rabier sense*;
(ii) *$h$ is orientable in the Benevieri–Furi sense*;
(iii) *$h$ is $*$-orientable.*

*Moreover, orientation in any one case induces canonically orientations in the other two cases.*

*Proof.* The equivalence (i) $\Leftrightarrow$ (ii) is proved in [2]. Alternatively, one can prove (i) $\Leftrightarrow$ (iii) as in the ensuing discussion.

The equivalence (ii) $\Leftrightarrow$ (iii) and the rest of the corollary is a special case of Theorem 1. □

We now illustrate why the $*$-orientability provides a convenient way to explain some of the important features in both [6] and [1], thus making it a useful link between the two notions. As cited in the beginning, the key feature of the Fitzpatrick–Pejsachowicz–Rabier approach is the introduction of the parity of a path in $\Lambda$, which involves the Leray–Schauder degree possessing a mod 2 value. In terms of our new setup, this parity can be defined using the determinant bundle as follows. Given a path $\gamma: [0,1] \to \Lambda$, the bundle $\det \operatorname{ind} h$ restricts trivially on $\gamma$. Any trivilization defines a bijection (orientation transport) between the orientation sets of the fibers of $\det \operatorname{ind} h$ over $\gamma(0)$ and $\gamma(1)$, which in the end is independent of the trivilization used. If $\gamma(0)$ and $\gamma(1)$ are regular points of $h$, then the fibers over them are canonically oriented and the bijection just defined is essentially a value in $\mathbf{Z}_2$, which is equal to the Fitzpatrick–Pejsachowicz–Rabier (FPR) parity along $\gamma$ (cf. [6, Prop. 1.5]). Putting it differently, we have given a geometric interpretation of the Leray–Schauder degree in the current context. This is interesting because the Leray–Schauder degree is defined using the eigenvalues of some linear operators, which is purely a functional analytic object.

Incidentally, these remarks show the equivalence between (i) and (iii) in Corollary 2. We continue to assume that $h$ is nondegenerate. Then $h$ is FPR-orientable if and only if the parity of any loop at a regular point $\lambda_0 \in R_h$ equals 1 (by [6, Prop. 1.7]). The latter in turn is equivalent to stating that the orientation transport is trivial over any loop at $\lambda_0$, which means exactly that the determinant bundle $\det \operatorname{ind} h$ is trivial.

As for the Benevieri–Furi approach, it is pointed out in [1] that the crucial property is the stability of their orientation. Namely, for a homotopy class of Fredholm families, $H: \Lambda \times [0,1] \to \Phi_0(E,F)$, the orientability of any section $H_t: \Lambda \times \{t\} \to \Phi_0(E,F)$ for some $t$ implies the orientability of the entire homotopy class $H$ (see [2, Thm. 3.14]). This property can be interpreted and verified easily using $*$-orientability: If $H_t$ is orientable, then the bundle $\det \operatorname{ind} H_t$ is trivial on $\Lambda \times \{t\}$. Since $\Lambda \times [0,1]$ contracts to $\Lambda \times \{t\}$, the determinant bundle $\det \operatorname{ind} H$ should be trivial as well. Hence the whole $H$ is orientable. Similarly, the relation between orientations of $H_t$ and $H$ can be verified using trivilizations of their respective determinant bundles.

Remark. A main technique in [2] is to introduce the double cover $\hat{\Phi}_0(E,F)$ of $\Phi_0(E,F)$ using pointwise orientations. This can be viewed as a principal $\mathbf{Z}_2$-bundle over $\Phi_0(E,F)$. Then the argument of Theorem 1 shows that $\det \operatorname{ind} h$ is the vector bundle associated with the pull-back principal bundle $h^*\hat{\Phi}_0(E,F)$ via $h: \Lambda \to \Phi_0(E,F)$. Hence a $*$-orientation of $h$ corresponds precisely to a section of $h^*\hat{\Phi}_0(E,F)$, namely a lifting $\hat{h}: \Lambda \to \hat{\Phi}_0(E,F)$ of $h$ in the notation of [2]. This provides another way to validate the main Definition 3.9 of [2]. Conversely, if one starts with the principal bundle $h^*\hat{\Phi}_0(E,F) \to \Lambda$ using the Benevieri–Furi orientations, then one has an alternative definition of the determinant bundle $\det \operatorname{ind} h$ as the associated vector bundle—in the case of zero Fredholm index. In general, if $h: \Lambda \to \Phi_n(E,F)$ has a positive index $n$, then one defines $\det \operatorname{ind} h$ to be $\det \operatorname{ind} h'$ with $h' = (h,0): \Lambda \to \Phi_0(E, F \oplus \mathbf{R}^n)$. Negative index $n$ can be dealt with similarly.

## 3. Fredholm Maps on Banach Manifolds

In this section we briefly examine how to define orientability and degree of a Fredholm map between two Banach manifolds using determinant bundles. In spirit this is quite similar to [1] and [6], and the interested reader is left to fill in the details.

Suppose $f: X \to Y$ is a smooth, index-0 Fredholm map between two Banach manifolds. Then the Fréchet derivative $Df(x): T_x X \to T_{f(x)} Y$ leads to a family of Fredholm operators parameterized by $X$, with varying Banach spaces. However, the determinant bundle of $Df$, $\det f = \det \operatorname{ind} Df \to X$, can be constructed as before without any change (unlike in [2], where additional care was needed for the manifold case). Then $f$ is called $*$-*orientable* if $\det f$ is a trivial bundle on $X$ and is called $*$-*oriented* if $\det f$ is, in addition, given a specified class of trivilizations.

REMARK. It is worth spelling out that the determinant line bundle is used here differently than in typical gauge theory, where the focus is on the determinant bundle over each individual set $f^{-1}(y)$ for a regular value $y$ (i.e., a moduli space corresponding to a parameter $y$). But our focus here is on the entire manifold $X$ in order to impose the orientability of $f$.

If $f$ is proper and $*$-oriented, then the degree can be defined as

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x),$$

where $y \in Y$ is a regular value and $\operatorname{sign} Df(x)$ is determined as follows. Since $x \in f^{-1}(y)$ is a regular point, it follows that $\ker Df(x) = \operatorname{coker} Df(x) = \{0\}$. Thus the fiber $\det f(x) = \det \operatorname{ind} Df(x) = \mathbf{R}$ has a canonical orientation as previously noted. The sign of $Df(x)$ is obtained by comparing this orientation with the global orientation already provided for $f$.

That $\deg f$ is independent of the choice of $y$ follows from the invariance of orientations under oriented homotopy discussed previously, much the same as in the situation of [6] and [2]. Any interested reader may check that other properties of the degree given in [1], [2], and [6] can be readily transplanted.

Of course, Theorem 1 and Corollary 2 continue to hold for the Fredholm map $f$. Hence the value of $\deg f$ remains the same for all three notions of orientability—under the condition that $f$ is nondegenerate.

It is interesting to compare this with the classical degree of Olum. Suppose $X$ and $Y$ are both finite dimensional. Denoting the orientation bundles of the manifolds by $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively, we have

$$\det f = \mathcal{O}_X \otimes f^* \mathcal{O}_Y \tag{6}$$

by using a fiberwise exact sequence similar to (4). It follows easily from (6) that $f$ is $*$-orientable if and only if $f$ is "orientation true" in the sense of [9]. Moreover, $\deg f$ is precisely the integer degree (twisted degree) of Olum when $f$ is oriented and proper.

As another application of (6), we verify that the nondegeneracy condition is indeed required in both Corollary 2 and its manifold version. To see that the equivalence between (i) and (iii) breaks down without this condition (cf. the remark after

Corollary 2), consider a constant map $c\colon X \to Y$, where $X$ is nonorientable and $Y$ is orientable. Since $c$ has no regular point, it is FPR-orientable by default (and the associated degree is 0). But $c$ is not $*$-orientable, since the determinant bundle $\det c = \mathcal{O}_X$ is nontrivial. (This simple example is also used in [2].) On the other hand, if $X$ is taken to be orientable as well, then $c$ is orientable in all three notions. Nonetheless, the correspondence between the orientation sets still fails: $c$ has one orientation in the Fitzpatrick–Pejsachowicz–Rabier sense but two $*$-orientations in our sense. At any rate, the case of degenerate maps is not so interesting because the degree will always be zero whenever it is defined.

Note that a formula similar to (6) carries over to a Fredholm map between two Banach manifolds, with $\mathcal{O}_X, \mathcal{O}_Y$ replaced by the classes of Fredholm structures on $X, Y$, as appeared in the context of Elworthy–Tromba [4].

Adapting the terminology of Hopf in the finite-dimensional case, we call $A(f) = |\deg f|$ the *absolute degree* of $f$ when $f$ is orientable and proper. It is an invariant under any homotopy, following from the oriented homotopy invariance of $\deg f$. We can also introduce the *geometric* degree $G(f)$: the smallest number of points in $f^{-1}(y)$ for any regular value $y \in Y$. Using a proof similar to that in Epstein [5], we generalize the Hopf–Olum theorem to Banach manifolds.

PROPOSITION 3. *Suppose $f\colon X \to Y$ is a smooth $*$-orientable proper Fredholm map, and suppose that $X$ admits partitions of unity. Then there is a homotopy $g$ of $f$ such that $A(f) = G(g) = A(g)$.*

*Proof.* This is actually simpler than Epstein's situation because we assume that $f$ is smooth. Take a regular value $y$ of $f$ so that $G(f) = \# f^{-1}(y)$. If all points in $f^{-1}(y)$ have the same sign under the orientation, then $G(f) = A(f)$ and we are done. Otherwise one can find two points, say $a, b$, in $f^{-1}(y)$, that have opposite signs. This means there is a path $\alpha$ joining $a, b$ with $-1$ parity. Then take any tubular neighborhood $N$ of $\alpha$ (since $X$ admits partitions of unity). Since $N$ is contractible, one can find a homotopy of $f$ that is constant outside $N$ and cancels out the pair $a, b$ inside $N$ (a special case of Whitney's lemma). The proof is finished by induction. $\square$

Typical examples of manifolds admitting partition of unity include paracompact Hilbert manifolds modeled on a separable Hilbert space.

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Department of Mathematics
University of Missouri
Columbia, MO 65211

sw@math.missouri.edu