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Isospectral Metrics and Potentials on Classical Compact Simple Lie Groups

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1. Introduction

Given a compact Riemannian manifold (M, g), the eigenvalues of the Laplace operator Δ form a discrete sequence known as the *spectrum* of (M, g). (In the case of M with boundary, we stipulate either Dirichlet or Neumann boundary conditions.) We say that two Riemannian manifolds are *isospectral* if they have the same spectrum. For a fixed manifold M, an isospectral deformation of a metric g_0 on M is a continuous family $\mathcal{F}$ of metrics on M containing g_0 such that each metric $g \in \mathcal{F}$ is isospectral to g_0. We say that the deformation is *nontrivial* if none of the other metrics in $\mathcal{F}$ are isometric to g_0 and that the deformation is *multidimensional* if $\mathcal{F}$ can be parameterized by more than one variable. For two functions $\phi, \psi \in C^\infty(M)$, we say that ϕ and ψ are isospectral *potentials* on (M, g) if the eigenvalue spectra of the Schrödinger operators $\hbar^2\Delta + \phi$ and $\hbar^2\Delta + \psi$ are equal for any choice of Planck's constant $\hbar$.

In this paper, we prove the existence of multiparameter isospectral deformations of metrics on $\mathrm{SO}(n)$ ($n = 9$ or $n \geq 11$), $\mathrm{SU}(n)$ ($n \geq 8$), and $\mathrm{Sp}(n)$ ($n \geq 4$). For these examples we follow a metric construction developed by Schueth, who had given one-parameter families of isospectral metrics on orthogonal and unitary groups. Our multiparameter families are obtained by a new proof of nontriviality that establishes a generic condition for nonisometry of metrics arising from the construction. We also show the existence of noncongruent pairs of isospectral potentials and nonisometric pairs of isospectral conformally equivalent metrics on $\mathrm{Sp}(n)$ for $n \geq 6$.

The industry of producing isospectral manifolds began in 1964 with Milnor's pair of 16-dimensional isospectral, nonisometric tori [M]. Several years later, in the early 1980s, new examples began to appear sporadically (e.g. [GW1; I; V]). These isospectral constructions were ad hoc and did not appear to be related until 1985, when Sunada began developing the first unified approach for producing isospectral manifolds. The method described a program for taking quotients of a given manifold so that the resulting manifolds were isospectral. Sunada's original theorem and subsequent generalizations [Be1; Be2; DG; P; Su] explained most of the previously known isospectral examples and led to a wide variety of new ones; see, for example, [BGG], [Bu], and [GWW].

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In 1993 Gordon produced the first examples of closed isospectral manifolds with different local geometry [G] and then, in a series of papers, generalized the construction to the following principle based on torus actions.

THEOREM 1.1. *Let T be a torus and suppose (M, g) and (M', g') are two principal T-bundles such that the fibers are totally geodesic flat tori. Suppose that, for any subtorus $K \subset T$ of codimension* 0 *or* 1, *the quotient manifolds $(M/K, \bar{g})$ and $(M'/K, \bar{g}')$, where $\bar{g}$ and $\bar{g}'$ are the induced submersion metrics, are isospectral. Then (M, g) and (M', g') are isospectral.*

Gordon's initial application of Theorem 1.1 was to give a sufficient condition for two compact nilmanifolds (discrete quotients of nilpotent Lie groups) to be isospectral. In 1997, Gordon and Wilson furthered the development of the submersion technique by constructing the first examples of *continuous* families of isospectral manifolds with different local geometry [GW2]. The base manifolds were products of n-dimensional balls with r-dimensional tori ($n \geq 5$, $r \geq 2$), realized as domains within nilmanifolds. Here Gordon and Wilson proved a general principle for local nonisometry within their construction. They were also able to exhibit specific examples of isospectral deformations of manifolds with boundary for which the eigenvalues of the Ricci tensor (which, in this setting, were constant functions on each manifold) deformed nontrivially. It was later proven in [G+] that the boundaries $S^{n-1} \times T^r$ of the manifolds in [GW2] were also examples of isospectral manifolds. These were closed manifolds that were not locally homogeneous. A general abstract principle was given for nonisometry but specific examples were also produced for which the maximum scalar curvature changed throughout the deformation, thereby proving that maximal scalar curvature is not a spectral invariant.

Expanding on the ideas in [G+], Schueth produced the first examples of simply connected closed isospectral manifolds; in fact, she even produced continuous families of such manifolds [S1]. Schueth's basic principle was to embed the torus T^2 into a larger, simply connected Lie group G (e.g., $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \simeq S^3 \times S^3$) and then extend the metric in order to find isospectral metrics on products of n-dimensional spheres with G (e.g., $S^4 \times S^3 \times S^3$). Since the torus was embedded in the group, the torus action on the manifold was the natural group action. Schueth's examples were not locally homogeneous. For these examples, the critical values of the scalar curvature changed throughout the deformation, proving that the manifolds were not locally isometric. Furthermore, by examining heat invariants related to the Laplacian on 1-forms, Schueth was able to prove that these examples were isospectral on functions but not on 1-forms.

Schueth continued to capitalize on the notion of embedding the torus in a larger group in [S2]. In this case, Schueth specialized Gordon's theorem (Theorem 1.1 here) to compact Lie groups (see Theorem 2.5) in order to produce one-dimensional isospectral deformations of each of $\mathrm{SO}(n) \times T^2$ ($n \geq 5$), $\mathrm{Spin}(n) \times T^2$ ($n \geq 5$), $\mathrm{SU}(n) \times T^2$ ($n \geq 3$), $\mathrm{SO}(n)$ ($n \geq 8$), $\mathrm{Spin}(n)$ ($n \geq 8$), and $\mathrm{SU}(n)$ ($n \geq 6$). Here the metrics were left-invariant and so the manifolds were homogeneous. As with many previous examples, Schueth's metrics were constructed from linear maps j into the Lie algebra of the Lie group in question. In order to prove nonisometry, Schueth expressed the norm of the Ricci tensor in terms of the associated linear map and chose her linear maps so that the norm of the Ricci tensor varied through the deformation.

These particular examples of Schueth's were the inspiration for the first part of this paper. We will use Schueth's specialization of Theorem 1.1 to produce our metrics. However, in this paper we will produce *multidimensional* families of metrics and will develop a general nonisometry principle for families of metrics arising from linear maps according to Schueth's construction. Furthermore, we will expand the class of Lie groups for which such families exist to include all of the classical compact simple Lie groups of sufficient dimension.

More recently, Gordon and Schueth have constructed conformally equivalent metrics $\phi_1 g$ and $\phi_2 g$ on spheres S^n and balls B^{n+1} ($n \geq 7$) and on $\mathrm{SO}(n)$ ($n \geq 14$), $\mathrm{Spin}(n)$ ($n \geq 14$), and $\mathrm{SU}(n)$ ($n \geq 9$) [GS]. They also showed that the conformal factors ϕ_1 and ϕ_2 were isospectral potentials for the Schrödinger operator $\hbar^2\Delta + \phi$ on each of these manifolds. In this paper, we extend their result to include $\mathrm{Sp}(n)$ for $n \geq 6$.

The outline of the paper is as follows. In Section 2 we describe the metrics and potentials to be used and state the theorems by Gordon and Schueth that we will apply. Next, in Section 3 we give our examples of multiparameter isospectral deformations of metrics on the classical compact simple Lie groups. Section 4 is devoted to proving the nontriviality of these examples. Finally, in Section 5 we give our examples of noncongruent isospectral potentials and nonisometric conformally equivalent isospectral metrics on $\mathrm{Sp}(n)$ for $n \geq 6$.

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2. Metric and Potential Constructions

In this section, we describe the metrics and potentials that are considered in the remainder of the paper. The constructions, which are due to Schueth and Gordon [S; GS], are based on linear maps.

Consider a Lie group G with Lie algebra $\mathfrak{g}$ and bi-invariant metric g_0. By *torus* we mean a nontrivial, compact, connected abelian Lie group. Suppose that $H < G$ is a torus with Lie algebra $\mathfrak{h}$ and that $K < G$ is a closed connected subgroup with Lie algebra $\mathfrak{k}$. Assume that $\mathfrak{h}$ is g_0-orthogonal to $\mathfrak{k}$ and that $[\mathfrak{h}, \mathfrak{k}] = 0$.

Notation 2.1. Given a linear map $j \colon \mathfrak{h} \to \mathfrak{k} \subset \mathfrak{g}$, we define $j^t \colon \mathfrak{g} \to \mathfrak{h}$ by $g_0(j^t(X), Z) = g_0(X, j(Z))$ for all $X \in \mathfrak{g}$ and $Z \in \mathfrak{h}$. In other words, j^t is the g_0-transpose of j. We then have an inner product g_j on $\mathfrak{g}$ given by $g_j = (\mathrm{Id} + j^t)^* g_0$. Let g_j also denote the left-invariant metric on G that is associated to this inner product.

Notice that g_j differs from g_0 only on $\mathfrak{k} \oplus \mathfrak{h}$, where we have used the linear map j to redefine orthogonality. In particular, j determines a subspace $S = \{X - j^t(X) \mid X \in \mathfrak{g}\}$ that is g_j-orthogonal to $\mathfrak{h}$ and such that g_j restricted to S is linearly isometric to g_0 restricted to $\mathfrak{k}$ via the map $X - j^t(X) \mapsto X$.

Recall that a Lie algebra is compact if it is the Lie algebra of a compact Lie group.

DEFINITION 2.2. Let $\mathfrak{g}$ be a compact Lie algebra with associated Lie group G, and let $\mathfrak{h}$ be a real inner product space. Suppose $j, j' : \mathfrak{h} \to \mathfrak{g}$ are linear maps. We say that j and j' are *isospectral,* denoted $j \sim j'$, if for each $z \in \mathfrak{h}$ there exists an $A_z \in G$ such that $j(z) = \mathrm{Ad}(A_z) j'(z)$. We say that j and j' are *equivalent,* denoted $j \simeq j'$, if there exist $A \in G$ and $C \in O(\mathfrak{h})$ such that $j(z) = \mathrm{Ad}(A) j'(C(z))$ for all $z \in \mathfrak{h}$.

REMARK 2.3. Note that in the case $G = \mathrm{SO}(n)$, $\mathrm{SU}(n)$, or $\mathrm{Sp}(n)$, the map $\mathrm{Ad}(A) : \mathfrak{g} \to \mathfrak{g}$ is given by matrix conjugation. Thus we may rewrite the isospectrality condition as $A_z j(z) A_z^{-1} = j'(z)$ and the equivalence condition as $A j(z) A^{-1} = j'(C(z))$.

REMARK 2.4. We use the definition of equivalence that was introduced in [GW2] for the case $\mathfrak{g} = \mathfrak{so}(n)$ and in [S2] for the case $\mathfrak{g} = \mathfrak{su}(n)$. This differs slightly from the one cited in [GS] by Gordon and Schueth, whose definition states that j and j' are equivalent if there exist a $C \in O(\mathfrak{h})$ and *any* automorphism ϕ of $\mathfrak{g}$ such that $j(z) = \phi(j'(C(z)))$ for all $z \in \mathfrak{h}$. This means that Gordon and Schueth's definition is less restrictive except in the cases of $\mathfrak{so}(n)$ (n odd) and $\mathfrak{sp}(n)$ where every automorphism of $\mathfrak{g}$ is an inner automorphism by some element of $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$, respectively.

The following theorem by Schueth is a specialization of Gordon's submersion theorem (Theorem 1.1).

THEOREM 2.5 [S2]. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let g_0 be a bi-invariant metric on G. Let $H < G$ be a torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Denote by $\mathfrak{u}$ the g_0-orthogonal complement of the centralizer $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\lambda, \lambda' : \mathfrak{g} \to \mathfrak{h}$ be two linear maps with $\lambda|_{\mathfrak{h}\oplus\mathfrak{u}} = \lambda'|_{\mathfrak{h}\oplus\mathfrak{u}} = 0$ that satisfy: For every $z \in \mathfrak{h}$ there exists an $A_z \in G$ such that A_z commutes with H and $\lambda'_z = \mathrm{Ad}(A_z)^* \lambda_z$, where $\lambda_z := g_0(\lambda(\cdot), z)$ and $\lambda'_z := g_0(\lambda'(\cdot), z)$. Denote by g_λ and $g_{\lambda'}$ the left-invariant metrics on G corresponding to the scalar products $(\mathrm{Id} + \lambda)^* g_0$ and $(\mathrm{Id} + \lambda')^* g_0$ on $\mathfrak{g}$. Then (G, g_λ) and $(G, g_{\lambda'})$ are isospectral.*

In particular, if $j, j' : \mathfrak{h} \to \mathfrak{k} \subset \mathfrak{g}$ are isospectral maps, then letting $\lambda = j^t$ and $\lambda' = j'^t$ allows us to conclude that the metrics g_j and $g_{j'}$ on G described previously are isospectral.

We have a similar theorem for producing pairs of isospectral potentials and pairs of conformally equivalent isospectral metrics.

THEOREM 2.6 [GS]. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, let P be a compact Lie subgroup with Lie algebra of the form $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{k}$ for some Lie algebra $\mathfrak{k}$, and let $H < G$ be a torus with Lie algebra $\mathfrak{h}$. Suppose that $[\mathfrak{p}, \mathfrak{h}] = 0$ and that $\mathfrak{h}$ is orthogonal to $\mathfrak{p}$ with respect to a bi-invariant metric g_0 on G. Let $j_1, j_2 : \mathfrak{h} \to \mathfrak{k}$ be isospectral linear maps as in Definition 2.2, and define $j : \mathfrak{h} \to \mathfrak{k} \oplus \mathfrak{k} = \mathfrak{p}$ by $j(Z) = (j_1(Z), j_2(Z))$. Denote by g_j the associated left-invariant* *metric on G. Let ϕ be a smooth function on G that is right invariant under H and invariant under conjugation by elements of P. Suppose there exists an isometric automorphism τ of (G, g_0) such that $\tau|_H = \mathrm{Id}$ and such that τ_* restricts to the map $(X,Y) \mapsto (Y,X)$ on $\mathfrak{k} \oplus \mathfrak{k} = \mathfrak{p} \subset \mathfrak{g}$. Then:*

(1) *ϕ and $\tau^*\phi$ are isospectral potentials on (G, g_j); and*
(2) *if, in addition, ϕ is positive, then ϕg_j and $(\tau^*\phi)g_j$ are conformally equivalent isospectral metrics on G.*

3. Examples of Isospectral Deformations of Metrics on Lie Groups

We now apply the material in Section 2 to produce examples of isospectral deformations of metrics on Lie groups. All of our examples arise from the following theorem. In Section 4 we prove the nontriviality of the examples.

THEOREM 3.1. *Suppose $\mathfrak{g}$ is one of $\mathfrak{so}(n)$ $(n = 5, n \geq 7)$, $\mathfrak{su}(n)$ $(n \geq 4)$, or $\mathfrak{sp}(n)$ $(n \geq 3)$ with associated group $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, or $\mathrm{Sp}(n)$, respectively. Suppose $\mathfrak{h}$ is the Lie algebra of the two-dimensional torus. Let L be the space of all linear maps $j\colon \mathfrak{h} \to \mathfrak{g}$. Then there exists a Zariski open set $\mathcal{O} \subset L$ such that each $j_0 \in \mathcal{O}$ is contained in a continuous d-parameter family of linear maps that are isospectral but pairwise not equivalent. Here d depends on $\mathfrak{g}$ as follows.*

$\mathfrak{g}$	d
$\mathfrak{so}(n)$	$d \geq n(n-1)/2 - \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+2\right)$
$\mathfrak{su}(n)$	$d \geq n^2 - 1 - \frac{n^2+3n}{2}$
$\mathfrak{sp}(n)$	$d \geq n^2 - n$

(Here $\left[\frac{n}{2}\right]$ denotes the largest integer less than or equal to $\frac{n}{2}$.) Note that for $\mathfrak{so}(n)$ we have $d > 1$ when $n = 5$ or $n \geq 7$. For $\mathfrak{su}(n)$, $d = 1$ when $n = 4$ and $d > 1$ when $n \geq 5$. For $\mathfrak{sp}(n)$, $d > 1$ when $n \geq 3$.

REMARK 3.2. This theorem was originally proven by Gordon and Wilson in [GW2] for the case of $\mathfrak{so}(n)$ with the associated group $O(n)$. However, for fixed $j_0 \in \mathcal{O}$, since the d-parameter family of linear maps that are isospectral to j_0 is continuous it follows that, for each $z \in \mathfrak{h}$, the family $\{j(z) \mid j$ is an element of the d-parameter family$\}$ is the orbit of $j_0(z)$ under the adjoint action of a continuous set of elements of $O(n)$. The identity is contained in this set and so the set is, in fact, contained in $\mathrm{SO}(n)$. Thus we have that each of Gordon and Wilson's families consists of maps that are isospectral via $\mathrm{SO}(n)$. We will use this in our Example 3.6. On the other hand, for any pair of maps within one of Gordon and Wilson's families, there is no element of $O(n)$ that makes them equivalent. This is stronger than pairwise nonequivalence via $\mathrm{SO}(n)$. Indeed, for $n = 5, 6, 7$ and $n \geq 9$, the automorphism group of $\mathfrak{so}(n)$ is contained in $\{\mathrm{Ad}(A) \mid A \in O(n)\}$. Thus we see that, except for $n = 8$, Gordon and Wilson's families consist of linear maps that are not equivalent even by the definition given in [GS] (cf. Remark 2.4). This will prove useful at the end of Section 4 when we prove nontriviality of our isospectral examples.

REMARK 3.3. Though Gordon and Wilson's original proof for $\mathfrak{so}(n)$ broke down in the case $n = 6$ $\big($since $n(n-1)/2 - \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+2\right) = 0$ for $n = 6\big)$, they explicitly exhibited one-parameter families of isospectral, nonequivalent linear maps $j\colon \mathfrak{h} \to \mathfrak{so}(6)$.

REMARK 3.4. In [S2], Schueth gives examples of one-dimensional families of isospectral, nonisometric metrics on $\mathrm{SO}(n)$ ($n \geq 9$), $\mathrm{Spin}(n)$ ($n \geq 9$), and $\mathrm{SU}(n)$ ($n \geq 6$) that arise from continuous one-dimensional families of isospectral linear maps from $\mathfrak{h}$ to $\mathfrak{so}(n)$ ($n \geq 5$) and $\mathfrak{su}(n)$ ($n \geq 3$), respectively. However, her proof of nonisometry differs from ours. Schueth's proof has the advantage of showing nonisometry in a geometric way by using the norm of the Ricci tensor, whereas ours has the advantage of giving a generic condition for linear maps to produce nonisometric metrics.

REMARK 3.5. Gordon and Wilson's proof extends in a straightforward way to $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$, making obvious adjustments depending on the Lie algebra. For full details see [Pr]. Here, for $\mathfrak{sp}(n)$ we have proved the case where $n \geq 3$. However, since $\mathfrak{sp}(2)$ is isomorphic to $\mathfrak{so}(5)$, the result is also true for $n = 2$.

Let T^r denote an r-dimensional torus.

EXAMPLE 3.6 (Isospectral Deformations of Metrics on $\mathrm{SO}(n)$). Observe that $\mathrm{SO}(n) \times T^r$ is contained as a subgroup of $\mathrm{SO}(n) \times \mathrm{SO}(2r)$ that is itself contained as a subgroup of $\mathrm{SO}(n+2r)$ in the form of diagonal block matrices. From Theorem 3.1 and Remarks 3.2 and 3.3 we have examples of one-dimensional families of pairwise isospectral linear maps $j\colon \mathfrak{h} \to \mathfrak{so}(6)$ and multidimensional families of pairwise isospectral linear maps $j\colon \mathfrak{h} \to \mathfrak{so}(n)$ for $n = 5$ or $n \geq 7$. Hence, according to the construction described in Section 2, this gives one-dimensional families of isospectral metrics on $\mathrm{SO}(10)$ and multidimensional families for $\mathrm{SO}(9)$ and $\mathrm{SO}(n)$ ($n \geq 11$). By Theorem 2.5, within each family the metrics are pairwise isospectral.

EXAMPLE 3.7 (Isospectral Deformations of Metrics on $\mathrm{Spin}(n)$). By lifting from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$, we may consider $\mathrm{Spin}(n) \times T^r$ a subgroup of $\mathrm{Spin}(n+2r)$. The orbits of $\mathrm{Ad}(\mathrm{Spin}(n))$ in $\mathfrak{so}(n)$ are equal to the orbits of $\mathrm{Ad}(\mathrm{SO}(n))$, so if j and j' are isospectral with respect to $\mathrm{SO}(n)$ then they are also isospectral with respect to $\mathrm{Spin}(n)$. Fix $j_0 \in \mathcal{O}$. By an argument similar to the one for $\mathrm{SO}(n)$, we conclude that there exist one-dimensional families of isospectral metrics on $\mathrm{Spin}(10)$ and multidimensional families of isospectral metrics on $\mathrm{Spin}(9)$ and $\mathrm{Spin}(n)$ for $n \geq 11$.

EXAMPLE 3.8 (Isospectral Deformations of Metrics on $\mathrm{SU}(n)$). Here we have $\mathrm{SU}(n) \times T^r$ contained as a subgroup of $\mathrm{SU}(n) \times \mathrm{SU}(r+1)$, which in turn is contained as a subgroup of $\mathrm{SU}(n+r+1)$. From Theorem 3.1, we have one-dimensional families of pairwise isospectral linear maps $j\colon \mathfrak{h} \to \mathfrak{su}(4)$ and multidimensional families of pairwise isospectral linear maps $j\colon \mathfrak{h} \to \mathfrak{su}(n)$ for $n \geq 5$. Thus we have one-dimensional isospectral deformations of metrics on SU(7) and multidimensional families of isospectral metrics on $\mathrm{SU}(n)$, $n \geq 8$.

EXAMPLE 3.9 (Isospectral Deformations of Metrics on $\mathrm{Sp}(n)$). $\mathrm{Sp}(n) \times T^r$ is contained as a subgroup of $\mathrm{Sp}(n) \times \mathrm{Sp}(r)$, which itself is contained as a subgroup of $\mathrm{Sp}(n+r)$. Theorem 3.1 and Remark 3.5 give us multidimensional families of pairwise isospectral linear maps $j\colon \mathfrak{h} \to \mathfrak{sp}(n)$ for $n \geq 2$. Thus we have multidimensional families of isospectral metrics on $\mathrm{Sp}(n)$ for $n \geq 4$.

4. Nonisometry of Examples

In this section we prove nontriviality of Examples 3.6–3.9. Here we let G_n denote one of $\mathrm{SO}(n)$, $\mathrm{Spin}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ and let $\mathfrak{g}_n$ denote the associated Lie algebra $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, or $\mathfrak{sp}(n)$. Recall that for Examples 3.6–3.9 we embedded $G_n \times T^2$ into a higher-dimensional group. In this section, we will refer to the higher-dimensional group as G_{n+p}, where $p = 4$ in the cases of $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$, $p = 3$ in the case of $\mathrm{SU}(n)$, and $p = 2$ in the case of $\mathrm{Sp}(n)$. Let $I_0(g_j)$ denote the identity component of the isometry group of (G_{n+p}, g_j), and let $I_0^e(g_j)$ denote the isotropy subgroup at e of $I_0(g_j)$. For $x \in G_{n+p}$, use L_x (resp. R_x) to denote left (resp. right) translation by x.

THEOREM 4.1 [OT]. *Let G be a compact, connected, simple Lie group and let g be a left-invariant Riemannian metric on G. Then, for each isometry f contained in the identity component of the group of isometries of (G, g), there exist $x, y \in G$ such that $f = L_x \circ R_y$.*

In particular, for $\alpha \in I_0^e(g_j)$, we have that there exists some $x \in G_{n+p}$ such that α is equal to conjugation of G_{n+p} by x. Since α fixes the identity, at the Lie algebra level we have that α_* is equal to $\mathrm{Ad}(x)$.

PROPOSITION 4.2. *Suppose G is a compact simple group with left-invariant metrics g and g', neither of which is bi-invariant. If $\mu\colon (G, g) \to (G, g')$ is an isometry such that $\mu(e) = e$, then μ is an automorphism of G.*

Proof. Since μ is an isometry, we have that $I_0^e(g_j)$ is isomorphic to $I_0^e(g_{j'})$ via conjugation by μ. Because G is compact, the isometry groups of (G, g) and (G, g') are also compact. Thus we may write the isometry group of (G, g) as $G_1 \times G_2 \times \cdots \times G_s \times T/Z$ and the isometry group of (G, g') as $G_1' \times G_2' \times \cdots \times G_t' \times T'/Z'$, where each $G_i^{(\prime)}$ is simple, $T^{(\prime)}$ is a torus, and $Z^{(\prime)}$ is central. Each isometry group contains a copy of G in the form of left translations. Furthermore, since neither g nor g' is bi-invariant, each isometry group contains (by Theorem 4.1) exactly one copy of G. Any isomorphism from the isometry group of (G, g) to the isometry group of (G, g') must carry simple factors to simple factors. Since G is the only simple factor of its dimension, it follows that any isomorphism carries G to G. This means that for every $x \in G$ there exists an $x' \in G$ such that $\mu L_x \mu^{-1} = L_{x'}$. Hence, for each $x, y \in G$,

$$\begin{aligned}\mu(xy) = \mu(L_{xy})(e) = \mu(L_x L_y)(e) &= \mu L_x \mu^{-1} \mu L_y(e) \\ &= L_{x'} \mu L_y(e) = L_{x'} L_{y'} \mu(e) = L_{x'} L_{y'}(e) \\ &= x'y' = \mu(x)\mu(y). \end{aligned} \tag{1}$$

□

REMARK 4.3. If $\mu\colon (G, g) \to (G, g')$ is an isometry that does not carry the identity to itself, then composing μ with $L_{\mu(e)^{-1}}$ yields an automorphism of G.

The following corollary is immediate.

COROLLARY 4.4. *Let G_{n+p} be one of* $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, $\mathrm{SU}(n+3)$, *or* $\mathrm{Sp}(n+2)$. *Given two nonzero linear maps j and j', suppose there exists an isometry $\mu\colon (G_{n+p}, g_j) \to (G_{n+p}, g_{j'})$, where g_j and $g_{j'}$ are as in Notation 2.1. Then $L_{\mu(e)^{-1}} \circ \mu$ is an automorphism of G_{n+p}.*

LEMMA 4.5. *Let G_{n+p} be one of* $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, $\mathrm{SU}(n+3)$, *or* $\mathrm{Sp}(n+2)$, *where T^2 is embedded (as before) in $G_p \subset G_{n+p}$. Let $j, j'\colon \mathfrak{h} \to \mathfrak{g}_n$ be nonzero linear maps and let g_j and $g_{j'}$ be as in Notation 2.1. Suppose there exists an isometry $\mu\colon (G_{n+p}, g_j) \to (G_{n+p}, g_{j'})$ such that $\mu(e) = e$ and $\mu(T^2) = T^2$. Then there is an element $C \in O(\mathfrak{h})$ such that $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$.*

Proof. From Corollary 4.4 we know that μ is an automorphism of G_{n+p}. Thus μ_* maps left-invariant vector fields to left-invariant vector fields; that is, $\mu_*\colon \mathfrak{g}_{n+p} \to \mathfrak{g}_{n+p}$. If μ maps T^2 to itself then it must isometrically map the Lie algebra $\mathfrak{h}$ to itself. This implies that there is an element $C \in O(\mathfrak{h})$ such that μ_* restricted to $\mathfrak{h}$ is equal to C.

Furthermore, since μ is an automorphism of G_{n+p}, if μ maps T^2 to itself in G_{n+p} then it must also isomorphically map the identity component of the centralizer of T^2 in G_{n+p} to itself. At the Lie algebra level, direct calculation shows that the centralizer of $\mathfrak{h}$ in

- $\mathfrak{so}(n+4)$ is $\mathfrak{so}(n) \oplus \mathfrak{h}$. Hence the identity component of the centralizer of T^2 in $\mathrm{SO}(n+4)$ is $\mathrm{SO}(n) \times T^2$ and the identity component of the centralizer of T^2 in $\mathrm{Spin}(n+4)$ is $\mathrm{Spin}(n) \times T^2$.
- $\mathfrak{su}(n+3)$ is $\mathfrak{su}(n) \oplus tu \oplus \mathfrak{h}$, where

$$u = \begin{bmatrix} i/n & & & & & \\ & \ddots & & & & \\ & & i/n & & & \\ & & & -i/3 & & \\ & & & & -i/3 & \\ & & & & & -i/3 \end{bmatrix} \in \mathfrak{su}(n+3). \tag{2}$$

Letting U denote the one-parameter subgroup associated to u, we have the identity component of the centralizer of T^2 in $\mathrm{SU}(n+3)$ is $\mathrm{SU}(n) \times U \times T^2$.

- $\mathfrak{sp}(n+2)$ is $\mathfrak{sp}(n) \oplus \mathfrak{h}$. Thus the identity component of the centralizer of T^2 in $\mathrm{Sp}(n+2)$ is $\mathrm{Sp}(n) \times T^2$.

In each case, the identity component of the centralizer of T^2 is the product of a simple group, G_n, with a torus. Therefore $\mu(G_n) = G_n$ and μ_* is a Lie algebra automorphism of $\mathfrak{g}_n$. As a result, for any $X \in \mathfrak{g}_n$ we have that $X - j^t(X) \in \mathfrak{h}^{\perp_{g_j}}$ is mapped to $\mu_* X - Cj^t(X)$.

On the other hand, since $\mu_* X \in \mathfrak{g}_n$ and $Cj^t(X) \in \mathfrak{h}$ and since $\mu_*\colon \mathfrak{h}^{\perp_{g_j}} \to \mathfrak{h}^{\perp_{g_{j'}}}$, it must be the case that $Cj^t(X) = j'^t(\mu_* X)$ for all $X \in \mathfrak{g}_n$. Otherwise, $Cj^t(X) = j'^t(\mu_* X) + Z$ for some nonzero $Z \in \mathfrak{h}$ depending on X, but in this case $\mu_* X - Cj^t(X) = \mu_* X - j'^t(\mu_* X) - Z$, which is not in $\mathfrak{h}^{\perp_{g_{j'}}}$.

Finally, taking transposes, we see that the condition $j^t(X) = C^{-1} j'^t(\mu_* X)$ for all $X \in \mathfrak{g}_n$ implies $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$. □

Remark 4.6. From the proof of Lemma 4.5, we see that μ restricted to G_n is an automorphism. Suppose μ restricted to G_n is an inner automorphism, so that μ_* restricted to $\mathfrak{g}_n$ is equal to $\mathrm{Ad}(A)$ for some $A \in G_n$. Then, by the proof, $j(z) = \mathrm{Ad}(A^{-1}) j'(Cz)$ for all $z \in \mathfrak{h}$. In other words, j and j' are equivalent.

Genericity Condition 4.7. We say that $j\colon \mathfrak{h} \to \mathfrak{g}_n$ is *generic* if there are only finitely many $A \in G_n$ such that $j(z) = \mathrm{Ad}(A) j(z)$ for all $z \in \mathfrak{h}$.

From the proofs found in [GW2] and [Pr], the linear maps j used in Examples 3.6–3.9 are generic.

Lemma 4.8. *Let G_{n+p} be one of* $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, $\mathrm{SU}(n+3)$, *or* $\mathrm{Sp}(n+2)$. *Let $j\colon \mathfrak{h} \to \mathfrak{g}_n$ be generic and let g_j be the associated metric on G_{n+p}. For G_{n+p} equal to* $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, *or* $\mathrm{Sp}(n+2)$, *let D be the group of isometries of (G_{n+p}, g_j) generated by the set $\{L_x \circ R_{x^{-1}} \mid x \in T^2\}$. For G_{n+p} equal to* $\mathrm{SU}(n+3)$, *let D be the group of isometries of (G_{n+p}, g_j) generated by the set $\{L_x \circ R_{x^{-1}} \mid x \in U \times T^2\}$, where U is as in the proof of Lemma 4.5. Then D is a maximal torus in $I_0^e(g_j)$.*

Proof. It is straightforward to check that $D \subset I_0^e(g_j)$. Recall that every element of $I_0^e(g_j)$ is of the form $L_x \circ R_{x^{-1}}$ for some $x \in G_{n+p}$. Let $C(G_{n+p})$ denote the finite center of G_{n+p}. We identify $I_0^e(g_j)$ with a subgroup of $G_{n+p}/C(G_{n+p})$ via the map that sends $L_x \circ R_{x^{-1}}$ to the coset of x in $G_{n+p}/C(G_{n+p})$. Under this correspondence, we consider D a subgroup of $G_{n+p}/C(G_{n+p})$.

Suppose that $\{y_t \mid t \in (-\varepsilon, \varepsilon)\}$ is a continuous family of elements of G_{n+p} with $y_0 = e$. Suppose furthermore that, for each t, $L_{y_t} \circ R_{y_t^{-1}}$ is an element of $I_0^e(g_j)$ that commutes with D. If $L_{y_t} \circ R_{y_t^{-1}}$ commutes with D then, under the identification of D with a subgroup of $G_{n+p}/C(G_{n+p})$, for each $x \in T^2$ (resp. $U \times T^2$) we have $y_t^{-1} x y_t = x z_t$ for some $z_t \in C(G_{n+p})$. But $C(G_{n+p})$ is discrete, so it must be the case that $z_t = e$ for all t and all $x \in T^2$. This implies for any t that, when y_t acts by isometry (i.e. conjugation) on G_{n+p}, it fixes T^2 pointwise. Thus the continuous family $\{y_t \mid t \in (-\varepsilon, \varepsilon)\}$ is contained in the identity component of the centralizer of T^2 in G_{n+p}.

Since y_t is contained in the identity component of the centralizer of T^2 for each t, it follows that:

- for $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, and $\mathrm{Sp}(n+2)$, y_t is equal to a product $A_t Z_t$; and
- for $\mathrm{SU}(n+3)$, y_t equals $A_t U Z_t$

for some $A_t \in G_n$ and $Z_t \in T^2$. In both cases, $(L_{y_t} \circ R_{y_t^{-1}})_*$ restricted to $\mathfrak{g}_n$ equals $\mathrm{Ad}(A_t)$. By the proof of Lemma 4.5, $j(z) = \mathrm{Ad}(A_t^{-1}) j(Cz)$ for some $C \in O(\mathfrak{h})$. But since $L_{y_t} \circ R_{y_t^{-1}}$ fixes T^2 pointwise, we have that C is equal to the identity and so

$$j(z) = \mathrm{Ad}(A_t^{-1}) j(z) \quad \text{for all } z \in \mathfrak{h}. \tag{3}$$

But by the genericity of j, there are only finitely many A_t for which equation (3) holds and thus only finitely many A_t such that $y_t = A_t Z_t$ (resp. $A_t U Z_t$) for some $Z_t \in T^2$. Because our family $\{y_t \mid t \in (-\varepsilon, \varepsilon)\}$ is continuous, it must be that A_t is the identity for all $t \in (-\varepsilon, \varepsilon)$. Therefore $\{y_t \mid t \in (-\varepsilon, \varepsilon)\} \subset T^2$. In other words, D is not contained in a higher-dimensional connected torus and hence is a maximal torus in $I_0^e(g_j)$. □

THEOREM 4.9. *Let G_{n+p} be one of* $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, $\mathrm{SU}(n+3)$, *or* $\mathrm{Sp}(n+2)$. *Let j and j' be generic linear maps such that $\mu\colon (G_{n+p}, g_j) \to (G_{n+p}, g_{j'})$ is an isometry. Then there exists an element $C \in O(\mathfrak{h})$ such that $j(z) = \mu_*^{-1} j'(Cz)$ for all $z \in \mathfrak{h}$. By Remark 4.6, if μ restricts to an inner automorphism of G_n, then j and j' are equivalent.*

Proof. Suppose that $\mu\colon (G_{n+p}, g_j) \to (G_{n+p}, g_{j'})$ is an isometry. We may assume that $\mu(e) = e$. By Lemma 4.5, it suffices to show that $\mu(T^2) = T^2$.

We know that $I_0^e(g_j)$ is isomorphic to $I_0^e(g_{j'})$ via conjugation by μ. According to Lemma 4.8, D is a maximal torus in $I_0^e(g_j)$ and so the isomorphism carries D to a maximal torus in $I_0^e(g_{j'})$. All maximal tori in a compact Lie group are conjugate; we may therefore assume (after possibly composing μ with an element of $I_0^e(g_{j'})$) that conjugation by μ carries D to the similarly defined set in $I_0^e(g_{j'})$.

Case I. For $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, and $\mathrm{Sp}(n+2)$, this implies that for any $a \in T^2$ we have $\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1} = L_b \circ R_{b^{-1}}$ for some $b \in T^2$. On the other hand, by Corollary 4.4, we know that μ is an automorphism of G_{n+p}. Thus, for any $x \in G_{n+p}$,

$$\mu \circ L_a \circ R_{a^{-1}} \circ \mu^{-1}(x) = \mu(a\mu^{-1}(x)a^{-1}) = \mu(a)x\mu^{-1}(a) = L_{\mu(a)} \circ R_{\mu(a^{-1})}(x).$$

In other words, $\mu(a) = bz$ for some $z \in C(G_{n+p})$. For each of $\mathrm{SO}(n+4)$, $\mathrm{Spin}(n+4)$, and $\mathrm{Sp}(n+2)$, $C(G_{n+p})$ is finite. Since μ is continuous and since $\mu(e) = e$, we have that $\mu(T^2) = T^2$.

Case II. For $\mathrm{SU}(n+3)$, a similar argument implies that $\mu(U \times T^2) = U \times T^2$ and thus, at the Lie algebra level, μ_* maps $tu \oplus \mathfrak{h}$ to $tu \oplus \mathfrak{h}$.

Now consider μ as an automorphism. The automorphism group of $\mathrm{SU}(n+3)$ is generated by the inner automorphisms and one outer automorphism, namely complex conjugation. At the Lie algebra level, conjugating an element $X \in \mathfrak{su}(n+3)$

by any element of $\mathrm{SU}(n+3)$ preserves the eigenvalues of X. In particular, u has eigenvalue i/n with multiplicity n and eigenvalue $-i/3$ with multiplicity 3. No other element of $tu \oplus \mathfrak{h}$ has the same eigenvalues. Therefore, each inner automorphism of $\mathfrak{su}(n+3)$ maps u to u. Similarly, at the Lie algebra level, the outer automorphism of $\mathrm{SU}(n+3)$ negates the eigenvalues of $X \in \mathfrak{su}(n+3)$. For each $t \in \mathbb{R}$, this sends tu to $-tu$. Thus the vector space spanned by u is fixed.

Since μ is an isometry, μ_* maps the g_j-orthogonal complement of the space spanned by u to the $g_{j'}$-orthogonal complement of the space spanned by u. Thus $\mu_*(\mathfrak{h}) = \mathfrak{h}$ and therefore $\mu(T^2) = T^2$. □

THEOREM 4.10. *Let $\mathfrak{g}_n$ be one of $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, or $\mathfrak{sp}(n)$. Suppose $j_0 \colon \mathfrak{h} \to \mathfrak{g}_n$ is contained in a family of generic linear maps that are pairwise nonequivalent. Then there are at most finitely other maps j contained in the family such that g_{j_0} and g_j are isometric.*

Proof. Suppose j and j' are two generic linear maps such that g_j and $g_{j'}$ are both isometric to g_{j_0}. By Theorem 4.9 we have Lie algebra automorphisms α, α' of $\mathfrak{g}_n$ and elements C, C' of $O(\mathfrak{h})$ such that

$$j(z) = \alpha j_0(Cz) \tag{4}$$

and

$$j'(z) = \alpha' j_0(C'z) \tag{5}$$

for all $z \in \mathfrak{h}$.

If α and α' are in the same coset of $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$, then they differ by an inner automorphism. But in this case j and j' are equivalent. Since our family consists of pairwise nonequivalent linear maps, j is equal to j'. The theorem now follows from the fact that, for each of $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, and $\mathfrak{sp}(n)$, $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ is finite. □

Finally we may apply Theorem 4.10 to Examples 3.6–3.9. For Examples 3.6 and 3.7, let j_0 be a linear map in the set $\mathcal{O}$ from Theorem 3.1 and let $\mathcal{F}_{\mathfrak{so}(n)}$ be a d-parameter family of linear maps that are isospectral but pairwise nonequivalent. Since the automorphism group of $\mathfrak{so}(n)$ ($n = 5, 6, 7, n \geq 9$) is contained in $\{\mathrm{Ad}(A) \mid A \in O(n)\}$, we have from the proof of Theorem 4.10 and Remark 3.2 that no two elements of $\mathcal{F}_{\mathfrak{so}(n)}$ give rise to isometric metrics on $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Thus we have an isospectral deformation of g_{j_0}. For $\mathfrak{so}(8)$, since the cardinality of $\mathrm{Aut}(\mathfrak{so}(8))/\mathrm{Aut}^0(\mathfrak{so}(8))$ is 3, for any element of $\mathcal{F}_{\mathfrak{so}(8)}$ there are at most two other elements that could give rise to isometric metrics on SO(12) or Spin(12). For fixed $j_0 \in \mathcal{O}$, we may choose $\mathcal{F}_{\mathfrak{so}(8)}$ small enough that no other element of $\mathcal{F}_{\mathfrak{so}(8)}$ produces a metric isometric to g_{j_0}, thereby obtaining an isospectral deformation of g_{j_0}.

The analysis of Examples 3.8 and 3.9 is similar. Recall that the automorphism group of $\mathfrak{su}(n)$ for $n \geq 4$ is generated by inner automorphisms and the outer automorphism that takes an element to its complex conjugate. By choosing j_0 in $\mathcal{O}$, we obtain a d-dimensional isospectral deformation of g_{j_0} such that, for any metric other than g_{j_0} within the deformation, there is at most one other isometric metric contained in the deformation. Finally, the automorphism group of $\mathfrak{sp}(n)$ consists entirely of inner automorphisms. Thus, for $n \geq 4$, we have multiparameter isospectral deformations of metrics on Sp(n) such that no two metrics in a given deformation are isometric.

Thus we have produced isospectral deformations of metrics on each of SO(n) ($n \geq 9$), Spin(n) ($n \geq 9$), SU(n) ($n \geq 7$), and Sp(n) ($n \geq 4$). Except for low dimensions, all of these deformations are multidimensional.

5. Examples of Isospectral Potentials and Conformally Equivalent Metrics on Sp(n)

Here we apply Theorem 2.6 to Sp(n). For a particular choice of ϕ we will find that our pairs of isospectral potentials are noncongruent and that our pairs of conformally equivalent isospectral metrics are nonisometric. As before, we let T^r denote an r-dimensional torus with Lie algebra $\mathfrak{h}$.

EXAMPLE 5.1. First, consider T^r as an embedded subtorus of Sp(r) and consider Sp(n) $\times$ Sp(n) $\times$ Sp(r) as a block-diagonal subgroup of Sp($2n+r$). (We are thinking of elements of Sp($2n+r$) as complex matrices with $(2n+r)^2$ 2×2 blocks.) Suppose τ is the automorphism of Sp($2n+r$) given via conjugation by the matrix

$$c = \begin{bmatrix} 0 & \mathrm{Id}_{2n} & 0 \\ \mathrm{Id}_{2n} & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{2r} \end{bmatrix}. \tag{6}$$

By Theorem 3.1 we have continuous families of isospectral linear maps from the Lie algebra $\mathfrak{h}$ of T^2 into $\mathfrak{sp}(n)$ for $n \geq 2$. Suppose that j_1 and j_2 are two elements of a particular family satisfying the following genericity conditions.

GENERICITY CONDITIONS 5.2.

(a) The kernel of the map $j = (j_1, j_2) \colon \mathfrak{h} \to \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$ is trivial.
(b) The image of j has trivial centralizer in $\mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$.

Notice that Condition 5.2(a) is a mild condition that is easily satisfied. Suppose Condition 5.2(b) is not satisfied by the maps j_1 and j_2. This would imply that there is at least a one-dimensional family of elements X in $\mathfrak{sp}(n)$ such that $[X, j_1(Z)] = 0$ for all $Z \in \mathfrak{h}$. But then it would follow that there exists a one-dimensional family of elements A in Sp(n) such that $\mathrm{Ad}(A) j_1(Z) = j_1(Z)$ for all $Z \in \mathfrak{h}$. Thus, if j_1, j_2 are generic in the sense of Condition 4.7, then Condition 5.2(b) is automatically satisfied.

Let ϕ be any smooth function on G that is right-invariant under T^2 and invariant under conjugation by elements of Sp(n) $\times$ Sp(n). Then, following the construction in Theorem 2.6, we have isospectral potentials ϕ and $\tau^*\phi$ on Sp$(2n+2, g_j)$ for $n \geq 2$. Furthermore, if ϕ is positive then ϕg_j and $(\tau^*\phi) g_j$ are conformally equivalent isospectral metrics on Sp($2n+2$).

We now choose ϕ so that ϕ and $\tau^*\phi$ are not congruent on Sp$(2n+2, g_j)$ and so that ϕg_j and $(\tau^*\phi) g_j$ are not isometric. We follow Gordon and Schueth's construction and suppose that each matrix in Sp($2n+r$) is written

$$X = \begin{bmatrix} A & B & C \\ D & E & F \\ H & J & L \end{bmatrix}, \tag{7}$$

where A, B, C, D are $2n \times 2n$ matrices, L is a $2r \times 2r$ matrix, C and F are $2n \times 2r$ matrices, and H and J are $2r \times 2n$ matrices. Let $c_1 > c_2 > 0$ and define ϕ by

$$\phi(X) := \exp[c_1 \operatorname{Re}(\det A) + c_2 \operatorname{Re}(\det E)]. \tag{8}$$

This function ϕ is almost exactly the same as the function ϕ used for $\mathrm{SO}(2m+2r)$ in [GS] except that here we have taken the real parts of the determinants of $\det A$ and $\det E$ to account for the fact that matrices in $\mathrm{Sp}(n)$ have complex entries. In their proof, Gordon and Schueth had to treat the cases of $\mathrm{SO}(2m+2r)$ and $\mathrm{SU}(2m+r+1)$ separately because of the elements $\begin{bmatrix} \alpha\,\mathrm{Id}_m & & \\ & \beta\,\mathrm{Id}_m & \\ & & \gamma\,\mathrm{Id}_{r+1} \end{bmatrix}$ in $\mathrm{SU}(2m+r+1)$ (cf. our proof of Lemma 4.5). The analogous element does not exist in $\mathrm{Sp}(2n+r)$. Thus the proofs that ϕ and $\tau^*\phi$ are not congruent and that ϕg_j and $(\tau^*\phi)g_j$ are not isometric follow almost exactly the proofs for $\mathrm{SO}(2m+2r)$ once we make the obvious adjustments for $\mathrm{Sp}(2n+r)$. Therefore, our examples are nontrivial.

References

[Be1] P. Berard, *Transplantation et isospectralité I,* Math. Ann. 292 (1992), 547–599.

[Be2] ———, *Transplantation et isospectralité II,* J. London Math. Soc. (2) 48 (1993), 565–576.

[BGG] R. Brooks, R. Gornet, and W. Gustafson, *Mutually isospectral Riemann surfaces,* Adv. Math. 138 (1998), 306–322.

[Bu] P. Buser, *Isospectral Riemann surfaces,* Ann. Inst. Fourier (Grenoble) 36 (1986), 167–192.

[DG] D. DeTurck and C. Gordon, *Isospectral deformations II: Trace formulas, metrics, and potentials,* Comm. Pure Appl. Math. 42 (1989), 1067–1095.

[G] C. Gordon, *Isospectral closed Riemannian manifolds which are not locally isometric,* J. Differential Geom. 37 (1993), 639–649.

[G+] C. Gordon, R. Gornet, D. Schueth, D. Webb, and E. N. Wilson, *Isospectral deformations of closed Riemannian manifolds with different scalar curvature,* Ann. Inst. Fourier (Grenoble) 48 (1998), 593–607.

[GS] C. Gordon and D. Schueth, *Isospectral potentials and conformally equivalent isospectral metrics on spheres, balls, and Lie groups,* J. Geom. Anal. 13 (2003), 300–328.

[GWW] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds,* Invent. Math. 110 (1992), 1–22.

[GW1] C. Gordon and E. N. Wilson, *Isospectral deformation of compact solvmanifolds,* J. Differential Geom. 19 (1984), 241–256.

[GW2] ———, *Continuous families of isospectral Riemannian metrics which are not locally isometric,* J. Differential Geom. 47 (1997), 504–529.

[I] A. Ikeda, *On lens spaces which are isospectral but not isometric,* Ann. Sci. École Norm. Sup. (4) 13 (1980), 303–315.

[M] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds,* Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 542.

[OT] T. Ochiai and T. Takahashi, *The group of isometries of a left invariant Riemannian metric on a Lie group,* Math. Ann. 223 (1976), 91–96.

[P] H. Pesce, *Représentations relativement équivalentes et variétés riemanniennes isospectrales,* Comment. Math. Helv. 71 (1996), 243–268.

[Pr] E. Proctor, *Isospectral metrics on classical compact simple Lie groups,* Ph.D. thesis, Dartmouth College, 2003.

[S1] D. Schueth, *Continuous families of isospectral metrics on simply connected manifolds,* Ann. of Math. (2) 149 (1999), 287–308.

[S2] ———, *Isospectral manifolds with different local geometries,* J. Reine Angew. Math. 534 (2001), 41–94.

[Su] C. Sutton, *Isospectral simply-connected homogeneous spaces and the spectral rigidity of group actions,* Comment. Math. Helv. 77 (2002), 701–717.

[V] M.-F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques,* Ann. of Math. (2) 112 (1980), 21–32.

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