Michigan Math. J. 53 (2005)

Families of Diffeomorphisms without Periodic Curves

Javier Ribón

1. Introduction

A formal curve is a reduced principal ideal $\hat{\gamma} = (\hat{f})$ of $\mathbb{C}[[x, y]]$. We say that $\hat{\gamma}$ is invariant by $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ if

$$(\hat{f}) = (\hat{f}) \circ \varphi$$

and that $\hat{\gamma}$ is p-periodic if it is invariant by $\varphi^{(p)}$ and not by $\varphi^{(j)}$ for $0 < j < p$.

Theorem 1. *There exists a germ of a diffeomorphism* $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ *that has no convergent periodic germs of curve.*

Moreover, we may choose φ inside each of the following classes.

- The formally linearizable germs of diffeomorphism.
- The germs of diffeomorphism whose linear part is the identity.

These germs of diffeomorphism have formal invariant curves. Although there are germs of diffeomorphism without formal invariant curves, we prove the following.

Theorem 2. *A formal diffeomorphism* $\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ *has at least one irreducible formal periodic curve.*

In [6] Hakim exhibits germs of diffeomorphisms of the type

$$\rho_\alpha = \left(\frac{x}{1+x}, ye^{-\alpha x} + x^2\right) \quad (\alpha \in \mathbb{C})$$

with divergent "strong" invariant curves, showing in this way that the argument in [2] does not work for germs of diffeomorphism. Nevertheless, the germs ρ_α preserve the foliation $dx = 0$, and this is essential in [6] for the proof of the divergence property. All the ρ_α have $x = 0$ as a periodic curve (in fact, a curve of fixed points). This situation is general, as our next result shows.

Theorem 3. *Let* φ *be an element of* $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ *preserving a formal* 1*-dimensional foliation* $\mathcal{F}$*. Then there exists a formal curve* γ *that is periodic by* φ *and invariant by* $\mathcal{F}$*. If* $\mathcal{F}$ *is convergent then* γ *can be chosen to be convergent.*

Received November 4, 2003. Revision received November 6, 2004.

The author is financed by "Secretaría de Estado de Educación y Universidades" Grant no. EX2002-0365.

The proof of Theorem 3 uses arguments of Camacho–Sad type ([2]; see also [4]) that are valid in the dicritical case. Theorem 2 is obtained as a consequence of the Jordan decomposition and construction of the logarithm for germs of diffeomorphism.

In order to prove Theorem 1, we consider polynomial families $\{\varphi_\lambda\}$ of elements of $\mathrm{Diff}(\mathbb{C}^2, 0)$. The formal periodic curves γ_λ of φ_λ will change in a polynomial way with respect to $\lambda \in \mathbb{C}^m$. We choose $\{\varphi_\lambda\}$ such that $\{\gamma_\lambda\}$ has at least one diverging element γ_{λ_0}. Using potential theory we deduce that, for almost all $\lambda \in \mathbb{C}^m$, the diffeomorphism φ_λ has no convergent invariant curves. More precisely, this property will be true in the complementary of a pluripolar set.

ACKNOWLEDGMENT. I thank the referee for helpful suggestions.

2. Preliminaries

Let us recall here some known results [3; 7] on the "Jordanization" of diffeomorphisms (directly issued from the corresponding ones for vector fields) and on the relationship between unipotent diffeomorphisms and nilpotent vector fields.

Denote by $\mathrm{Diff}(\mathbb{C}^n, 0)$ the group of germs of convergent diffeomorphisms at $0 \in \mathbb{C}^n$ and by $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ the group of formal diffeomorphisms. Call m and $\hat{m}$ the maximal ideals of the rings $\mathbb{C}\{x_1, \dots, x_n\}$ and $\mathbb{C}[[x_1, \dots, x_n]]$, respectively. Any $\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ induces a $\mathbb{C}$-linear automorphism

$$\begin{aligned} \varphi_k \colon m/m^{k+1} &\to m/m^{k+1}, \\ f + m^{k+1} &\mapsto f \circ \varphi + m^{k+1}. \end{aligned}$$

We can express $\varphi_k = \varphi_k^s \circ \varphi_k^u = \varphi_k^u \circ \varphi_k^s$ in a unique way as the composition of two elements φ_k^s and φ_k^u in $\mathrm{GL}(m/m^{k+1})$ such that:

- φ_k^s is semisimple;
- φ_k^u is unipotent.

This Jordan multiplicative decomposition is compatible with the filtration in the space of jets, and we deduce that $\{\varphi_k^s\}_{k \geq 1}$ and $\{\varphi_k^u\}_{k \geq 1}$ induce (respectively) the $\mathbb{C}$-automorphisms φ^s and φ^u of $\hat{m}$.

The set $D_k = \{\varphi_k : \varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)\}$ is an algebraic subgroup of $\mathrm{GL}(m/m^{k+1})$ since it corresponds to the $A \in \mathrm{GL}(m/m^{k+1})$ such that $A(ab) = A(a)A(b)$ for all $(a, b) \in (m/m^{k+1})^2$. The existence of a Jordan decomposition for algebraic groups [3] implies that φ_k^s and φ_k^u are also in D_k. We deduce that φ^s and φ^u act over $\hat{m}$ as formal diffeomorphisms. That is, there exist unique φ_s and φ_u in $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ such that

$$\varphi^s(f) = f \circ \varphi_s \quad \text{and} \quad \varphi^u(f) = f \circ \varphi_u \quad \text{for all } \ f \in \hat{m}.$$

The equality $\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$ holds because it holds in each $\mathrm{GL}(m/m^{k+1})$. Note also that $(\varphi_s)_k = \varphi_k^s$ and $(\varphi_u)_k = \varphi_k^u$ for all $k \geq 1$.

A formal diffeomorphism $\beta \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is semisimple (resp. unipotent) if β_k is semisimple (resp. unipotent) for all $k \geq 1$. The following proposition follows from this discussion.

PROPOSITION 1. *Any formal diffeomorphism* $\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ *can be written in a unique way in the form* $\varphi = \varphi_s \circ \varphi_u$, *where* $\varphi_s \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ *is semisimple,* $\varphi_u \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ *is unipotent, and* $\varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$.

Working jet by jet, we can prove that β is semisimple if and only if it is formally linearizable and β_1 is semisimple. We also have that β is unipotent if and only if β_1 is unipotent. Moreover, the semisimple–unipotent decomposition of the iteration $\varphi^{(p)}$ of φ is given by $\varphi^{(p)} = \varphi_s^{(p)} \circ \varphi_u^{(p)}$.

Unipotent diffeomorphisms are strongly related to nilpotent vector fields, as we shall see. Denote by $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ the set of formal unipotent diffeomorphisms and by $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ the set of formal vector fields singular at 0. A formal vector field is a derivation acting on the group of formal power series; we denote by $\hat{X}^{(i)}(\hat{h})$ the power series obtained by applying i times $\hat{X}$ to the formal power series $\hat{h}$. More precisely, we have $\hat{X}^{(0)}(\hat{h}) = \hat{h}$ and $\hat{X}^{(i+1)}(\hat{h}) = \hat{X}(\hat{X}^{(i)}(\hat{h}))$ for all $i \in \mathbb{N}$. We say that $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^n, 0)$ is nilpotent if its linear part is nilpotent. Denote by $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ the set of formal nilpotent vector fields. For any $\hat{X} \in \hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$, the formal flow

$$\exp(t\hat{X}) = \left(\sum_{i=0}^{\infty} t^i \frac{\hat{X}^{(i)}(x_1)}{i!}, \ldots, \sum_{i=0}^{\infty} t^i \frac{\hat{X}^{(i)}(x_n)}{i!} \right)$$

is well-defined and the components belong to $\mathbb{C}[t][[x_1, \ldots, x_n]]$. Moreover, the exponential application

$$\hat{X} \mapsto \exp(1\hat{X}) = \exp(\hat{X})$$

induces a bijection between $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ and $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$. The logarithm $\log \varphi$ of an element $\varphi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ is by definition the only element in $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ whose exponential is φ.

3. Diffeomorphisms Preserving a Foliation

In this section we prove Theorem 3 by using a version (valid for the dicritical case) of the argument in [4]. Let M be a 2-dimensional holomorphic manifold and consider a singular foliation $\mathcal{F}$ on M. By definition, $\mathcal{F}$ is locally represented at each point $Q \in M$ by a germ of a 1-differential form ω_Q whose coefficients do not have a common factor; the 1-forms ω_Q and ω_P are collinear in the intersection of their domains of definition (i.e., $\omega_Q \wedge \omega_P \equiv 0$). The singular locus of $\mathcal{F}$ is the set of points $Q \in M$ such that $\omega(Q) = 0$, where ω defines $\mathcal{F}$ locally; it is a set of isolated points since the coefficients of ω do not have a common factor. Given a blow-up $\pi : M' \to M$ with center at $P \in M$, we define the foliation $\pi^*\mathcal{F}$ by dividing locally $\pi^*\omega$ by a suitable power of a local equation of the exceptional divisor $\pi^{-1}(P)$.

We shall consider a normal crossings divisor $D \subset M$ such that its irreducible components are invariant by $\mathcal{F}$. Eventually we will take $D = \emptyset$. Moreover, we need to enlarge the definition of the foliation $\mathcal{F}$ to obtain "formal foliations" at a point P or "transversally formal foliations" along D at a point $P \in D$. A formal foliation at $P \in M$ is simply defined by $\hat{\omega} = 0$, where $\hat{\omega} = \hat{a}(x, y)\, dx + \hat{b}(x, y)\, dy$ is a formal differential 1-form at P. A transversally formal foliation along D at $P \in D$ is given by $\hat{\omega}$ as before but now such that $\hat{a}$ and $\hat{b}$ belong to the formal completion of the ring of convergent series with respect to the ideal defining D at P.

If $\pi : M' \to M$ is the blow-up with center at $P \in M$ and if $\hat{\mathcal{F}}$ is a formal foliation at P, then $\pi^*\hat{\mathcal{F}}$ is naturally a transversally formal foliation along $\pi^{-1}(P)$ at every $Q \in \pi^{-1}(P)$ (under the assumption that $\pi^{-1}(P)$ is invariant by $\pi^*\hat{\mathcal{F}}$). In this paper we first consider convergent foliations; later on we generalize the results to formal foliations at P, and since our arguments involve blow-ups we arrive at the transversally formal (also called semiconvergent) setting.

Take M, D, and $\mathcal{F}$ as before and consider a point $P \in M$. We shall deal with triples $(\mathcal{F}, D;\, P)$ that implicitly represent the objects locally at P (e.g., if $P \notin D$ then our triple is $(\mathcal{F}, \emptyset;\, P)$). We denote by $i_P(\mathcal{F}, S)$ the Camacho–Sad index of $\mathcal{F}$ at P along S. By definition $(\mathcal{F}, D;\, P)$ satisfies the property (*) if one of the following conditions holds (see [4]):

(*1) $D = \emptyset$;

(*2) $D = \{S\}$ and $i_P(\mathcal{F}, S) \notin \mathbb{Q}_{\geq 0}$;

(*3) $D = \{S_+, S_-\}$ and there is a real number $a > 0$ such that $i_P(\mathcal{F}, S_+) \in \mathbb{Q}_{\leq -a}$ and $i_P(\mathcal{F}, S_-) \notin \mathbb{Q}_{\geq -1/a}$.

Suppose now that α is a formal diffeomorphism at P. We say that the 4-tuple $(\mathcal{F}, \alpha, D;\, P)$ is *good* if: $\alpha^*\mathcal{F} = \mathcal{F}$; $\alpha(S) = S$ for every irreducible component S of D; and $(\mathcal{F}, D;\, P)$ satisfies property (*).

LEMMA 1. *Assume that $(\mathcal{F}, \alpha, D;\, P)$ is good and that $\mathcal{F}$ is either regular or irreducible at P. Then there exists a germ of curve $\Gamma \not\subset D$ at P that is invariant by $\mathcal{F}$ and periodic by α.*

Proof. If $\mathcal{F}$ is regular at P, then there is only one germ of curve Γ invariant by $\mathcal{F}$ and the index $i_P(\mathcal{F}, \Gamma)$ is zero. Necessarily (*1) holds and then $\Gamma \not\subset D$. Moreover Γ is invariant by α (by uniqueness) since the set of invariant curves of $\mathcal{F}$ is always invariant by α. If $\mathcal{F}$ is irreducible at P then property (*3) doesn't hold, because the product of the two indices is either 0 or 1. At an irreducible singularity there are two formal invariant curves, S_1 and S_2. At least one of them (suppose it is S_1) is convergent; moreover, if S_2 is divergent then $i_P(\mathcal{F}, S_1) = 0$. These properties are a consequence of Briot–Bouquet's theorem. Thus, there exists a convergent invariant curve $\Gamma \not\subset D$ of $\mathcal{F}$. Since α induces a bijection in the set of formal invariant curves, it follows that $\alpha^{(2)}$ fixes Γ. □

Consider a good 4-tuple $(\mathcal{F}, \alpha, D;\, P)$ and let π be the blow-up of M with center at P. Then π can be considered as a morphism $\pi : (M', \pi^{-1}(P)) \to (M, P)$ of germs of holomorphic manifolds; furthermore, if we choose a point $Q \in \pi^{-1}(P)$ we may also use $\pi : (M', Q) \to (M, P)$ to denote the corresponding local blow-up. We shall denote by $D' \subset M'$ the normal crossings divisor obtained as the union of the irreducible components of $\pi^{-1}(D \cup \{P\})$ that are invariant by $\pi^*\mathcal{F}$. Hence D' is the strict transform $\tilde{D}$ of D if $\pi^{-1}(P)$ is not invariant, and $D' = \tilde{D} \cup \pi^{-1}(P)$ if

$\pi^{-1}(P)$ is invariant. Note also that the formal diffeomorphism α induces a holomorphic automorphism in the exceptional divisor $\pi^{-1}(P)$. If $Q \in \pi^{-1}(P)$ is invariant by $\alpha^{(q)}$ for some $q \in \mathbb{Z}_{>0}$, then $\alpha^{(q)}$ localizes to a formal diffeomorphism at Q.

LEMMA 2. *Assume that $(\mathcal{F}, \alpha, D; P)$ is good. Then there exist a $Q \in \pi^{-1}(P)$ and a $q > 0$ such that $\alpha^{(q)}(Q) = Q$ and $(\pi^*\mathcal{F}, \alpha^{(q)}, D'; Q)$ is good.*

Proof. Assume first that $\pi^{-1}(P)$ is invariant by $\pi^*\mathcal{F}$. By the arguments in [4], there is a point $Q \in \pi^{-1}(P)$, singular for $\pi^*\mathcal{F}$, such that $(\pi^*\mathcal{F}, D'; Q)$ satisfies either (*2) or (*3). All the (finitely many) points in the orbit of Q by α are also singular points of $\pi^*\mathcal{F}$, and α induces a bijection on them. Thus Q is q-periodic by α for some $q \in \mathbb{N}$.

Suppose that $\pi^{-1}(P)$ is not invariant by $\pi^*\mathcal{F}$. If $D = \emptyset$ then $D' = \emptyset$ and, since α defines a holomorphic automorphism on the projective line $\pi^{-1}(P)$, there is at least one fixed point $Q \in \pi^{-1}(P)$ and we get a good 4-tuple at Q. If $D \neq \emptyset$ then we denote by $Q_S \in \pi^{-1}(P)$ the point in the strict transform of a component S of D (note that D has one or two components). The point Q_S is fixed by α because both $\pi^{-1}(P)$ and S are preserved by α. Let us consider two cases: $D = \{S\}$ and $D = \{S_+, S_-\}$. In the first case we take $Q = Q_S$, since $D' = \tilde{S}$ is the strict transform of S and $i_Q(\pi^*\mathcal{F}, \tilde{S}) = i_P(\mathcal{F}, S) - 1 \notin \mathbb{Q}_{\geq 0}$. In the second case we apply the same argument to Q_{S_+} (where S_+, S_- are chosen as in property (*3)). $\square$

Let us prove Theorem 3 supposing that $\mathcal{F}$ is a germ of foliation. We consider the good 4-tuple $(\mathcal{F}, \varphi, \emptyset; 0)$. Lemma 2 allows us to construct a sequence

$$(\mathcal{F}, \varphi, \emptyset; 0) \xleftarrow{\pi} (\mathcal{F}_1, \varphi_1, D_1; P_1) \xleftarrow{\pi} \cdots \xleftarrow{\pi} (\mathcal{F}_r, \varphi_r, D_r; P_r) \xleftarrow{\pi} \cdots$$

of good 4-tuples. By [11] there exists a $j \geq 0$ such that $\mathcal{F}_j$ is regular or irreducible at P_j. By Lemma 1 we have a curve $\tilde{\Gamma}_j \not\subset D_j$ invariant by $\mathcal{F}_j$ and periodic by φ_j. The blow-down Γ_j of $\tilde{\Gamma}_j$ is a curve; it is invariant by $\mathcal{F}$ and periodic by φ.

REMARK 1. So far we have supposed that φ is formal but $\mathcal{F}$ is convergent. Theorem 3 also holds for $\mathcal{F}$ formal. Here we explain briefly why the ingredients in the proof can be applied to the formal case.

Seidenberg's desingularization theorem is valid for formal foliations. In fact, desingularization algorithms do not make a difference between convergent and divergent forms. Moreover, the desingularization process depends only on a finite jet and hence any formal 1-form shares the desingularization with a germ of foliation.

The arguments in [4], excepting Briot–Bouquet's theorem, are valid if $\mathcal{F}$ is divergent. The proof relies in the behavior of the Camacho–Sad index by blow-up. The three main properties are:

- $\sum_{P \in D \cap \mathrm{Sing}\, \pi^*\mathcal{F}} i_P(\mathcal{F}, D) = -1$ for a blow-up π whose divisor D is invariant by $\pi^*\mathcal{F}$;
- $i_P(\mathcal{F}, S) = i_{\pi^{-1}(P)}(\pi^*\mathcal{F}, \pi^{-1}(S))$ if P is not the center of the blow-up π; and

- $i_{P'}(\pi^*\mathcal{F}, \tilde{S}) = i_P(\mathcal{F}, S) - 1$ if P is the center of π. In this case, $\tilde{S}$ is the strict transform of S and $P' = \tilde{S} \cap \pi^{-1}(P)$.

It is easy to verify that all these properties persist in the semiconvergent case. Besides these formulas the other arguments in [4] are combinatorial. Furthermore, Camacho–Sad's indexes along components of a divisor associated to a desingularization process depend only on a finite jet of the form defining $\mathcal{F}$.

The last required ingredient is a formal version of Lemma 1. For $\mathcal{F}$ regular this lemma is trivial because (*1) always holds. Otherwise, the singularity is irreducible and condition (*3) does not hold. Since there are two formal invariant curves through P then at least one of them is not a component of the divisor. That formal curve is periodic. Therefore, the proof is even simpler than in the convergent case.

4. Formal Periodic Curves

In order to prove Theorem 2, we will use the following results.

PROPOSITION 2. *Let φ be an element of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$. A formal irreducible curve $\hat{\gamma}$ is invariant by φ if and only if $\hat{\gamma}$ is invariant by both φ_s and φ_u.*

Proof. The necessary condition is trivial.

Since φ fixes $\hat{\gamma}$, it follows that $j^1\varphi$ fixes the tangent cone of $\hat{\gamma}$. The tangent cone is a line with multiplicity; it can not contain two different lines, because otherwise $\hat{\gamma}$ would not be irreducible. We suppose without loss of generality that the support of the tangent cone of $\hat{\gamma}$ is the line $y = 0$. We deduce that $(1, 0)$ is an eigenvector of $j^1\varphi$. The formal diffeomorphisms $j^1\varphi_s$ and $j^1\varphi_u$ share also the eigenvector $(1, 0)$ by Jordan's theorem.

Let π be the blow-up of the origin. We put

$$P = \pi^{-1}(0) \cap \overline{\pi^{-1}([y=0] \setminus \{(0,0)\})}.$$

We will consider the expression $(x, t) \to (x, xt)$ of π in the first chart. The diffeomorphisms $\varphi, \varphi_s, \varphi_u$ can be lifted to $\widehat{\mathrm{Diff}}(\widetilde{\mathbb{C}^2}, P)$ because $(1, 0)$ is an eigenvector for the linear parts of all three. If we denote the liftings by a tilde then

$$\tilde{\varphi} = \widetilde{(\varphi_s)} \circ \widetilde{(\varphi_u)} = \widetilde{(\varphi_u)} \circ \widetilde{(\varphi_s)}.$$

We will prove next that the blow-up is compatible with the semisimple–unipotent decomposition, that is,

$$(\tilde{\varphi})_s = \widetilde{(\varphi_s)} \quad \text{and} \quad (\tilde{\varphi})_u = \widetilde{(\varphi_u)}.$$

This is equivalent to proving that the lifting of a semisimple map is semisimple and the lifting of a unipotent map is unipotent.

If $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ is unipotent and if $(1, 0)$ is an eigenvector of $j^1\sigma$, then $j^1\sigma = (x + \lambda y, y)$ for some $\lambda \in \mathbb{C}$. The lifting $\tilde{\sigma}$ holds $j^1\tilde{\sigma} = (x, \mu x + t)$ for some $\mu \in \mathbb{C}$. The linear part $j^1\tilde{\sigma}$ is unipotent and then $\tilde{\sigma}$ is unipotent too.

If $\sigma \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ is semisimple then there exists a formal conjugation η such that $\eta \circ \sigma \circ \eta^{(-1)} = (\lambda_1 x, \lambda_2 y)$ for some λ_1 and λ_2 in $\mathbb{C}^*$. The linear space $j^1\eta(y = 0)$ is an eigenspace of $(\lambda_1 x, \lambda_2 y)$. As a consequence we can precompose η with a linear transformation η' such that $\eta' \circ j^1\eta$ fixes $y = 0$ while

$$(\eta' \circ \eta) \circ \sigma \circ (\eta' \circ \eta)^{(-1)}$$

is still in diagonal form. We denote $\eta' \circ \eta$ by η_1; the formal diffeomorphism η_1 can be lifted to $\widehat{\mathrm{Diff}}(\widetilde{\mathbb{C}^2}, P)$ because $j^1\eta_1$ fixes $y = 0$. We have

$$\tilde{\eta}_1 \circ \tilde{\sigma} \circ \tilde{\eta}_1^{(-1)} = (\lambda_1 x, (\lambda_2/\lambda_1)t).$$

Thus the formal diffeomorphism $\tilde{\sigma}$ is semisimple.

So far we have shown that the property we want to prove is invariant by blow-up. Up to a finite number of blow-ups we can suppose that the curve $\hat{\gamma}$ is smooth—that is, for every irreducible equation $\hat{f}$ of $\hat{\gamma}$ we have $d\hat{f}(0,0) \neq 0$. Up to a formal change of coordinates we can suppose that $\hat{\gamma} \equiv [y = 0]$. The set F_k of elements in $\mathrm{GL}(m/m^{k+1})$ holding

$$\begin{cases} A(ab) = A(a)A(b) & \text{for all } a \text{ and } b \text{ in } m/m^{k+1}, \\ y \mid A(y), \end{cases}$$

coincides with the set of actions induced in m/m^{k+1} by the elements of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ preserving $y = 0$. That is a group, and so is F_k as the image by a homomorphism of a group. The condition $y \mid A(y)$ is equivalent to k algebraic equations corresponding to the vanishing of the coefficients of $x, \ldots, x^k$ in $A(y)$. Hence the set F_k is an algebraic subgroup of $\mathrm{GL}(m/m^{k+1})$, and we deduce that

$$y \circ \varphi_s(x, 0) \in (x^{k+1}) \quad \text{and} \quad y \circ \varphi_u(x, 0) \in (x^{k+1})$$

for all $k \geq 1$. We obtain that $y = 0$ is invariant by φ_s and φ_u, as we wanted to prove. $\square$

PROPOSITION 3. *Given $\varphi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^2, 0)$ and $\hat{\gamma} = (\hat{f})$ irreducible, we have that $\hat{\gamma}$ is invariant by φ if and only if $\hat{f}$ divides* $\log\varphi(\hat{f})$.

Proof. Consider the equation

$$\hat{f} \circ \exp(t \log \varphi) = \sum_{i=0}^{\infty} t^i \frac{(\log \varphi)^{(i)}(\hat{f})}{i!}.$$

If $\hat{f}$ divides both $(\log\varphi)^{(i)}(\hat{f})$ and $(\log\varphi)(\hat{f})$ then it also divides $(\log\varphi)^{(i+1)}(\hat{f})$. For $t = 1$, we can deduce that $(\hat{f}) = (\hat{f} \circ \varphi) = (\hat{f}) \circ \varphi$. Conversely, let us consider a Puiseux parameterization $\hat{\gamma}(s) = (\hat{\gamma}_1(s), \hat{\gamma}_2(s))$ of $\hat{\gamma}$. The series $\hat{f} \circ \exp(t \log\varphi) \circ \hat{\gamma}(s)$ belongs to $\mathbb{C}[t][[s]]$; it is zero for $t \in \mathbb{Z}$ and hence identically zero. Now the equality

$$(\log\varphi)(\hat{f}) = \lim_{t \to 0} \frac{\hat{f} \circ \exp(t \log\varphi) - \hat{f}}{t}$$

implies that $[(\log\varphi)(\hat{f})] \circ \hat{\gamma}(s) \equiv 0$ and then $\hat{f}$ divides $(\log\varphi)(\hat{f})$. $\square$

COROLLARY 1. *Let φ be an element of $\widehat{\mathrm{Diff}}_u(\mathbb{C}^2, 0)$. Then any periodic formal curve $\hat{\gamma} = (\hat{f})$ of φ is invariant by φ.*

Proof. Since $\hat{\gamma}$ is invariant by $\varphi^{(p)} = \exp(p \log \varphi)$, it follows that $\hat{f}$ divides $(p \log \varphi)(\hat{f})$ and hence also divides $(\log \varphi)(\hat{f})$. □

COROLLARY 2. *Let φ be an element of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$. A formal irreducible curve $\hat{\gamma}$ is p-periodic by φ if and only if $\hat{\gamma}$ is p-periodic by φ_s and invariant by φ_u.*

Proof. If $\hat{\gamma}$ is p-periodic by φ then $\hat{\gamma}$ is fixed by $\varphi^{(p)}$. Proposition 2 implies that $\hat{\gamma}$ is periodic by φ_s and φ_u. The formal curve $\hat{\gamma}$ is fixed by φ_u; since φ_s and φ_u commute, the periods of φ and φ_s coincide.

If $\hat{\gamma}$ is periodic by φ_s and invariant by φ_u, then $\hat{\gamma}$ is periodic by φ. By the first part of the proof, the periods for φ and φ_s are equal. □

Now let us prove Theorem 2. Consider the decomposition of φ into semisimple and unipotent parts:

$$\varphi = \varphi_s \circ \exp(\log \varphi_u) = \exp(\log \varphi_u) \circ \varphi_s.$$

If φ is semisimple, then φ is formally conjugated to the first jet $j^1\varphi$ that has at least two invariant lines through the origin. If $\log \varphi_u \not\equiv 0$ then the foliation induced by $\log \varphi_u$ is not trivial and is invariant by φ_s. We finish by using the formal version of Theorem 3.

Note that the foliation $\hat{\mathcal{F}}$ induced by $\log \varphi_u$ is represented by any 1-form $\hat{\omega}$ such that $\hat{\omega}(\log \varphi_u) = 0$ and whose coefficients do not share a common factor. The singular set of $\log \varphi_u$ can be bigger than the singular set of $\hat{\mathcal{F}}$ because the coefficients of $\log \varphi_u$ are not necessarily free of having a common factor. Anyway, in such a case our proof can be much simplified; the singular set of $\log \varphi_u$ is a nonempty finite union of irreducible formal curves. Since it is also invariant by φ_s, Theorem 2 becomes trivial.

EXAMPLE 1. Not all the diffeomorphisms in $\mathrm{Diff}(\mathbb{C}^2, 0)$ have invariant curves. Consider $\alpha(x, y) = (iy, ix)$ and $\beta(x, y) = (xe^{xy}, ye^{-xy})$, where

$$\log \beta = xy\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right).$$

Define $\varphi = (iye^{-xy}, ixe^{xy}) = \alpha \circ \beta = \beta \circ \alpha$. This is the decomposition of φ into semisimple and unipotent parts. The periodic curves of φ are invariant by β. The only fixed curves of $\log \beta$ are $x = 0$ and $y = 0$. Since α permutes them, the germ φ has only two periodic irreducible formal curves, both of which are convergent and 2-periodic.

5. Semisimple Families

In this section we present a family $\{\sigma_\lambda^{\alpha,\beta}\}_{\lambda\in\mathbb{C}}$ of semisimple diffeomorphisms constructed for certain α, β in such a way that $\sigma_\lambda^{\alpha,\beta}$ does not have any periodic curve for λ outside a Borel polar set. Note that any Borel polar set has Lebesgue measure 0, has Hausdorff dimension 0, and is totally disconnected [10].

We shall fix complex numbers α and β such that:

(i) $\alpha \neq \beta$ and $|\alpha| = |\beta| = 1$;
(ii) $\alpha \neq \alpha^k \beta^l \neq \beta$ for all $(k,l) \in \mathbb{N} \times \mathbb{N}$ such that $k + l \geq 2$;
(iii) $\alpha^k \neq \beta^l$ if $k \geq 1$ and $l \geq 1$; and
(iv) the sets $\{k \in \mathbb{N} : |\alpha^k - \beta| \leq 1/k!\}$ and $\{k \in \mathbb{N} : |\beta^k - \alpha| \leq 1/k!\}$ are infinite.

Such α and β can be obtained by working with continuous fractions.

Consider $\sigma \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $j^1\sigma = (\alpha x, \beta y)$. The conditions (i) and (ii) imply that σ is formally linearizable. By (iii) we know that $(\alpha x, \beta y)$ has only the formal periodic curves $x = 0$ and $y = 0$. We deduce that σ has two formal periodic curves that are invariant, smooth, and tangent to $x = 0$ and $y = 0$, respectively.

We define $\sigma_\lambda = \sigma_\lambda^{\alpha,\beta} = \big(\alpha x + (1-\lambda)\sum_{k=2}^{\infty} y^k, \beta y + \lambda \sum_{k=2}^{\infty} x^k\big)$. Let $y = \hat{f}_\lambda(x)$ and $x = \hat{h}_\lambda(y)$ be the two formal periodic curves of σ_λ.

The series $\hat{f}_\lambda(x) = \sum_{k=2}^{\infty} f_k(\lambda) x^k$ is the only solution of the equation

$$(y - \hat{f}_\lambda(x)) \circ \sigma_\lambda \circ (x, \hat{f}_\lambda(x)) = 0.$$

This is equivalent to saying that

$$\hat{f}_\lambda\left(\alpha x + (1-\lambda)\frac{\hat{f}_\lambda(x)^2}{1 - \hat{f}_\lambda(x)}\right) = \beta \hat{f}_\lambda(x) + \lambda \frac{x^2}{1-x}.$$

Proceeding by induction, we obtain that $f_k(\lambda)$ is a polynomial in λ for $k \geq 2$.

Remark 2. In the case $\lambda = 1$ we have an equation of the type

$$\hat{f}_1 \circ \theta(x) = r(x, \hat{f}_1) \quad (\theta \in \mathrm{Diff}(\mathbb{C}, 0),\ r \in \mathbb{C}\{x, y\}),$$

with $\theta(x) = \alpha x$ and $r(x,y) = \beta y + x^2/(1-x)$. Note that θ does not depend on $\hat{f}_1$: this is possible because σ_1 preserves the foliation $dx = 0$, as in [6]; the nature of $\hat{f}_1$ can be obtained through the study of $\theta(x)$. In our context, $\hat{f}_1(x) = \sum_{k=2}^{\infty} (\alpha^k - \beta)^{-1} x^k$ is divergent by condition (iv).

Put $\eta_\mu = \sigma_{1/\mu}$ and $x = x'\mu$, $y = y'\mu$; we have

$$\eta_\mu(x', y') = \left(\alpha x' + (\mu - 1)\frac{y'^2}{1 - \mu y'}, \beta y' + \frac{x'^2}{1 - \mu x'}\right).$$

Then $\eta_\mu(x', y')$ has a unique formal invariant curve $y' = \sum_{k=2}^{\infty} g_k(\mu) x'^k$ for every $\mu \in \mathbb{C}$. As before, a process of induction proves that $g_k(\mu)$ is a polynomial for all $k \geq 2$. Now the equality $f_k(1/\lambda)\lambda^{k-1} = g_k(\lambda)$ implies that $\deg f_k \leq k - 1$ for all $k \geq 2$. We next use the following proposition.

Proposition 4 [8; 9]. *Let $\hat{g}(t) = \sum_{k=0}^{\infty} g_k(x_1, \dots, x_m) t^k$ be such that $g_k \in \mathbb{C}[x_1, \dots, x_n]$ and $\deg g_k \leq Ak + B$ for some real numbers A and B and all $k \geq 0$. If $\hat{g}(t)$ does not converge in a neighborhood of $t = 0$, then it diverges for $(x_1, \dots, x_m) \in \mathbb{C}^m$ outside a pluripolar Borel set.*

Since $\hat{f}_1(x)$ is divergent, the formal invariant curve of σ_λ tangent to $y = 0$ is divergent for all $\lambda \in \mathbb{C}$ outside a Borel polar set. We proceed in the same way for the other invariant curve; since the countable union of Borel polar sets is polar, we deduce our next proposition.

PROPOSITION 5. *The germ σ_λ does not have any convergent periodic curve for $\lambda \in \mathbb{C}$ outside a Borel polar set.*

6. Unipotent Families

In this section we consider the family

$$\beta_\lambda = (x + y^2, y + x^2 + \lambda\Delta(x)), \quad \lambda \in \mathbb{C},$$

of unipotent diffeomorphisms with the following conditions on $\Delta(x)$:

(a) $\Delta(x) \in \mathbb{C}\{x\} \cap (x^3)$ and $\Delta(\alpha x) = \alpha^2\Delta(x)$, where $\alpha = e^{(2\pi i)/3}$;
(b) $f(x + x^2) = (1 - 2x)f(x) + \Delta(x)$ has no solutions in $\mathbb{C}\{x\}$.

PROPOSITION 6. *The formal periodic curves of β_λ are all divergent except for λ in a Borel polar set.*

Proof. The logarithm

$$\hat{X}_\lambda = \left(y^2 + \sum_{j+k\geq 3} a_{jk}(\lambda)x^j y^k\right)\frac{\partial}{\partial x} + \left(x^2 + \sum_{j+k\geq 3} b_{jk}(\lambda)x^j y^k\right)\frac{\partial}{\partial y}$$

of β_λ (where a_{jk} and b_{jk} are entire functions for all $j + k \geq 3$) defines a formal foliation $\hat{\omega}_\lambda = x^2dx - y^2dy + \text{h.o.t.}$ Since $j^2\hat{\omega}_\lambda = d(x^3 - y^3)/3$, it follows that $\pi^*\hat{\omega}_\lambda$ has three irreducible singularities in $\pi^{-1}(0)$ (this property holds for $d(x^3 - y^3)/3$ and is stable for foliations having the same 2-jet). We deduce that β_λ has three formal invariant curves that are smooth and tangent to $y = x$, $y = \alpha x$, and $y = \alpha^2 x$ (respectively). We will use γ_λ to denote the invariant curve of β_λ tangent to $y = x$.

The germ $(\alpha x, \alpha^2 y)$ commutes with β_λ (condition (a)) and therefore induces a permutation in the set of formal invariant curves of β_λ for all $\lambda \in \mathbb{C}$, acting on tangents as follows:

$$(y = x) \xrightarrow{(\alpha x,\, \alpha^2 y)} (y = \alpha x) \xrightarrow{(\alpha x,\, \alpha^2 y)} (y = \alpha^2 x) \xrightarrow{(\alpha x,\, \alpha^2 y)} (y = x).$$

Thus the orbit of γ_λ by $(\alpha x, \alpha^2 y)$ is equal to the total set of invariant curves of β_λ. Moreover, the divergence of γ_λ ($\lambda \in \mathbb{C}$) would imply the divergence of all the formal invariant curves of β_λ. Hence it is enough to prove that γ_λ diverges outside a polar set.

The curve γ_λ has a unique equation of the form

$$y - x - \hat{f}(x, \lambda) = 0, \qquad \hat{f}(x) = \sum_{j=2}^{\infty} P_j(\lambda)x^j = \sum_{k=1}^{\infty} f_k(x)\lambda^k.$$

The functions $P_j(\lambda)$ are holomorphic for all $j \geq 2$. Now we define

$$\eta_\mu = \beta_{1/\mu} = \left(x + y^2, y + x^2 + \frac{\Delta(x)}{\mu}\right) \quad \text{for } \mu \in \mathbb{C}.$$

Let us make the transformations $x = x'\mu$ and $y = y'\mu$, which yields a change of coordinates for all $\mu \neq 0$. We obtain

$$\eta_\mu(x', y') = \left(x' + \mu y'^2, y' + \mu x'^2 + \frac{\Delta(\mu x')}{\mu}\right).$$

We now have that $\eta_\mu(x', y') \in \mathrm{Diff}(\mathbb{C}^2, 0)$ for all $\mu \in \mathbb{C}$ and $\eta_0(x', y') = (x', y')$, so the logarithm of $\eta_\mu(x', y')$ is equal to

$$\mu\left(\left(y'^2 + \sum_{j+k\geq 3} A_{jk}(\mu)x'^j y'^k\right)\frac{\partial}{\partial x'} + \left(x'^2 + \sum_{j+k\geq 3} B_{jk}(\mu)x'^j y'^k\right)\frac{\partial}{\partial y'}\right),$$

where A_{jk} and B_{jk} are entire functions for $j + k \geq 3$. Analogous to $\beta_\lambda(x, y)$, the germ $\eta_\mu(x', y')$ has a unique formal invariant curve of the form

$$y' = x' + \sum_{k=2}^{\infty} Q_k(\mu)x'^k,$$

where $Q_k(\mu)$ is an entire function for all $k \geq 2$. The curves

$$y = x + \sum_{j=2}^{\infty} P_j(\lambda)x^j \quad \text{and} \quad \frac{y}{\mu} = \frac{x}{\mu} + \sum_{k=2}^{\infty} \frac{Q_k(\mu)}{\mu^k} x^k$$

are equal if $\mu = 1/\lambda$. The polynomial

$$P_k(\lambda) = Q_k(1/\lambda)\lambda^{k-1}$$

has degree at most $k - 1$ for $k \geq 2$. By Proposition 4 it is enough to prove that $\hat{f}(x, \lambda)$ is not convergent in a neighborhood of $x = 0$.

Suppose $\hat{f}(x, \lambda)$ converges in a neighborhood of $x = 0$; then $f_k(x)$ converges for all $k \geq 1$. The series $\hat{f}(x, \lambda)$ satisfies the equation

$$(y - x - \hat{f}(x, \lambda)) \circ (x + y^2, y + x^2 + \lambda\Delta(x)) \circ (x, x + \hat{f}(x, \lambda)) = 0,$$

which is equivalent to

$$\hat{f}(x + (x + \hat{f}(x, \lambda))^2, \lambda) = \hat{f}(x, \lambda)(1 - 2x) - \hat{f}(x, \lambda)^2 + \lambda\Delta(x). \tag{1}$$

The series $f_1(x)$ is the solution of a linear equation obtained by deriving equation (1) with respect to λ and evaluating at $\lambda = 0$. We obtain

$$f_1(x + x^2) = f_1(x)(1 - 2x) + \Delta(x).$$

Condition (b) now implies the divergence of $f_1(x)$. □

Let us look for functions $\Delta(x)$ satisfying conditions (a) and (b).

Proposition 7. $\Delta(x) = x^5/(\varepsilon^3 i + x^3)$ *satisfies conditions* (a) *and* (b) *for* $\varepsilon > 0$ *small enough.*

Proof. Condition (a) clearly holds. We will show that

$$f(x+x^2) = f(x)(1-2x) + \Delta(x) \tag{2}$$

does not have solutions in $\mathbb{C}\{x\}$. The diffeomorphism $x \mapsto x+x^2$ has a repelling petal P^- centered in the direction $\mathbb{R}^+$ and an attracting one P^+ centered in $\mathbb{R}^-$. If $\varepsilon > 0$ is small enough then $\varepsilon i \in P^+ \cap P^-$, and the other roots of $\varepsilon^3 i + x^3 = 0$ ($\alpha\varepsilon i$ and $\alpha^2\varepsilon i$) do not belong to the orbit of εi by $x \mapsto x+x^2$. Consider such an ε. Suppose that f belongs to $\mathbb{C}\{x\}$; then f is defined in a open neighborhood V of the origin. A good choice of V makes $V \cap P^-$ connected. The functional equation (2) allows us to extend f to $V \cup P^-$ as a meromorphic function. The function $\Delta(x)$ has a pole in εi and no other poles in the orbit of εi. We deduce that f has a pole of order 1 in a neighborhood of $\varepsilon i + \varepsilon i^2$, and then in the neighborhood of every point in the positive orbit of εi by $x+x^2$. Since εi belongs to P^+ it follows that f has infinitely many poles in a neighborhood of 0, which is impossible. $\square$

REMARK 3. If $\Delta(x) \in (x^3)$ is a polynomial of odd degree (e.g., $\Delta = x^3$), then equation (2) has no convergent solution. This can be proved by using the study of the diffeomorphism $x \mapsto x+x^2$ in [1]. The main steps in the proof are as follows.

1. If $\hat{f} \in \mathbb{C}\{x\}$ is a solution of (2) then $\hat{f}$ is an entire function.
2. If $\hat{f}$ is an entire function then $\hat{f}$ is a polynomial.
3. No polynomial satisfies equation (2).

In particular, if $\Delta(x) = x^5$ then β_λ has no convergent periodic curves outside of a Borel polar set.

Tangent to the identity diffeomorphisms are quite special, and for them pluripolarity does not optimally describe the nature of the set of parameters in which invariant curves converge. Sectors of convergence for the invariant curves and summability properties are expected. It is natural to think that the sets of convergence are analytic outside the ramification places. Nevertheless, our approach is very accurate for a wide class of problems, including small divisors (semisimple case).

In Section 5 we obtained parameters of divergence by using fibration-preserving examples, in which functional equations are simple. In the last example the functional equations became linear by differentiation, a suppler technique than the fibration method and one that can be applied in a more general context.

7. Diffeomorphisms without Formal Invariant Curves

The semisimple or unipotent diffeomorphisms always have formal invariant curves. Here we present an example without formal invariant curves. Consider

$$\varphi_\lambda = (\alpha x + \alpha y^2, \alpha^2 y + \alpha^2 x^2 + \alpha^2 \lambda \Delta(x)).$$

We have

$$\varphi_\lambda = (\alpha x, \alpha^2 y) \circ \beta_\lambda = \beta_\lambda \circ (\alpha x, \alpha^2 y)$$

if $\Delta(\alpha x) = \alpha^2 \Delta(x)$. Using the results from Section 6, together with Proposition 2 and Corollary 2, we prove our final proposition.

PROPOSITION 8. *Suppose that* $\Delta(x)$ *satisfies conditions* (a) *and* (b). *Then* φ_λ *has exactly three formal periodic* (*in fact,* 3-*periodic*) *curves that are divergent for all* $\lambda \in \mathbb{C}$ *outside of a Borel polar set.*

Consider the set

$$E = \{\eta \in \operatorname{Diff}(\mathbb{C}^2, 0) : j^2\eta = (\alpha x + \alpha y^2, \alpha^2 y + \alpha^2 x^2)\}.$$

The Jordan decomposition of $\xi = (\alpha x + \alpha y^2, \alpha^2 y + \alpha^2 x^2)$ is equal to

$$\xi = (\alpha x, \alpha^2 y) \circ (x + y^2, y + x^2) = (x + y^2, y + x^2) \circ (\alpha x, \alpha^2 y).$$

Because the Jordan decomposition is compatible with the filtration in the space of jets, we have

$$j^2\eta_s = (\alpha x, \alpha^2 y) \quad \text{and} \quad j^2\eta_u = (x + y^2, y + x^2)$$

for all $\eta \in E$. The formal diffeomorphism η_u has three formal invariant curves, whose tangents are the lines $y = x$, $y = \alpha x$, and $y = \alpha^2 x$. Since η_s commutes with η_u and $j^1\eta_s = (\alpha x, \alpha^2 y)$, the germ η has exactly three formal periodic curves, all of them 3-periodic. The proofs are analogous to those in the previous section.

The space E is an infinite-dimensional affine space, and the vector space associated to E is $(x, y)^3 \times (x, y)^3$. We choose $\eta_0 \in E$ such that all the periodic curves of η_0 are divergent. The space E is then the union of all the lines through η_0. We define the set

$$L_C = \{\eta \in L : \eta \text{ has at least one convergent periodic curve}\}$$

for any line L passing through η_0. We can prove (as in the previous sections) that either $L = L_C$ or L_C is a Borel polar set. Then L_C is polar because $\eta_0 \notin L_C$. We conclude that not having convergent periodic curves is a generic property in E. The same result can be obtained for the set of polynomial elements of E by choosing η_0 polynomial.

References

[1] P. Ahern and J.-P. Rosay, *Entire functions in the classification of differentiable germs tangent to the identity, in one or two variables,* Trans. Amer. Math. Soc. 347 (1995), 543–572.

[2] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields,* Ann. of Math. (2) 115 (1982), 579–595.

[3] F. Cano and D. Cerveau, *Desingularization of nondicritical holomorphic foliations and existence of separatrices,* Acta Math. 169 (1992), 1–103.

[4] J. Cano, *Construction of invariant curves for singular holomorphic vector fields,* Proc. Amer. Math. Soc. 125 (1997), 2649–2650.

[5] C. Chevalley, *Théorie des groupes de Lie,* vol. 2: *Groupes algébriques,* Hermann, Paris, 1951.

[6] M. Hakim, *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity,* Duke Math. J. 92 (1998), 403–428.
[7] J. Martinet, *Normalisation des champs de vecteurs holomorphes (d'après A.-D. Brjuno),* Lecture Notes in Math., 901, Springer-Verlag, New York, 1981.
[8] R. Pérez-Marco, *A note on holomorphic extensions,* preprint, 2000, http://xxx.lanl.gov/abs/math.DS/0009031.
[9] ———, *Total convergence or general divergence in small divisors,* Comm. Math. Phys. 223 (2001), 451–464.
[10] T. Ransford, *Potential theory in the complex plane,* London Math. Soc. Stud. Texts, 28, Cambridge Univ. Press, Cambridge, U.K., 1995.
[11] A. Seidenberg, *Reduction of singularities of the differential equation* $A\,dy = B\,dx$, Amer. J. Math. 90 (1968), 248–269.

Department of Mathematics
University of California – Los Angeles
Los Angeles, CA 90095

jribon@agt.uva.es