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Nonstability of the AK Invariant

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1. Introduction

In this paper we again try to understand (see [BM-L1; BM-L2]) what properties of a surface are related to the stability of its AK invariant. We call an AK invariant *stable* if

$$AK(S \times \mathbb{C}) = AK(S).$$

We provide an example of a surface with a nonstable AK invariant and nontrivial fundamental group. We also compute the invariant for the cylinders over surfaces endowed with a $\mathbb{C}$-action that has one singular fiber and where this fiber has reduced components.

Until now the surfaces $S_n = \{x^n y = p(x,z)\}$, where $p(0,v)$ does not have multiple roots, were the only examples of smooth surfaces with unstable invariant [BM-L1; D; Fie; W]. In particular, the Danielewski surfaces

$$S_{n,m} = \{x^n y = z^m - 1\}$$

are of this kind. All these surfaces have the following common features:

(i) $\pi_1(S_{n,m}) = 1$;
(ii) $S_{n,m}$ admits a fixed point–free $\mathbb{C}$-action;
(iii) fibering of $S_{n,m}$ that corresponds to a $\mathbb{C}$-action has only one nonconnected fiber;
(iv) all components of any fiber are reduced.

As far as nonstability of the invariant is concerned we show that (i) may be replaced by

(i′) $\pi_1(S_{n,m})$ is finite cyclic.

Condition (ii) is not important. However, (iii) seems to be important (see Section 3) and we believe that it cannot be relaxed. Condition (iv) is not crucial: we provide an example where it is not satisfied.

On the other hand if (iii) and (iv) are satisfied (the so-called generalized Danielewski surfaces [Du]) and if $AK(S) \neq \mathbb{C}$, then the invariant is nonstable.

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Let us call the fundamental group "small" if it is finite cyclic and "large" otherwise. In Section 2 we review some known facts about surfaces with large fundamental group and recall that their invariant is stable.

In Section 3 we give an example of a surface with small (but not trivial) fundamental group and unstable invariant. To obtain this example we describe a homological $\mathbb{Q}$-plane S with a $\mathbb{C}$-action and with $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$ (where m is prime) as a quotient of a surface S_n by the cyclic transformation group. In the case $AK(S) = \mathbb{C}$ this was done by Miyanishi and Masuda [MaMi1; MaMi2].

In Section 4 we prove that, for a surface S with properties (iii) and (iv),

$$AK(S \times \mathbb{C}) = \mathbb{C}.$$

2. Surfaces with Large Fundamental Group

DEFINITION 1. Let V be an affine variety and let $G(V)$ be the group generated by all $\mathbb{C}$-actions on V. Then $AK(V) \subset \mathcal{O}(V)$ is a *subring* of all regular $G(V)$-invariant functions on V.

We want to know when

$$AK(S \times \mathbb{C}) = AK(S). \tag{1}$$

If the logarithmic Kodaira dimension $\bar{k}(S) \geq 0$ then (1) holds by a theorem of Fujita and Iitaka [FuI]. So, let S be a surface with logarithmic Kodaira dimension $\bar{k}(S) = -\infty$. Miyanishi and Sugie [MiSu; Su] proved that the equality $\bar{k}(S) = -\infty$ is equivalent to the existence of a cylinder-like subset (c.l.s.).

DEFINITION 2. A surface S has a *cylinder-like subset* U (c.l.s. U) if it has a Zariski open subset $U \subset S$ that is isomorphic to the product $B \times \mathbb{C}$ of a smooth affine curve B and the complex line.

Any c.l.s. provides a line pencil on a surface S (see [Be; Mi1; MiSu]), that is, a morphism $\rho \colon S \to C$ into a smooth curve C ($C \supset B$) such that the fiber $\rho^{-1}(z)$ for a general $z \in C$ is isomorphic to $\mathbb{C}$. The pencils are different if their general fibers do not coincide. Any line pencil ρ over an affine curve C on a surface S corresponds to a $\mathbb{C}$-action φ_ρ on S such that its general orbit: (a) coincides with the general fiber of the pencil (see Lemma 1); and (b) corresponds to a locally nilpotent derivation (l.n.d.) ∂_ρ of the ring $\mathcal{O}(S)$ of regular functions on S such that $\partial_\rho f = 0$ if and only if f is φ_ρ-invariant [KM-L; M-L3; Mi1; S].

DEFINITION 3. Two $\mathbb{C}$-actions are *equivalent* if they induce the same fibering ρ (i.e., if they have the same general orbits).

DEFINITION 4. An affine surface is *affine rational* if there exists a dominant regular map from $\mathbb{C}^n$ into S for some $n \in \mathbb{N}$.

If in Definition 2 the surface S is affine rational, then $C \cong \mathbb{C}$ or $C \cong \mathbb{P}^1$.

For a pencil ρ over C, one can find a closure $\bar{S}$ of S and the extension $\bar{\rho}\colon \bar{S} \to \bar{C}$ such that (a) the general fiber $\bar{\rho}^{-1}(z) \cong \mathbb{P}^1$ and (b) any (-1) curve contained in a fiber intersects S. For any $z \in C$, the fiber $\rho^{-1}(z) = \bigcup C_i'$ is a union of smooth disjoint curves $C_i' \cong \mathbb{C}^1$ [Mi1, Lemma 4.4.1].

DEFINITION 5 [Fu]. Let $F_z = \overline{\rho^{-1}(z)} = \sum_{i=1}^{i=m} n_i C_i$, where the C_i are connected (and irreducible) components. If $m = 1$ and $n_1 = 1$, then the fiber is called *nonsingular.* A singular fiber either is nonconnected or has $m = 1$ and $n_1 > 1$. If $F_z = \sum_{i=1}^{i=m} C_i$ (i.e. $n_i = 1$) then the fiber is called *reduced.* The number $\mu(z) =$ g.c.d.(n_i) is called the *multiplicity* of a fiber. If a fiber has no points in S then, by definition, $\mu = \infty$.

The topology of surfaces with fibrations has been studied by Fujita. Let k be the number of fibers with multiplicity $\mu > 1$ (see Definition 5). We shall use the following facts from [Fu].

I. If the fundamental group $\pi_1(S)$ is infinite, then there is no dominant morphism $\phi\colon \mathbb{C}^n \to S$.
II. If the fundamental group $\pi_1(S)$ is finite, then $k \le 3$.
III. If $\pi_1(S)$ is finite non-abelian, then $k = 3$ and for all three fibers $1 < \mu < \infty$.
IV. If $\pi_1(S)$ is finite abelian and nontrivial, then either (a) $k = 3$, $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\mu = 2$ for all three singular fibers; or $k = 2$ and $\pi_1(S) \cong \mathbb{Z}/m\mathbb{Z}$, where $m > 1$ is the g.c.d. of two multiplicities.
V. If $\pi_1(S)$ is trivial, then either $k \le 1$ or $k = 2$ and $m = 1$.

In order to use this information for computing $AK(S \times \mathbb{C})$, we need the following lemmas.

LEMMA 1 (see also [MaMi2, Lemma 1.1]). *Assume that on a smooth affine surface S there is a $\mathbb{C}$-action ϕ. Let $\rho\colon S \to C$ be a regular map of S onto a smooth curve C such that the general fiber is an orbit of ϕ. Then C is affine.*

Proof. Since any $\mathbb{C}$-action on S defines an l.n.d. in $\mathcal{O}(S)$ (see [S]), this lemma follows from Lemma 1 of [M-L1], according to which there should be a regular nonconstant ϕ-invariant function in $\mathcal{O}(S)$. This function is constant along the fibers of ρ and so belongs to $\mathcal{O}(C)$. But then C cannot be compact. □

LEMMA 2. *Let A be an affine commutative domain. If $AK(A) = A$ then we have $AK(A[x]) = AK(A)$.*

Proof. Let us assume that ∂ is a nonzero locally nilpotent derivation of $A[x]$ that is not identically zero on A. Let $\mathrm{def}(r) = \deg(\partial(r)) - \deg(r)$, where "deg" denotes the degree relative to x. Let $m = \max\{\mathrm{def}(r) \mid r \in A[x]\}$. Then $m < \infty$.

Indeed, A is finitely generated. Let $y_1, \dots, y_k$ be a generating set of A. If $a \in A$ then $a = p(y_1, \dots, y_k)$, where p is a polynomial. Hence $\partial(a) = \sum_i \partial(y_i) p_i$, where the p_i are partial derivatives of p relative to y_i. Because all p_i are elements of A, we may conclude that $\mathrm{def}(a) = \deg(\partial(a)) \le \max\{\deg(\partial(y_i))\}$ is uniformly bounded. Now let $r \in A[x]$. Then $r = \sum_{i=0}^{n} r_i x^i$ where $r_i \in A$, so $\partial(r) = \sum_{i=0}^{n} \partial(r_i)x^i + \sum_{i=0}^{n} i r_i x^{i-1}\partial(x)$. Therefore,

$$\deg(\partial(r)) \le \max\{n + \max\{\deg(\partial(r_i))\}, n - 1 + \deg(\partial(x))\}$$

and so $\operatorname{def}(r) = \deg(\partial(r)) - n$ is uniformly bounded. By our definition of m we can write $\partial(a) = \sum_{i=0}^{m} \partial_i(a)x^i$ for any element $a \in A$. We also can write $\partial(x) = \sum_{i=0}^{m+1} e_i x^i$, where $e_i \in A$. Let us define a new derivation ϵ on $A[x]$ by $\epsilon(a) = \partial_m(a)x^m$ and $\epsilon(x) = e_{m+1}x^{m+1}$. It is easy to check that it is, in fact, a derivation and that it must be locally nilpotent, for otherwise ∂ cannot be locally nilpotent. Let $d(r)$ denote the degree of $r \in A[x]$ relative to ϵ (see [FLN; KM-L]; this degree is defined by the formula $d(r) = \max\{d \mid \epsilon^d(r) \neq 0\}$ and is nonnegative for nonzero r). Then $d(x) - 1 = d(e_{m+1}) + (m + 1)d(x)$. Since this equality is not possible for nonnegative $d(x)$ and $d(e_{m+1})$, we may conclude that $e_{m+1} = 0$. Hence x is a constant for ϵ and ∂_m is a locally nilpotent derivation on A, which implies that $AK(A) \neq A$. □

Together these facts yield the following proposition.

PROPOSITION 1. *Let S be an affine smooth surface with rational closure and let $W = S \times \mathbb{C}$. Then*:

(a) $AK(W) = AK(S)$ *if* $\pi_1(S)$ *is infinite*;
(b) $AK(W) = AK(S) = \mathcal{O}(S)$ *if* $\pi_1(S)$ *is finite and non-abelian*;
(c) $AK(W) = AK(S) = \mathcal{O}(S)$ *if* $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. We prove the proposition case by case.

(a) If $\pi_1(S)$ is infinite, then S is not affine rational [Fu]. On the other hand, if $AK(S) = \mathbb{C}$ then the orbit $G(S)$ contains an image of $\mathbb{C}^2$ under a dominant map [BM-L1]. Therefore, $AK(S) \neq \mathbb{C}$ and only two cases are possible.

- If $AK(S) = \mathcal{O}(S)$ (no actions at all), then by Lemma 2 we have $AK(W) = AK(S) = \mathcal{O}(S)$.
- If $AK(S) \neq \mathcal{O}(S)$ then $AK(S) = \mathcal{O}(C)$, where C is a base of the corresponding fibration (see Definitions 2 and 3).

The general orbit O_W of $G(W)$ contains $O_S \times \mathbb{C}$, where O_S is an orbit of a $\mathbb{C}$-action on S. On the other hand, O_W contains an image of $\mathbb{C}^2$ under a dominant map [BM-L1] and there is no dominant map from $\mathbb{C}^2$ into S. Hence, the image of O_W under the projection of W onto S is not dense and contains an orbit O_S; that is, it coincides with O_S. Therefore, $AK(W) = \mathcal{O}(C) = AK(S)$.

(b) If $\pi_1(S)$ is finite and nonabelian then, by [Fu], $k = 3$ and there is no fiber with $\mu = \infty$ for any linear pencil. By Lemma 1 we obtain that there are no $\mathbb{C}$-actions on such surfaces (another proof of this fact was explained to the authors by R. V. Gurjar in a private communication); by Lemma 2, $AK(W) = AK(S) = \mathcal{O}(S)$.

(c) $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ holds only if $k = 3$ and $\mu = 2$, so there are no $\mathbb{C}$-actions on these surfaces either. □

3. Surfaces with Small Fundamental Group

The surfaces with finite cyclic fundamental group are much more complicated from the AK-invariant point of view. We shall investigate three examples of surfaces S_i ($i = 1, 2, 3$) with $\pi_1(S_i) = \mathbb{Z}/2\mathbb{Z}$ and the AK invariant of all three possible types.

In the discussion of Examples 2 and 3 we shall use the following properties of locally nilpotent derivations. Let A be a ring with a filtration and let $\bar{A}$ be the corresponding graded ring. It is known (see [KM-L, Sec. 5]) that:

A. under some mild conditions (which are satisfied in our examples), a nonzero locally nilpotent derivation ∂ of A induces a nonzero locally nilpotent derivation $\bar{\partial}$ on $\bar{A}$, so that the kernel $\bar{A}^{\bar{\partial}}$ of $\bar{\partial}$ contains $\overline{A^{\partial}}$;

B. a nonzero locally nilpotent derivation ∂ of A allows us to define on A a function $d(a) = \max\{d \mid \partial^d(a) \neq 0\}$ (here a is a nonzero element of A)—this function is called the *degree function relative to* ∂ since it possesses all usual properties of a degree function [FLN];

C. if a nonzero product is a ∂ constant then all the factors are also ∂ constants (see [FLN]);

D. $[\mathrm{Frac}(A)]^{\partial} = \mathrm{Frac}(A^{\partial})$. Indeed, if $f = \alpha g$ and $\partial\alpha = 0$, then $\partial^k f = \alpha\partial^k g$ for all k ([M-L2], Lemma 1 of O. Hadas).

Example 1. $S_1 \subset \mathbb{C}^5$ is defined by the equations

$$y^2 = x(z-1), \quad t^2 = u(z-1), \quad z(z-1) = yt;$$
$$yu = tz, \quad yz = xt, \quad xu = z^2.$$

Observe that S_1 has two nonequivalent actions, which correspond to the following locally nilpotent derivations:

$$\begin{array}{ll} \partial x = 0, & \partial u = 0, \\ \partial y = x, & \partial t = u, \\ \partial z = 2y, & \partial z = 2t, \\ \partial t = 3z-2, & \partial y = 3z-2, \\ \partial u = 4t, & \partial x = 4y. \end{array}$$

Therefore, $AK(S_1) = \mathbb{C}$.

The fibering on S_1 is defined by the regular function x. There are two types of fibers.

1. The general fiber, $x \neq 0$. It is isomorphic to $\mathbb{C}^1$ with coordinate y:

$$z = y^2/x + 1, \quad t = \frac{y(y^2/x+1)}{x}, \quad u = \frac{(y^2/x+1)^2}{x}.$$

2. $x = 0$. Then $y = 0$, $z = 0$, and $u = -t^2$; that is, the fiber is isomorphic to $\mathbb{C}^1$. Take as local coordinates $h = y/(z-1)$ and t. Then

$$z = ht, \quad u = t^2/(ht-1), \quad y = h(ht-1), \quad x = h^2(ht-1),$$

which shows that x has a zero of the second order. Hence the multiplicity of the fiber is 2 and $\pi_1(S_1) = \mathbb{Z}/2\mathbb{Z}$.

The surface is smooth because it has two fixed point–free $\mathbb{C}$-actions. Actually this surface is described in Theorem 2.6 of [MaMi2]. The map

$$(\alpha, \beta, \gamma) \to \left(x = \frac{\alpha^2}{\delta},\ y = \frac{\alpha\gamma}{\delta},\ z = \frac{\alpha\beta}{\delta},\ t = \frac{\gamma\beta}{\delta},\ u = \frac{\beta^2}{\delta}\right)$$

is an isomorphism of S_1 and $\mathbb{P}^2 \setminus C$, where (α, β, γ) are homogeneous coordinates in $\mathbb{P}^2$, $\{\delta = \alpha\beta - \gamma^2\}$, and C is a conic $\{\delta = 0\}$.

EXAMPLE 2.

$$S_2 = \{\alpha(\alpha\beta + 1) = \gamma^2\} \subset \mathbb{C}^3,$$

where α, β, γ are coordinates in $\mathbb{C}^3$. Here S_2 has an action corresponding to the l.n.d. $\partial\alpha = 0$, $\partial\beta = 2\gamma$, $\partial\gamma = \alpha^2$. The fibering on S_2 is defined by the regular function α, and again there are two types of fibers.

1. The general fiber, $\alpha \neq 0$. It is isomorphic to $\mathbb{C}^1$ with coordinate γ:

$$\beta = \frac{\gamma^2 - \alpha}{\alpha^2}.$$

2. $\alpha = 0$. Then $\gamma = 0$, and local coordinates may be chosen as $h = \beta/(\alpha\beta + 1)$ and γ:

$$\alpha\beta = \gamma^2 h, \quad \beta = h(h\gamma^2 + 1), \quad \alpha = \frac{\gamma^2}{h\gamma^2 + 1};$$

this shows that α has a zero of the second order, the multiplicity of the fiber is 2, and $\pi_1(S_2) = \mathbb{Z}/2\mathbb{Z}$.

Let us show that S_2 has just one class of l.n.d.s up to equivalence (i.e., all other l.n.d.s define the same fibration). The ring $\mathcal{O}(S_2) = R$ can be embedded in $\mathbb{C}[a, b]$ by mapping α to a^2, γ to $a(1 + a^3 b)$, and β to $ab(2 + a^3 b)$. One l.n.d. on R is defined by $\partial(a) = 0$ and $\partial(b) = 1$. Assume that there is another nonzero l.n.d. ϵ that does not send a to zero. Let f be from its kernel. Let us take the lexicographic order $b \gg a$ on $\mathbb{C}[a, b]$ and the induced order on R. We denote by $\bar{u}$ the leading form of an element u and denote by $\bar{R}$ the ring that is generated by the leading forms of elements of R.

Clearly $\bar{f}$ is $a^i b^j$ where $j > 0$. Since $R = \mathbb{C}[\alpha, \beta] + \gamma\mathbb{C}[\alpha, \beta]$, we see that $\bar{R}$ is generated by a^2, a^4b^2, and a^4b. Therefore, ϵ defines a nonzero l.n.d. $\bar{\epsilon}$ on $\bar{R}$ and $\bar{f} \in \ker(\bar{\epsilon})$ (see property A). Replacing f by f^2 if necessary, we may assume that j is even and so $\bar{f} = (a^4b^2)^k$ (since $\bar{f}$ is $\bar{\epsilon}$-constant, if a factorization of $\bar{f}$ contains two different generators of $\bar{R}$ then $\bar{\epsilon}$ is the zero derivation by property C).

Let us consider the degree function induced by $\bar{\epsilon}$ on $\bar{R}$. Since $\deg(a^4b^2) = 0$, we may assume that $\deg(a) = d$ and $\deg(b) = -2d$. Therefore $\deg(a^2) = 2d$, $\deg(a^4 b) = 2d$, and elements of $\bar{R}$ have only even degrees. Since application of $\bar{\epsilon}$ decreases degree by 1, this is impossible. So ϵ must be the zero derivation. It follows that $AK(S_2) = \mathbb{C}[x]$.

We derive the same conclusion by considering the "minimal" complement of this surface: the divisor in the complement is not a "zigzag" (i.e., it is not a linear chain; see [BM-L2; Be; G]).

A third way of reasoning is as follows: our surface is a hypersurface. On the other hand, if it had a trivial AK invariant then it should be isomorphic to $W = \mathbb{P}^2 \setminus C$, where C is a smooth conic [MaMi2, Thm. 2.6]. But this is impossible, because W has nontrivial canonical class.

EXAMPLE 3. $S_3 \subset \mathbb{C}^7$ is defined by

$$uv = z(z-1), \tag{2}$$
$$v^2 z = uw, \tag{3}$$
$$z^2(w-1) = xu^2, \tag{4}$$
$$u^2(z-1) = tv, \tag{5}$$
$$(z-1)^2(t-1) = yv^2, \tag{6}$$
$$u^2v^2 = wt, \tag{7}$$
$$yz^2 = u^2(t-1), \tag{8}$$
$$x(z-1)^2 = v^2(w-1), \tag{9}$$
$$v^4x = w^2(w-1), \tag{10}$$
$$u^4y = t^2(t-1), \tag{11}$$
$$v^3 = (z-1)w, \tag{12}$$
$$u^3 = tz. \tag{13}$$

The equations (7)–(13) are consequences of the equations (2)–(6). The surface S_3 is smooth, because the rank of the Jacobi matrix of equations (2)–(13) is maximal everywhere. It has an automorphism $\alpha\colon (u, v, z, t, w, x, y) \to (-v, -u, 1-z, w, t, y, x)$.

Now let us consider the fibering over $\mathbb{P}^1$ defined by the rational function $f = u/z = (z-1)/v$. It is defined everywhere, since z and $z-1$ do not vanish simultaneously. We have three cases.

Case 1: The fiber $f = k \neq 0$ is isomorphic to $\mathbb{C}^1$ with coordinate z. Then

$$u = kz, \quad v = \frac{z-1}{k}, \quad w = \frac{v^2 z}{u} = \frac{v^2}{k},$$
$$x = \frac{w-1}{k^2}, \quad t = u^2 k, \quad y = (t-1)k^2.$$

Case 2: Assume that $k = 0$. Then $u = 0$ and $z - 1 = 0$ so $z = 1$. We have $v = 0$ by equation (3), $w = 1$ by (4), $y = 0$ by (8), and $t = 0$ by (7). The local parameters near this fiber are two functions: $g = x/w^2$ (which is defined, since $w = 1$ in the neighborhood of this fiber) and v. Indeed, we can make the following statements.

- (10) gives $gv^4 = w - 1$; that is, $w = 1 + gv^4$ and $x = gw^2 = g(1 + gv^4)^2$.
- (3) gives $f = u/z = v^2/w = v^2/(1 + gv^4)$, which is defined because $v = 0$ along this fiber (and hence the denominator does not vanish in a neighborhood of the fiber).
- By definition, $z - 1 = fv = v^3/(1 + gv^4)$; that is, $z = (v^3/(1 + gv^4)) + 1$. Hence

$$u = fz = \frac{v^2}{1 + gv^4}\left(\frac{v^3}{1 + gv^4} + 1\right),$$

$$t = \frac{u^2(z - 1)}{v} = u^2 f = \frac{u^2 v^2}{1 + gv^4},$$

$$y = \frac{(z - 1)^2(t - 1)}{v^2} = (t - 1)f^2 = \frac{(t - 1)v^4}{(1 + gv^4)^2}.$$

In these local coordinates $f = v^2/(1 + gv^4)$ has order 2, so this component has multiplicity 2.

Case 3: The fiber $f = \infty$, $f' = 1/f = 0$. We have $v = 0$ and $z = 0$, and from equations (5) and (6) we obtain $u = 0$ and $t = 1$, respectively. From (9) and (7) we get $x = 0$ and $w = 0$. The local parameters in the neighborhood of this fiber are $h = y/t^2$ and u. Thus:

- from (11) we have $t = u^4 h + 1$;
- from (5) we have

$$f' = \frac{u^2}{t} = \frac{u^2}{u^4 h + 1},$$

$$z = uf' = \frac{u^3}{u^4 h + 1}, \qquad v = f'(z - 1) = \frac{(z - 1)u^2}{u^4 h + 1},$$

$$w = \frac{v^2 z}{u} = \frac{v^2 u^2}{u^4 h + 1}, \qquad x = \frac{z^2(w - 1)}{u^2} = \frac{(w - 1)u^4}{(u^4 h + 1)^2}.$$

Once more, $f' = u^2/(u^4 h + 1)$ has order 2. Hence this fibration has two fibers of multiplicity 2, and $\pi_1(S_3) = \mathbb{Z}/2\mathbb{Z}$.

We want to show now that $AK(S_3) = \mathcal{O}(S_3)$. We do this in nine steps as follows.

1. Express all variables as rational functions in u and z:

$$v = \frac{z(z - 1)}{u}, \qquad w = \frac{v^2 z}{u},$$

$$x = \frac{z^2(w - 1)}{u^2}, \quad t = \frac{u^2(z - 1)}{v}, \quad y = \frac{u^2(t - 1)}{z^2}.$$

2. Take $u = a(a^4 c - 1)$ and $z = a^3(a^4 c - 1)$. Then all variables are polynomials in a and c.

3. We consider two rings: $R = \mathcal{O}(S_3)$, and R_1 as generated by $(a, a^4 c - 1, c(a^4 c - 2)) = \left(\sqrt{z/u}, \sqrt{u^3/z}, y\right)$. Then $R \subset R_1$.

4. Take weights $w(c) = 1$ and $w(a) = 1+\rho$, where ρ is a small irrational number. Then the corresponding ring $\bar{R}$ contains only monomials. It certainly contains the leading forms of all generators: a^5c, a^7c, a^9c, a^4c^2, a^8c^2, $a^{20}c^2$, $a^{24}c^2$. The ring $\overline{R_1}$ is generated by a, a^4c, a^4c^2 because each of its elements contains only monomials of this kind.

5. Assume that $\bar{R}$ has a nonzero l.n.d. ∂. Then $\bar{R}$ contains a ∂-constant monomial v and a monomial v_1 that has degree 1 relative to this derivation. We choose a monomial $v_0 \in \mathrm{Frac}(\bar{R})$ in such a way that:

(a) $v = v_0^s$ for some natural s;
(b) v_0 has the minimal positive weight among all the monomials meeting condition (a).

Then $\bar{R}$ belongs to the algebra generated by $v_0^{\pm 1}$ and v_1.

6. In the plane we consider a set S (respectively, S_1) of all the points (n,m) such that the monomial $a^nc^m \in \bar{R}$ (resp., $a^nc^m \in \overline{R_1}$). Evidently, $S \subset S_1$ and both contain the point $(4,2)$; both sets are contained in the first quadrant. Moreover, S_1 is the cone generated by the vectors $(1,0)$, $(4,1)$, $(4,2)$. Thus, the line $L = \{(2t,t)\} = \{x - 2y = 0\}$ is the boundary line for set S and for set S_1 (both sets have two boundary lines of which L is one). Also, $S_1 \cap L = \{n(4,2), n \in \mathbb{N}\}$.

7. Since all monomials of $\bar{R}$ have nonnegative ∂-degree, v_0 corresponds to a point p_0 in a boundary line. Indeed, in computing $\partial(a^nc^m)$ we add to a point (n,m) the vector that depends on ∂ but does not depend on the point.

8. Assume that $p_0 \in L$. Then $\deg_\partial(a) = 1$, for otherwise even $\overline{R_1}$ would not contain an element of degree 1. On the other hand, $\bar{R}^\partial$ is generated by a^4c^2 in this case. It follows that $\mathrm{Frac}(\bar{R}^\partial) = [\mathrm{Frac}(\bar{R})]^\partial$ is generated by a^4c^2. Hence $\bar{R} \subset \mathrm{span}(a, (a^4c^2)^{\pm 1})$ (see step 5) and cannot contain the monomials with odd powers of c. But $\bar{R} \subset \overline{R_1}$ and contains a^5c, a^7c, a^9c, a^4c^2, a^8c^2, $a^{20}c^2$, and $a^{24}c^2$. Thus, $\partial(a^4c^2) = 0$ is impossible.

9. The automorphism α sends a^4c^2 to $a^{24}c^2$, so this monomial generates the second boundary line for S. Consequently, if $\partial(a^4c^2) \neq 0$ then $\partial(a^{24}c^2) = 0$ and there is another l.n.d. ∂' such that $\partial'(a^4c^2) = \alpha\partial\alpha(a^4c^2) = 0$, which is impossible (step 8).

Thus, $AK(S_3) = \mathcal{O}(S_3)$.

Next, we want to build a surface S such that

- $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$,
- $AK(S) = \mathbb{C}[x]$,
- $AK(S \times \mathbb{C}) = \mathbb{C}$.

We start with a description of the surfaces with only two singular fibers: one fiber of multiplicity $\mu = \infty$, and one fiber of multiplicity m that consists of the single curve $C \cong \mathbb{C}^1$. These surfaces are $\mathbb{Q}$-planes (see e.g. [Mi3]). It was proved in [MaMi2] that any $\mathbb{Q}$-plane S with $AK(S) = \mathbb{C}$ is a quotient of the Danielewski surface $S_{1,m}$ by an action of the group $\mathbb{Z}/m\mathbb{Z} = \pi_1(S)$. We want to use a similar description for $\mathbb{Q}$-planes S with $AK(S) = \mathbb{C}[x]$.

PROPOSITION 2. *Let S be a smooth $\mathbb{Q}$-plane admitting a $\mathbb{C}$-action. Let $\pi_1(S) = \mathbb{Z}/m\mathbb{Z}$, where m is prime. Then $S \cong V/\mathcal{G}$, where*

- *V is a hypersurface in $\mathbb{C}^3$ with coordinates (u, y, v): $V = \{u^k y = v^m - v_1^m + u\tilde{q}(u, v)\}$ for $\tilde{q}(u, v)$ a polynomial of degree less than m relative to v and $v_1 \in \mathbb{C}$; and*
- *$\mathcal{G}$ is a group of transformations generated by $g(u, y, v) = (u\varepsilon, y\varepsilon^{-k}, v\varepsilon^{-\alpha})$ for some $\alpha, k \in \mathbb{N}$ and $\varepsilon = e^{2i\pi/m}$.*

Proof. Let ∂ be an l.n.d. corresponding to a $\mathbb{C}$-action on S. From the general properties of $\mathbb{Q}$-planes and $\mathbb{C}$-actions (see e.g. [Mi3; BM-L2, Prop. 1]), it follows that there are two functions $x \in \mathcal{O}(S)$ and $z \in \mathcal{O}(S)$ such that:

- a fiber $\{x = c \neq 0\} = F_c \cong \mathbb{C}$ is reduced;
- the fiber $\{x = 0\} = F_0 \cong \mathbb{C}$ has multiplicity m;
- for $c \neq 0$ the function $z|_{F_c}$ is linear (i.e., it takes any finite value at exactly one point);
- $z|_{F_0} = 0$;
- $\partial x = 0$, $\partial z = x^n$, and n is the minimum possible for z with these properties.

Now consider a surface U defined as the normalization of $S \times_{\mathbb{C}} \Gamma$, where Γ denotes the m-ramified cover of $\mathbb{C}$ with the only branch point at $\gamma = 0$ over $x = 0$ (similar to [MaMi2]). Let $\pi : U \to S$ and $p : U \to \Gamma$ be the natural projections. This surface U has the following properties:

(a) there is a function $u \in \mathcal{O}(U)$ such that $\pi^* x = u^m$;
(b) U is an unramified m-sheeted cover of S;
(c) the fiber $\mathcal{E}_0 = \{u = 0\} = \bigcup_{i=1}^m E_i$ is a union of m reduced components $E_i \simeq \mathbb{C}^1$;
(d) U is affine [Sa, Sec. II.5, Thm. 4];
(e) the general fiber $E_c = \{u = c\} \simeq \mathbb{C}$.

Indeed, we can make the following statements.

- In property (a), $u = p^*\gamma$.
- Properties (b) and (c) are proved in [BaPV, Sec. III.9] for proper maps. Since both properties are local, the same proof is valid in our case as well. The only modification needed in the proof is to use $\pi^*\phi_f$, where $\phi_f = 0$ is a local equation of F_0 in an appropriate neighborhood of a point $f \in F_0$, instead of the function g (the global equation of the multiple fiber there).
- Property (e) is valid because $E_c \simeq F_{c^m}$ by construction.

Property (e) enables the existence of a $\mathbb{C}$-action φ on U with general orbit $\{u = \text{const.}\}$. Let ∂_1 be a corresponding l.n.d.: $\partial_1(\pi^*(f)) = \pi^*(\partial f)$ for $f \in \mathcal{O}(S)$; $\partial_1(u) = 0$. Since all the fibers $\{u = \text{const.}\}$ are reduced, we have $\pi_1(U) = 0$. It follows that U is the universal covering of S and that $S = U/\mathcal{G}$, where $\mathcal{G}$ is the group of deck transformations. Any transformation $g \in \mathcal{G}$ sends the fiber $E_{u_0} = \{u = u_0\}$ to the fiber $E_{u_0\varepsilon^k} = \{u = u_0\varepsilon^k\}$, where $k \in N$ and $\varepsilon = e^{2i\pi/m}$. In particular, it moves a component E_i of $\mathcal{E}_0$ to the component $E_j \subset \mathcal{E}_0$.

Since $z|_{F_0} = 0$, we have $\pi^*z|_{\mathcal{E}_0} = 0$. But all the components E_i are reduced; hence $\pi^*z = u^\alpha \cdot v$ for some $\alpha \in \mathbb{N}$ and $v \in \mathcal{O}(U)$ such that $v|_{\mathcal{E}_0} \not\equiv 0$. Further on we shall use z and x instead of π^*z and π^*x when no confusion may arise.

LEMMA 3.

(a) $(\alpha, m) = 1$.
(b) *v is linear along each fiber $E_{u_0} = \{u = u_0 \neq 0\}$.*
(c) *$v|_{E_i} = v_i$ is a constant, and $v_i \neq v_j$ if $i \neq j$.*
(d) $\partial_1 v = u^{mn-\alpha}$.
(e) *$g^*(\partial_1 h) = \partial_1(g^*h)$ for any $h \in \mathcal{O}(U)$.*

Proof. (a) Since $z|_{F_0} = 0$, there are coprime numbers r and k and a function $\psi \in \mathcal{O}(S)$ such that $z^k = x^r \cdot \psi$ and $\psi|_{F_0} \not\equiv 0$. It follows that $u^{\alpha k}v^k = u^{mr}\pi^*(\psi)$, that is, $\alpha k = mr$. Since m is prime, the following two cases are possible.

(1) $\alpha = mr$, $k = 1$, and $z = x^r \cdot \psi$, with $\partial z = x^r \cdot \partial\psi = x^n$ and $\partial\psi = x^{n-r}$. Then ψ is linear along a general fiber and constant along F_0 [BM-L2, Prop. 1]; that is, n is not minimal.
(2) $\alpha = r$ and $k = m$. But then $(\alpha, m) = (k, r) = 1$.

(b) $v = z/u^\alpha$ is linear along each fiber $\{u = c \neq 0\}$ because z is linear along a fiber $\{x = \text{const.}\}$.

(c) Take now E_1 and g a generator of G such that $g^*(u) = u\varepsilon$, and let $E_2 = g(E_1)$, $E_3 = g(E_2)$, …, $E_m = g(E_{m-1})$. Then

$$g^*(u) = u \cdot \varepsilon, \qquad g^*(v) = g^*(z) \cdot g^*(u)^{-\alpha} = z \cdot u^{-\alpha}\varepsilon^{-\alpha} = v\varepsilon^{-\alpha}.$$

If v were not constant along a component $E_1 \subset \mathcal{E}_0$ then it would be nonconstant along each E_i, $i = 1, \dots, m$. This means that v would take any value $v_0 \in \mathbb{C}$ at least m times in $\mathcal{E}_0$, which contradicts the fact that v is linear along the general fiber. (Indeed, $\{v = v_0\}$ intersects each fiber at most at one point).

Thus, $v|_{E_i} = v_i = \text{const}$. Then

$$v_2 = \varepsilon^{-\alpha}v_1,\ v_3 = \varepsilon^{-2\alpha}v_1,\ \dots,\ v_m = \varepsilon^{-(m-1)\alpha}v_1;$$

that is, $v_i \neq v_j$.

(d) $\partial_1 v = \partial_1(z/u^\alpha) = x^n/u^\alpha = u^{mn-\alpha}$.

(e) Any function $h \in \mathcal{O}(U)$ may be represented as $h = q(u, v)/u^l$ for some polynomial q, since U is affine rational. The property may then be obtained by a direct computation. □

LEMMA 4. *The surface U admits a fixed point–free $\mathbb{C}$-action equivalent to φ.*

Proof. We must prove that $\partial_1 = u^\beta\tilde{\partial}$, where $\beta \geq 0$ and $\tilde{\partial}$ is an l.n.d. of $\mathcal{O}(U)$ such that, for any $i = 1, \dots, m$, if $h \in \mathcal{O}(U)$ and $h|_{E_i} \neq \text{const.}$ then $\tilde{\partial}h|_{E_i} \not\equiv 0$.

Let $\partial_1 h|_{E_i} \equiv 0$ for all $h \in \mathcal{O}(U)$. That is, let $\partial_1 h = u^{n_i(h)}\psi$, where ψ is rational on U and where $\psi|_{E_i} \not\equiv 0$ and $\psi|_{E_i} \neq \infty$. Let n_i be the minimal value of $n_i(h)$ for $h \in \mathcal{O}(U)$. Since the action of group $\mathcal{G}$ permutes the components, it follows that $n_1 = n_2 = \cdots = n_m$ and that $\tilde{\partial} = \partial_1/u^{n_1}$ is defined and corresponds to a fixed point–free $\mathbb{C}$-action. $\square$

LEMMA 5. *Let $\tilde{\partial} v = u^k$. Then there exists a polynomial $p(u,v) = p_0(v) + up_1(u,v)$ such that the function $y = p(u,v)/u^k \in \mathcal{O}(U)$ is linear along each E_i, $i = 1,\dots,m$.*

Proof. The polynomial $p_0(v) = v^m - v_1^m$ vanishes along $\mathcal{E}_0$: $p_0(v) = u^{s_1} \cdot y_1$, where $y_1 \in \mathcal{O}(U)$ and $y_1|_{E_i} \not\equiv 0$ for at least one value of i. Since $g^*(y_1) = y_1\varepsilon^{-s_1}$, it follows that y_1 is either constant or nonconstant along all E_i simultaneously and that $y_1|_{E_i} \not\equiv 0$ for all i.

We have

$$u^{s_1} y_1 = p_0(v),$$
$$u^{s_1} \tilde{\partial} y_1 = p_0'(v) u^k,$$
$$\tilde{\partial} y_1 = p_0'(v) u^{k-s_1} \in \mathcal{O}(U);$$

it follows that $k \geq s_1$. Consider two cases.

Case I: $y_1|_{E_i} \neq$ const. Then $\tilde{\partial} y_1 = p_0'(v) u^{k-s_1} \not\equiv 0$; that is, $k - s_1 = 0$, $\tilde{\partial} y_1|_{E_i} =$ const., and y_1 is linear along each component E_i.

Case II: $y_1|_{E_1} = c_1 =$ const. Then $y_1|_{E_i} = c_1\varepsilon^{-s_1 i}$. Let $h < m$ be the natural number such that $s_1 \equiv \alpha h \pmod{m}$ and $p_1(v) = c_1(v/v_1)^h$. Then $y_1 - p_1(v)|_{\mathcal{E}_0} \equiv 0$ and $y_1 - p_1(v) = u^{s_2} y_2$, where $y_2 \in \mathcal{O}(U)$ and $y_2|_{E_i} \not\equiv 0$ for at least one value of i.

Since $g^*(y_2) = y_1\varepsilon^{-s_1-s_2}$, we have that y_1 is either constant or nonconstant on all E_i simultaneously and that $y_2|_{E_i} \not\equiv 0$ for all i. Now

$$u^{s_1+s_2} y_2 = p_0(v) - u^{s_1} p_1(v),$$
$$\tilde{\partial} y_2 = (p_0'(v) - u^{s_1} p_1'(v)) u^{-(s_1+s_2)+k},$$

so $k \geq (s_1 + s_2)$.

If $y_2|_{E_i} \neq$ const. then it is linear along each E_i; if not, we continue the process. After r steps we obtain

$$u^{s_1+s_2+\cdots+s_r} y_r = p_0(v) - u^{s_1} p_1(v) - \cdots - u^{s_1+s_2+\cdots+s_{r-1}} p_{r-1}(v).$$

Then, either

$$s_1 + s_2 + \cdots + s_r < k, \quad y_r|_{E_i} = \text{const.}$$

or

$$s_1 + s_2 + \cdots + s_r = k, \quad y_r|_{E_i} \neq \text{const.},$$

whence

$$\tilde{\partial} y_r = (p_0'(v) - u^{s_1} p_1'(v) - \cdots - u^{s_1+s_2+\cdots+s_{r-1}} p_{r-1}'(v))$$

and y_r is linear along the components E_i.

So, after a finite number of steps we will derive a function y, linear along each E_i, with

$$u^k y = p_0(v) + u\tilde{q}(u,v).$$

Moreover, by construction, $g^*y = y\varepsilon^{-k}$ and $q(u,v)$ has degree less than m relative to v. $\square$

We may now proceed with the proof of Proposition 2. Let $\theta\colon U \to \mathbb{C}^3$ be defined as $\theta(p) = (u, y, v)$ for a point $p \in U$. This regular map is bijective: indeed, the function u distinguishes the fibers, v is linear along the general fiber and distinguishes components of the singular fiber, and y is linear along the components of the singular fiber.

The surface $V = \theta(U)$ is given by

$$\{u^k y = v^m - v_1^m + u\tilde{q}(u,v)\}.$$

Since V is a smooth surface bijection, θ is an isomorphism. $\square$

Example 4. We use Proposition 2 for the case $m = 2$, $s = 2$. The Danielewski surface

$$S_{2,2} = \{x^2 y_2 = z_2^2 - 1\}$$

is the universal cover of the surface S_2 (see Example 2). Consider the following map:

$$(x, y_2, z_2) \to (\alpha = x^2,\ \beta = y_2,\ \gamma = xz_2).$$

This map glues the points $s = (x, y_2, z_2)$ and $Ts = (-x, y_2, -z_2)$ and is an unramified double covering. Obviously,

$$\alpha^2\beta + \alpha = \gamma^2.$$

The surface S_2 has the fundamental group $\mathbb{Z}/2\mathbb{Z}$ and precisely one singular fiber $\{\alpha = 0\}$, which is a double straight line. All actions on this surface are equivalent: $AK(S_2) = \mathbb{C}[x]$.

Now we want to compute the invariant of its cylinder $Z = S_2 \times \mathbb{C}$. The universal cover of this cylinder is $Y = S_{2,2} \times \mathbb{C} \cong S_{1,2} \times \mathbb{C}$ [D; Fie], where $S_{1,2} = \{xy_1 = z_1^2 - 1\}$.

Lemma 6 (communicated by P. Russell).

$$Y = \left\{\begin{aligned} xy_1 &= z_1^2 - 1,\\ x^2 y_2 &= z_2^2 - 1,\\ xu &= z_1 - z_2,\\ y_1 - xy_2 &= u(z_1 + z_2). \end{aligned}\right.$$

In Y the surface $S_{2,2}$ is defined as $u = \text{const.}$ and the surface $S_{1,2}$ as $w = u^3x + 3u^2z_2 + 3uxy_2 + z_2y_2 = \text{const.}$

Proof. Let $A = \mathbb{C}[x, y_1, y_2, z_1, z_2, u]$, where $xy_1 = z_1^2 - 1$, $x^2y_2 = z_2^2 - 1$, and $xu = z_1 - z_2$. Then $A = \mathbb{C}[x, y_2, z_2][u]$, since $z_1 = xu + z_2$ and $y_1 = u^2x + 2uz_2 + xy_2$.

Similarly, $A = \mathbb{C}[x, y_1, z_1][w]$, where $w = u^3x + 3u^2z_2 + 3uxy_2 + z_2y_2$, since $xw = y_1z_1 - 2u$ and $wz_1 = y_2 - u^2 + y_1^2$. Hence $2u = -xw + z_1y_1$, $y_2 = wz_1 + u^2 - y_1^2$, and $2z_2 = x^2w - z_1^3 + 3z_1$. $\square$

The cylinder $Z = S_2 \times \mathbb{C}$ is a quotient of Y by the transformation T', which preserves $S_{2,2}$ (i.e., the value of u) and coincides on $S_{2,2}$ with T:

$$T'\colon (x, y_2, z_2, y_1, z_1, u) \to (-x, y_2, -z_2, -y_1, -z_1, u).$$

On the other hand, $Z = \{(\alpha, \beta, \gamma, t) : \alpha^2\beta + \alpha = \gamma^2\} \subset \mathbb{C}^4$ and the map $Y \to Z$ is defined as

$$\alpha = x^2, \quad \beta = y_2, \quad \gamma = xz_2, \quad t = u.$$

By Lemma 6 we have

$$\begin{aligned} u &= \tfrac{1}{2}(-xw + z_1 y_1), \\ y_2 &= wz_1 + u^2 - y_1^2, \\ z_2 &= \tfrac{1}{2}(x^2 w - z_1^3 + 3z_1). \end{aligned}$$

Any l.n.d. defined on $\mathcal{O}(S_{1,2})$ may be extended to an l.n.d. on $\mathcal{O}(Y)$. In particular this applies to the l.n.d. defined by

$$\partial y_1 = 0, \quad \partial z_1 = y_1, \quad \partial x = 2z_1, \quad \partial w = 0.$$

Then

$$\begin{aligned} \partial u &= \tfrac{1}{2}(-2z_1 w + y_1^2), \\ \partial y_2 &= (wy_1 + u(y_1^2 - 2wz_1)), \\ \partial z_2 &= \tfrac{1}{2}(4xwz_1 - 3z_1^2 y_1 + 3y_1), \end{aligned}$$

and ∂ is invariant under T'; that is, $\partial T' = T'\partial$. For example,

$$T'(\partial x) = T'(2z) = -2z = \partial T'(x),$$

and we can check similarly for the remaining generators.

As a result this l.n.d. can be pushed down to $\mathcal{O}(Z)$, which is the quotient of Y by T'. This means that in Z there is an l.n.d. for which α is not invariant. Hence $AK(Z) = \mathbb{C}$, though $AK(S_2) = \mathbb{C}[x]$.

4. Cylinders over Surfaces with Reduced Fibers and Primitive Danielewski–Feiseler Quotient

We want to show in this section that property (ii) of Danielewski surfaces (see Section 1) is not essential for nonstability of the AK invariant of a surface. The Danielewski construction represents a surface admitting a $\mathbb{C}$-action with reduced components of fibers as an affine bundle over the Danielewski–Fieseler quotient.

For a given c.l.s. $U \subset S$ or a corresponding fibering ρ, the Danielewski–Fieseler (DF) quotient $X = S/_U = S/_\rho$ is a (nonseparated) prevariety X such that the points of X are in one-to-one correspondence with the connected components of $\rho^{-1}(c)$ for all $c \in C$. The precise and detailed definition is given in [D; Fie].

We describe this quotient in the following way.

Definition 6. A quotient X of a smooth surface S by a pencil ρ is an algebraic prevariety X included in the commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "\rho"'] & & X \arrow[dl, "\sigma"] \\ & C & \end{tikzcd}$$

and having the following properties.

- The maps π, σ, ρ are regular.
- σ is an isomorphism of $X \setminus \{x_{i,j}\}$ $(i = 1, \dots, r)$ onto $C \setminus \{c_1, \dots, c_r\}$, where the set of points $\{x_{i,j}\} \in X$ is finite.
- Let $F_{c_i} = \rho^{-1}(c_i) = \bigcup_{j=1}^{t} C_{ij}$ be the union of t connected (necessarily irreducible; [Mi1, Lemma 4.4.1]) components C_{ij}. Then $\pi|_{C_{ij}} = x_{ij}$ and $x_{ij} \neq x_{ik}$ if $j \neq k$. There is a one-to-one correspondence between points $x_{ij} \in \sigma^{-1}(c_i)$ and components $C_{ij} \subset \rho^{-1}(c_i)$.

We call X *primitive* if $C \cong \mathbb{C}$ and there is only one singular fiber.

Theorem 1. *Let S be a smooth affine surface and let $\alpha\colon S \times \mathbb{C} \to S$ be a $\mathbb{C}$-action on S for which all the components of all the fibers are reduced. Assume that the DF quotient $X = S/\alpha$ is primitive. Then $AK(\mathcal{O}(S \times \mathbb{C})) \cong \mathbb{C}$.*

The proof of this theorem is based on the following theorem of Kaliman and Zaidenberg [KZ].

Theorem KZ. *Assume that there exists a dominant morphism $f\colon X \to S$ of a smooth quasiprojective variety X to a smooth quasiprojective variety S, and assume that a general fiber $f^{-1}(s)$, $s \in S$, is isomorphic to $\mathbb{C}^2$. Then there exists a Zariski open subset S_0 of S such that $f^{-1}(S_0) \cong S_0 \times \mathbb{C}^2$. Moreover, if we denote by ϕ the isomorphism $f^{-1}(S_0) \to S_0 \times \mathbb{C}^2$ and by $p\colon S_0 \times \mathbb{C}^2 \to S_0$ the projection to the first factor, then $p^{-1}(s) = \phi(f^{-1}(s))$ for any $s \in S_0$.*

Proof of Theorem 1. Since X is primitive, the base of the fibering $C \cong \mathbb{C}$ and we may assume that the multiple point is $c = 0$. Let $\sigma^{-1}(0) = \{x_1, \dots, x_k\}$. Consider a surface $S_1 \subset \mathbb{C}^3$ defined by

$$xy = (z-1)(z-2)\cdots(z-k) = p(z),$$

where $\{x, y, z\}$ are coordinates in $\mathbb{C}^3$. Here S_1 is a smooth affine surface with two $\mathbb{C}$-actions β and δ. The orbits of β and δ are the curves $R_x = \{x = \text{const.}\}$ and $R_y = \{y = \text{const.}\}$ (respectively), and the corresponding l.n.d.s of $\mathcal{O}(S_1)$ are defined as follows.

I. β: $\partial x = 0$, $\partial z = x$, and $\partial y = p'(z)$.
II. δ: $\partial y = 0$, $\partial z = y$, and $\partial x = p'(z)$.

Note that the DF quotients $S_1/_\beta \cong S_1/_\delta \cong X$.

Now consider the commutative diagram

$$\begin{tikzcd}
 & S \times \mathbb{C} \cong S \times_X S_1 = W. \arrow[dl, "p"'] \arrow[dr, "p_1"] & & \\
S \arrow[dr, "\pi_\alpha"'] & & S_1 \arrow[dl, "\pi_\beta"] \arrow[dr, "\pi_\delta"] & \\
 & X \arrow[d, "\sigma"'] & & X \arrow[d, "\sigma_1"'] \\
 & \mathbb{C} & & \mathbb{C}
\end{tikzcd}$$

All the fibers of π_β are reduced, and the action β is fixed point–free. By Danielewski and Fieseler [D; Fie], this means that $W = S \times_X S_1$ has the following properties:

(a) W is a $\mathbb{C}$ bundle over S;
(b) (W, p) and (W, p_1) are locally trivial fiber bundles over S and S_1, respectively.

Therefore,

(a′) $W \cong S \times \mathbb{C}$, since any $\mathbb{C}^+$-bundle over an affine surface S is trivial.
(b′) Consider a map $\mu \colon S \times \mathbb{C} \to \mathbb{C}$, where $\mu = \sigma_1 \circ \pi_\delta \circ p_1$. The general fiber $P_y = \mu^{-1}(y)$, $y \in \mathbb{C}$, is a locally trivial fiber bundle (restriction of (W, p_1)) over a curve $R_y = (\sigma_1 \circ \pi_\delta)^{-1}(y) \subset S_1$. The fiber of this bundle is reduced and isomorphic to $\mathbb{C}$, and $R_y \cong \mathbb{C}$ for a general y. Thus, $P_y \cong \mathbb{C}^2$ [MiSu] for a general y.

According to Theorem KZ, there exist a Zariski open subset $Z \subset S \times \mathbb{C}$ and a $U \subset \mathbb{C}$ such that $Z = \mu^{-1}(U)$, $Z \cong U \times \mathbb{C}^2$, and μ is the projection on the first factor in this product. By [Mi2, Lemma 2.2] there exist two commuting l.n.d.s on the ring $\mathcal{O}(S \times \mathbb{C})$ such that the general orbit of the group generated by the corresponding $\mathbb{C}$-actions coincides with P_y. The image $\sigma \circ \pi_\beta \circ p_1(P_y) = \sigma \circ \pi_\beta(R_y) = \mathbb{C}$. Thus, there is a $\mathbb{C}$-action ϵ on $S \times \mathbb{C}$ whose orbit is not contained in a fiber $Q_z = p_1^{-1} \circ \pi_\beta^{-1} \circ \sigma^{-1}(z)$, $z \in \mathbb{C}$.

We obtained three $\mathbb{C}$-actions on $S \times \mathbb{C}$: α', induced by α; ω, acting along the fibers of the map $p \colon W \to S$; and ϵ. The general orbit of the group generated by α' and ω is $Q_z = p_1^{-1} \circ \pi_\beta^{-1} \circ \sigma^{-1}(z)$. The orbit of the third action ϵ is not contained in Q_z. Hence, the general orbit of the group generated by all three actions is three-dimensional, and $AK(\mathcal{O}(S \times \mathbb{C})) \cong \mathbb{C}$. □

EXAMPLE 5. Consider a surface

$$S_5 = \begin{cases} x^2 y_2 = z_2^2 - 1, \\ (z_2 - 1) y_2 = xt, \\ x y_2^2 = (z_2 + 1)t. \end{cases}$$

This surface is obtained from $S_{2,2} = \{x^2 y_2 = z_2^2 - 1\}$ by the following affine modification: the point $\{x = 0, z_2 = -1, y_2 = 0\}$ is blown up, and the curve $\{x = 0, z_2 = -1, y_2 \neq 0\}$ is taken off. Since the graph of the complement of this surface is not linear [Be; G], the surface S_5 admits only one class of $\mathbb{C}$-actions; that is, any l.n.d. is proportional to the following one:

$$\partial x = 0, \quad \partial z_2 = x^3, \quad \partial y_2 = 2 z_2 x, \quad \partial t = 3 z_2^2 - 2 z_2 - 1.$$

The corresponding fibering has one singular fiber $\{x = 0\}$ with two connected reduced components $\{x = 0, z_2 = 1, t = 0\}$ and $\{x = 0, z_2 = -1, y_2 = 0\}$. This l.n.d. vanishes along the curve $\{x = 0, z_2 = 1, t = 0\}$; that is, there are fixed points for any $\mathbb{C}$-action.

According to Theorem 1, $AK(\mathbb{C} \times S_5) = \mathbb{C}$. We will show this explicitly, using the same formulas as in Example 4 and Lemma 6.

Let $B = \mathbb{C}[x, y_1, y_2, z_1, z_2, u, t]$, where $x y_1 = z_1^2 - 1$, $x^2 y_2 = z_2^2 - 1$, $(z_2 - 1) y_2 = x t$, $x y_2^2 = (z_2 + 1) t$, and $x u = z_1 - z_2$. Then $B = \mathbb{C}[x, y_2, z_2, t][u]$, since $z_1 = x u + z_2$ and $y_1 = u^2 x + 2 u z_2 + x y_2$. Therefore, $B = \mathcal{O}(S_5 \times \mathbb{C})$, and ∂ can be extended on B by $\partial u = 0$.

Substituting into the identity $x t = (z_2 - 1) y_2$ expressions for y_2 and z_2 through x, y_1, z_1, w (see Example 4), we obtain

$$\begin{aligned} x t &= (z_1 - x u - 1) y_2 = -x u y_2 + (z_1 - 1)\big[w z_1 - y_1^2 + \tfrac{1}{4}(z_1 y_1 - x w)^2\big] \\ &= -x u y_2 + \tfrac{1}{4}(z_1 - 1)(x^2 w^2 - 2 x y_1 z_1 w) + (z_1 - 1)\big[w z_1 - y_1^2 + \tfrac{1}{4} z_1^2 y_1^2\big]. \end{aligned}$$

To simplify this expression, take

$$s = t + u y_2 - \tfrac{1}{4}(z_1 - 1)(x w^2 - 2 y_1 z_1 w).$$

Then

$$x s = \tfrac{1}{4}(z_1 - 1)(4 w z_1 - 4 y_1^2 + z_1^2 y_1^2).$$

Next,

$$\begin{aligned} 4 x s &= (z_1 - 1)(4 w z_1 - 4 y_1^2 + z_1^2 y_1^2) \\ &= (z_1^2 - 1)(y_1^2 (z_1 - 1) + 4 w) - (z_1 - 1)(4 w + 3 y_1^2) \\ &= x y_1 [y_1^2 (z_1 - 1) + 4 w] - (z_1 - 1)(4 w + 3 y_1^2). \end{aligned}$$

Introducing now

$$r = -4 s + y_1 [y_1^2 (z_1 - 1) + 4 w],$$

we obtain $x r = (z_1 - 1)(4 w + 3 y_1^2)$. Since

$$(z_1 + 1) x r = (z_1^2 - 1)(4 w + 3 y_1^2) = x y_1 (4 w + 3 y_1^2),$$

we also have $(z_1 + 1) r = y_1 (4 w + 3 y_1^2)$. Clearly $B = \mathbb{C}[x, y_1, y_2, z_1, z_2, u, r]$, and (as in Example 4) $B = \mathbb{C}[x, y_1, z_1, r, w]$.

Let us define now an l.n.d. of B by $\tilde{\partial} y_1 = \tilde{\partial} r = 0$, $\tilde{\partial} z_1 = 4 y_1$, $\tilde{\partial} x = 8 z_1$, and $\tilde{\partial} w = r$. It acts on the cylinder, and $\tilde{\partial} x \neq 0$. A third l.n.d., ∂', can be derived from the presentation $B = \mathbb{C}[x, y_2, z_2, t][u]$:

$$\partial' u = 1, \qquad \partial' x = \partial' z_2 = \partial' y_2 = \partial' t = 0.$$

In the notation of Theorem 1, the map $\mu(x, y_2, z_2, u, t) = y_1$ and the fiber of this map is $\mathbb{C}^2$ with coordinates (z_1, r). Indeed, for $y_1 = c$ we have

$$\begin{aligned} x &= \frac{z_1^2 - 1}{c}, \\ w &= \frac{(z_1 + 1)r - 3c^2}{4c}, \\ u &= \tfrac{1}{2}(-xw + z_1 c), \\ y_2 &= wz_1 + u^2 - c^2, \\ z_2 &= z_1 - xu. \end{aligned}$$

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