

A Cantor Set in the Unit Sphere in \mathbb{C}^2 with Large Polynomial Hull

BURGLIND JÖRICKÉ

An old question of Walter Rudin, asked in connection with Banach algebras and approximation by polynomials, concerns how massive the polynomial hulls of Cantor sets may be. In contrast to expectation (note that Cantor sets have topological dimension 0), it has been shown by Rudin, Vitushkin, and Henkin that the mentioned polynomial hull can be rather massive. Rudin himself constructed a Cantor set in \mathbb{C}^2 whose polynomial hull (and even its rational hull) contains an analytic variety of dimension 1 [12, Thm. 5; 7, Thm. III.2.5]. Later Vitushkin [14] and Henkin [8] gave examples of Cantor sets with interior points in the polynomial hull.

The problem received new attention in connection with interest in topology on strictly pseudoconvex boundaries and hulls of their subsets, as well as in connection with removable singularities of CR functions. In particular, it was asked whether Cantor sets in the unit sphere in \mathbb{C}^2 are polynomially convex. The expectation was that, for subsets of the sphere, the situation would change dramatically as for the case with some other problems. For example, totally real discs in \mathbb{C}^2 are not necessarily polynomially convex [11], but if contained in the sphere they are so [9]. Further, the polynomial hull of a compact set in \mathbb{C}^2 of finite 1-dimensional Hausdorff measure is not necessarily an analytic variety [1], but if the set is contained in the sphere it is so [14]. Moreover, the question about polynomial hulls of Cantor sets in the sphere has some relation to a still open conjecture of Vitushkin on the existence of a lower bound for the diameter of the largest boundary component of a relatively closed complex curve in the ball passing through the origin.

In [6], the slightly more general question was raised of whether Cantor sets in boundaries ∂G of strictly pseudoconvex domains G in \mathbb{C}^2 are convex with respect to the space of holomorphic functions in G that are continuous in the closure \bar{G} of the domain G . The question was answered affirmatively in [6] for some class of Cantor sets. The main tools used for this in [6] are the theorem of Bedford and Klingenberg [2], a characterization of tame Cantor sets in \mathbb{R}^n due to Bing, and the nontrivial fact that a C^2 manifold that is homeomorphic to \mathbb{R}^3 is C^2 diffeomorphic to \mathbb{R}^3 (see references in [6]). With the mentioned tools in mind, it was a natural step to obtain the following result.

THEOREM A. *Tame Cantor sets in the unit sphere in \mathbb{C}^2 are polynomially convex.*

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The theorem follows directly from [5] and the mentioned result of Bing, and it was proved independently a bit later by Lawrence [10].

Recall that a compact set $E \subset \mathbb{R}^n$ is a tame Cantor set if there is a homeomorphism of \mathbb{R}^n that carries E to the middle-third Cantor set in a coordinate line. Bing’s result is that this is equivalent to the following separation property.

For each pair of different points p and q in E and any $\varepsilon > 0$, there exists a set $\bar{b}^n \subset \mathbb{R}^n$ that is homeomorphic to a closed n -ball of diameter less than ε and with boundary $\partial b^n = S^{n-1}$ disjoint from E with $p \in b^n$ and $q \notin \bar{b}^n$.

Theorem A can be stated in a slightly more general form. For a domain $G \subset \mathbb{C}^n$, denote by $A(G)$ the algebra of analytic functions in G that are continuous in \bar{G} . For a compact subset K of \bar{G} we consider its $A(G)$ -hull,

$$A(G)\text{-hull}(K) = \{z \in \bar{G} : |f(z)| \leq \max_K |f| \text{ for all functions } f \in A(G)\}.$$

In case $G = \mathbb{C}^n$, we obtain the polynomial hull

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \max_K |p| \text{ for all polynomials } p\}$$

of the compact subset K of \mathbb{C}^n .

The rational hull of the compact K is defined by replacing, in the last condition, polynomials by rational functions that are analytic in a neighborhood of the compact K .

THEOREM B. *Let G be a bounded strictly pseudoconvex domain in \mathbb{C}^2 with smooth boundary. Let $K \subset \partial G$ be a Cantor set with the following separation property.*

For each pair of distinct points p and q in K , there exists a smooth 2-sphere $S^2 \subset \partial G \setminus K$ that separates p and q (i.e., p and q are in different connected components of $\partial G \setminus S^2$).

Then K is $A(G)$ -convex (i.e., $A(G)\text{-hull}(K) = K$).

Theorem A follows from Theorem B by the result of Bing and the mentioned fact about homeomorphic and diffeomorphic \mathbb{R}^3 (for more detail see [6]). Note that it is not necessary to require in Theorem B that the 2-spheres bound 3-balls in ∂G or that these balls have small size. In the description of tame Cantor sets, the condition that the b^3 have small diameter is important. There are wild Cantor sets in \mathbb{R}^3 [3] wherein each pair of distinct points can be separated by spheres (but not by spheres of small size).

The proof of Theorem B follows from the theorem of Bedford and Klingenberg. Here is a sketch of its proof for the reader’s convenience (see [6; 5; 10]).

If K were not $A(G)$ -convex, then by Zorn’s lemma there would be a minimal compact subset K' of K containing a given point of G in its $A(G)$ -hull. Then $A(G)\text{-hull}(K')$ is connected by Rossi’s maximum principle. By the separation property there is a smooth 2-sphere $S^2 \subset \partial G \setminus K'$ that divides K' and hence also ∂G . By [2] (perhaps after perturbing the 2-sphere), S^2 bounds a Levi-flat 3-ball $B^3 \subset G$ such that \bar{B}^3 is equal to the envelope of holomorphy of S^2 and to $A(G)\text{-hull}(S^2)$. By a theorem of Alexander and Stout (see the references in [13]), \bar{B}^3 divides \bar{G} . The envelope of holomorphy of $\partial G \setminus K'$ equals $\bar{G} \setminus A(G)\text{-hull}(K')$ (see

the references in [13]), so $\bar{\mathcal{B}}^3$ does not meet $A(G)$ -hull(K'). Hence $A(G)$ -hull(K') is contained in a connected component of $\bar{G} \setminus \bar{\mathcal{B}}^3$, which is impossible.

The mentioned results suggest, for instance, that Cantor sets in the unit sphere in \mathbb{C}^2 are polynomially convex. The purpose of this paper is to construct Cantor sets in the unit sphere with large polynomial hull.

For a subset A of \mathbb{C}^n and a positive number r , we denote by rA the set $\{rz : z \in A\}$. Let \mathbb{B}^n denote the unit ball in \mathbb{C}^n and $\partial\mathbb{B}^n$ its boundary.

THEOREM. *For any positive number $\beta < 1$, there exists a Cantor set E contained in $\partial\mathbb{B}^2$ whose polynomial hull \hat{E} contains the closed ball $\overline{\beta\mathbb{B}^2}$.*

The theorem implies, for example, that there are continuous arcs in the sphere (i.e., homeomorphic images of the unit interval of the real line) whose polynomial hull contains big balls.

The basis of the proof is the following observation. Consider two circles on the sphere, each of them the intersection of the sphere with a complex line. Then the polynomial hull of their union is equal to the union of their polynomial hulls. However, the polynomial hull of the union of their ε -neighborhoods ($\varepsilon > 0$) is essentially larger than the union of the hull of their ε -neighborhoods. More precisely, the following lemma holds.

MAIN LEMMA. *Let f and g be complex affine functions on \mathbb{C}^2 with $|\nabla f| = |\nabla g| = 1$. Suppose the sets $\{f = 0\} \cap \partial\mathbb{B}^2$ and $\{g = 0\} \cap \partial\mathbb{B}^2$ are disjoint circles. Then there exist positive numbers $a = a(f, g)$ and $r' = r'(f, g) < 1$ such that, for any positive ε , the following inclusion holds:*

$$\{|f \cdot g| \leq a\varepsilon\} \cap (\overline{\mathbb{B}^2} \setminus r'\overline{\mathbb{B}^2}) \subset (\{|f| \leq \varepsilon\} \cup \{|g| \leq \varepsilon\}) \cap (\overline{\mathbb{B}^2} \setminus r'\overline{\mathbb{B}^2}). \tag{1}$$

By a complex affine function f we mean a mapping $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ of the form $f(z) = f_0 + f_1 \cdot z_1 + f_2 \cdot z_2$ for complex numbers f_0, f_1 , and f_2 .

COROLLARY 1. *With f, g, a, ε , and r' as before, the inclusion*

$$\{|fg| \leq a\varepsilon\} \cap r\overline{\mathbb{B}^2} \subset ((\{|f| \leq \varepsilon\} \cap r\partial\mathbb{B}^2) \cup ((\{|g| \leq \varepsilon\} \cap r\partial\mathbb{B}^2)))^\wedge \tag{2}$$

holds for any $r \in [r', 1]$.

Proof. (1) implies in particular that, for any $r \in [r', 1]$,

$$\{|fg| \leq a\varepsilon\} \cap r\partial\mathbb{B}^2 \subset ((\{|f| \leq \varepsilon\} \cap r\partial\mathbb{B}^2) \cup ((\{|g| \leq \varepsilon\} \cap r\partial\mathbb{B}^2)) \tag{3}$$

holds. The polynomial hull of the left-hand side is $\{|fg| \leq a\varepsilon\} \cap r\overline{\mathbb{B}^2}$, hence (2) holds. □

REMARK 1. Each set on the right-hand side of (3) is the intersection of the sphere $r\partial\mathbb{B}^2$ with the closed ε -neighborhood of a circle (i.e., a solid torus in the 3-sphere if ε is small). Suppose the complex lines $\{f = 0\}$ and $\{g = 0\}$ intersect inside \mathbb{B}^2 . Then the left-hand side of (2) contains, for example, a ball of radius $\sqrt{a} \cdot \sqrt{\varepsilon}$

around the intersection point. The radius is much larger than ε for small enough $\varepsilon > 0$. The proof of the theorem is based on this observation.

Proof of the Main Lemma. Since the circles are disjoint there exist a and r' such that also the sets $\{|f| \leq a\} \cap (\mathbb{B}^2 \setminus r'\mathbb{B}^2)$ and $\{|g| \leq a\} \cap (\mathbb{B}^2 \setminus r'\mathbb{B}^2)$ are disjoint. If $z \in \mathbb{B}^2 \setminus r'\mathbb{B}^2$ and $|f(z) \cdot g(z)| \leq a \cdot \varepsilon$, then either $|f(z)| > a$ and then $|g(z)| \leq \varepsilon$, or $|f(z)| \leq a$ and then $|g(z)| > a$ and hence $|f(z)| \leq \varepsilon$. \square

Note that the constants a and r' can be chosen so that they serve also for pairs of complex affine functions close to f and g .

Preparation of Proof of the Theorem. We will find the required set E of the form $E = \bigcap_{N=1}^\infty E_N$ for a decreasing family of closed sets $E_N \subset \partial\mathbb{B}^2$ such that $\beta\mathbb{B}^2 \subset \hat{E}_N$ for each N . Then $\hat{E} \supset \overline{\beta\mathbb{B}^2}$. Indeed, let $z \in \overline{\beta\mathbb{B}^2}$. For any fixed polynomial p and for any N , there exists a point $z_N \in E_N$ for which $|p(z)| \leq |p(z_N)|$. If (for fixed p) z^* is an accumulation point of the z_N , then $z^* \in E$ and by continuity $|p(z)| \leq |p(z^*)|$. This holds for arbitrary polynomials p , hence $z \in \hat{E}$.

Each set E_N will be the finite union of disjoint solid tori of the form just described. We introduce the following notation. For a complex affine function f with $|\nabla f| = 1$ and a positive number σ , we denote $T_f(\sigma) = \{z \in \partial\mathbb{B}^2 : |f(z)| \leq \sigma\}$. After a unitary change of coordinates in \mathbb{C}^2 , we may assume that $|f|$ has the form $|z_1 - (1 - s)|$ for a real number $s = s_f$. The mentioned unitary transformation takes $T_f(\sigma)$ to the set $T^s(\sigma) = \{z \in \partial\mathbb{B}^2 : |z_1 - (1 - s)| \leq \sigma\}$. This is a solid torus if $s < 1$ and $\sigma < s$. For $r < 1$, denote ${}^rT_f(\sigma) = \{|f| \leq \sigma\} \cap r\partial\mathbb{B}^2$; similarly, we write ${}^rT^s(\sigma)$. Denote, finally, the complex lines (the symmetry axes of the respective tori) by $\ell_f = \{f = 0\}$ and $\ell^s = \{z_1 = 1 - s\}$.

Note that the number σ together with the unitary invariant s of a torus $T_f(\sigma)$ contained in $\partial\mathbb{B}^2$ determines its diameter. If s and σ are small then the diameter of $T^s(\sigma)$ (and hence of each unitarily equivalent torus) is also small.

By the following easy lemma, it is enough to cover bidiscs by polynomial hulls of suitable Cantor sets.

LEMMA 1. *For any $\beta \in (0, 1)$ there is a number $q \in (0, 1)$ such that $\overline{\beta\mathbb{B}^2}$ can be covered by a finite union of bidiscs of the form $q(\overline{\mathcal{D}}_1 \times \overline{\mathcal{D}}_2)$, where $\overline{\mathcal{D}}_j$ are closed discs in \mathbb{C} centered at zero such that $\partial\mathcal{D}_1 \times \partial\mathcal{D}_2 \subset \partial\mathbb{B}^2$.*

The following proposition allows us to cover bidiscs of Lemma 1 by the polynomial hull of finite unions of disjoint solid tori that are arbitrarily thin tubular neighborhoods of (in general not small) circles.

PROPOSITION 1. *Let \mathcal{D}_1 and \mathcal{D}_2 be discs in \mathbb{C} centered at the origin and such that $\partial\mathcal{D}_1 \times \partial\mathcal{D}_2 \subset \partial\mathbb{B}^2$. Let $\gamma > 0$ and $q \in (0, 1)$. There exist two numbers $s_1, s_2 \in (0, 1)$ and a positive number ε' , all depending only on $\mathcal{D}_1, \mathcal{D}_2, q$, and γ , such that for each $\varepsilon \in (0, \varepsilon')$ there exist two families of solid tori $T_j(\varepsilon) = T_{f_j}(\varepsilon)$ and $T_k^*(\varepsilon) = T_{g_k}(\varepsilon)$ with each family containing finitely many tori that are unitarily equivalent to $T^{s_1}(\varepsilon)$ and $T^{s_2}(\varepsilon)$, respectively, and that possess the following properties.*

- All thicker tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are pairwise disjoint and contained in the γ -neighborhood of $\partial\mathcal{D}_1 \times \partial\mathcal{D}_2$.
- The polynomial hull of the union satisfies the relation

$$q(\overline{\mathcal{D}}_1 \times \overline{\mathcal{D}}_2) \subset \left(\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon) \right)^\wedge. \tag{4}$$

REMARK 2. For some $r' < 1$ depending only on \mathcal{D}_1 , \mathcal{D}_2 , q , and γ , the tori in Proposition 1 can be chosen such that, for all $r \in [r', 1]$, the inclusion

$$q(\overline{\mathcal{D}}_1 \times \overline{\mathcal{D}}_2) \subset \left(\bigcup_j {}^r T_j(\varepsilon) \cup \bigcup_k {}^r T_k^*(\varepsilon) \right)^\wedge \tag{4_r}$$

also holds.

Let \mathbb{D} be the unit disc in \mathbb{C} and let $\mathbb{T} = \partial\mathbb{D}$.

Proof of Proposition 1. Let $\mathcal{D}_1 = R_1\mathbb{D}$ and $\mathcal{D}_2 = R_2\mathbb{D}$. Increasing q , we may assume that $q < 1$ is as close to 1 as needed. Let ζ_1^j be equidistributed points on $qR_1\mathbb{T}$ with the distance of nearest points being a number between $B\varepsilon$ and $(B + 1)\varepsilon$, where B is any constant, $B \geq 5$, and $\varepsilon > 0$ is small enough. For each B such points can be found if ε is small. Similarly, let ζ_2^k be equidistributed points on $qR_2\mathbb{T}$ with distance between closest points in $[B\varepsilon, (B + 1)\varepsilon]$. Define

$$f_j(z) = z_1 - \zeta_1^j, \quad g_k(z) = z_2 - \zeta_2^k.$$

All sets $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are nonempty tori if ε is small enough, and they are pairwise disjoint for such ε . Disjointness is clear for tori of the same family (since $B \geq 5$) and follows from the fact that $\{f_j = 0\} \cap \{g_k = 0\}$ is contained in \mathbb{B}^2 in the other case.

For any j and k there is a unitary transformation that takes $|f_j|$ to $|z_1 - (1 - s_1)|$ for $s_1 = 1 - R_1q$ and $|g_k|$ to $|z_2 - (1 - s_2)|$ for $s_2 = 1 - R_2q$. It follows that the $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ and that the $T_k^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$. Moreover, by Corollary 1 there exist constants a and r' depending only on s_1 and s_2 (since all pairs $(|f_j|, |g_k|)$ are unitarily equivalent to $(|z_1 - (1 - s_1)|, |z_2 - (1 - s_2)|)$) such that

$$\{|f_j g_k| \leq a\varepsilon\} \cap r\mathbb{B}^2 \subset ((\{|f_j| \leq \varepsilon\} \cup \{|g_k| \leq \varepsilon\}) \cap r\partial\mathbb{B}^2)^\wedge$$

for all $r \in [r', 1]$ and all j and k . The left-hand side contains the bidisc

$$\{|f_j| \leq \sqrt{a\varepsilon}\} \cap \{|g_k| \leq \sqrt{a\varepsilon}\} = (\zeta_1^j, \zeta_2^k) + \sqrt{a\varepsilon} (\mathbb{D} \times \mathbb{D})$$

of radius $\sqrt{a}\sqrt{\varepsilon}$ around the intersection point $\ell_{f_j} \cap \ell_{g_k} = (\zeta_1^j, \zeta_2^k)$ (with r' close to 1 and ε small enough). If $\sqrt{a}\sqrt{\varepsilon} > (B + 1)\varepsilon$, then we obtain (running over all pairs (j, k)) that the right-hand side of (4_r) contains the product of the circles $qR_1\mathbb{T}$ and $qR_2\mathbb{T}$; hence (4_r) holds.

Finally, initially taking q close to 1 (and $\varepsilon > 0$ small), we ensure that all $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are contained in a small neighborhood of $\partial\mathcal{D}_1 \times \partial\mathcal{D}_2$. \square

The following proposition allows us to cover small enough neighborhoods of given complex affine discs (intersections of complex lines with suitable balls) by polynomial hulls of unions of disjoint tori with small diameter contained in the sphere. Propositions 2, 3, and 4 will be stated for tori related to the function $z_1 - (1 - s)$ for some $s \in (0, 1)$. They hold also for $z_1 - (1 - s)$ replaced by any complex affine function f such that $|f|$ is unitarily equivalent to $|z_1 - (1 - s)|$.

PROPOSITION 2. *Let $s \in (0, 1)$. For any small $\delta > 0$, any $q \in (0, 1)$ close to 1, and any small $s' > 0$, one can find two positive numbers $s_1 < s'$ and $s_2 < s'$ as well as a positive number $\varepsilon' = \varepsilon'(s, s_1, s_2, \delta)$ such that, for any positive $\varepsilon < \varepsilon'$, there exist two finite families of complex affine functions f_j and g_k with the following properties.*

The related solid tori $T_j(2\varepsilon) \stackrel{\text{def}}{=} T_{f_j}(2\varepsilon)$ and $T_k^(2\varepsilon) \stackrel{\text{def}}{=} T_{g_k}^*(2\varepsilon)$ are all pairwise disjoint and contained in $\{|z_1 - (1 - s)| \leq \delta\}$. The $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ and the $T_k^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$. Moreover,*

$$\overline{q\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \varepsilon\} \subset \left(\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon) \right)^\wedge. \tag{5}$$

The following stronger assertion holds. There exists an $r' < 1$ such that, for any $r \in [r', 1]$, the inclusion

$$\overline{q\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \varepsilon\} \subset \left(\bigcup_j {}^rT_j(\varepsilon) \cup \bigcup_k {}^rT_k^*(\varepsilon) \right)^\wedge \tag{5_r}$$

also holds.

Note that the number of tori and their symmetry axes will be chosen together with ε .

REMARK 3. Proposition 2 describes a subset of the sphere whose connected components have small diameter but whose polynomial hull contains, for instance, the disc $\{z_1 = 0, |z_2| \leq q\}$ ($q < 1$). This statement should be contrasted with the following still open conjecture of Vitushkin, which appeared in connection with a problem of E. Kallin on polynomial convexity of finite unions of disjoint balls.

CONJECTURE (Vitushkin). *Let X be a relatively closed complex manifold of dimension 1 in \mathbb{B}^2 , smooth up to the boundary and transversal to $\partial\mathbb{B}^2$. Suppose X contains the origin. Then X has a boundary component of diameter bounded from below by an absolute constant.*

The link between Vitushkin’s conjecture and Remark 3 (or the theorem, respectively) is the following open problem.

PROBLEM. Let $K \subset \mathbb{C}^2$ be compact, $K \neq \hat{K}$, and suppose that $z \in \hat{K} \setminus K$. Under which conditions does the ε -neighborhood K_ε of K (for every $\varepsilon > 0$) contain the boundary of a Riemann surface passing through z ?

Note that Proposition 2 does not control the ratio of the numbers ε (the width of the tubular neighborhood of ℓ^s that is covered by the polynomial hull of the union of small tori) and δ (the width of the tubular neighborhood that contains the small tori). Propositions 3 and 4 will allow such control, after which the inductive construction of the E_N can be easily provided.

Plan of Proof of Proposition 2. The functions f_j in Proposition 2 will be chosen so that ℓ_{f_j} will pass through a point p_j , where the p_j are equidistributed on the circle $\{z_1 = 1 - s\} \cap R\partial\mathbb{B}^2$ for a suitable $R < 1$ and close to 1. Moreover, the ℓ_{f_j} are complex tangent to the sphere $R\partial\mathbb{B}^2$. The ℓ_{g_j} pass through points p_j^* , which are equidistributed on the same circle. Furthermore, the p_j^* are obtained from p_j via turning by a fixed angle ψ in the z_2 -direction and the ℓ_{g_j} are obtained via turning the complex tangents to the sphere $R\partial\mathbb{B}^2$ at p_j^* by a fixed angle ν . Lemma 2 states that (for suitable p_j, R, ε) the tubes $\{|f_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint. (Hence the tori $T_j(2\varepsilon) = \{|f_j| \leq 2\varepsilon\} \cap \partial\mathbb{B}^2$ are pairwise disjoint and not linked with each other in $\partial\mathbb{B}^2$.) Moreover, with a suitable choice of the angle ν , the same is true for the tubes $\{|g_k| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ and the tori $T_k^*(2\varepsilon)$.

On the other hand, each complex line ℓ_{f_j} intersects several lines ℓ_{g_k} inside \mathbb{B}^2 (equivalently, the corresponding circles $\ell_{f_j} \cap \partial\mathbb{B}^2$ and $\ell_{g_k} \cap \partial\mathbb{B}^2$ are linked in $\partial\mathbb{B}^2$). Lemma 3 allows us to choose the angle ψ in such a way that the tubes $\{|f_j| \leq 2\varepsilon\}$ and $\{|g_k| \leq 2\varepsilon\}$ intersect at points that are not contained in the sphere $\partial\mathbb{B}^2$; in other words, the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint. An application of the main lemma will then give the proposition.

THE TORI $T_j(\varepsilon)$ AND $T_j^*(\varepsilon)$. The $T_j(\varepsilon)$ will be determined by the following parameters: $s \in (0, 1)$, a small number $t > 0$, a (large) natural number \mathcal{N} , and a small number $\varepsilon > 0$. The $T_j^*(\varepsilon)$ will be determined by $s, t, \mathcal{N}, \varepsilon$, and a small number ν .

Consider the intersection

$$\ell^s \cap \partial\mathbb{B}^2 = \{(1 - s, R_2 e^{i\phi}) : \phi \in [0, 2\pi)\} \quad (R_2^2 = 2s - s^2).$$

Denote by C_t the slightly smaller concentric circle $C_t = \{(1 - s, R_2(1 - t)e^{i\phi}) : \phi \in [0, 2\pi)\} = \ell^s \cap R\partial\mathbb{B}^2$, where R and t are related by the equality $(2t - t^2)(2s - s^2) = 1 - R^2$.

Let p_j be equidistributed points on C_t ,

$$p_j = (1 - s, R_2(1 - t)e^{i\phi_j}), \quad \text{where } \phi_j = \frac{2\pi}{\mathcal{N}}j, \quad j = 0, \dots, \mathcal{N} - 1.$$

Define the constant B by the following relation: the distance between the nearest of the equidistributed points, $|p_j - p_{j-1}|$, equals $B \cdot \varepsilon$. Denote by ℓ_j the complex lines through p_j that are tangent to $R\partial\mathbb{B}^2$,

$$\ell_j = \{p_j + v_j \cdot \zeta : \zeta \in \mathbb{C}\}, \quad v_j = ((1-t)R_2, -(1-s)e^{i\phi_j}), \quad (6)$$

and let f_j be annihilating functions of the ℓ_j with gradient of norm 1.

For a (small) number ψ , denote by p_j^* the points

$$p_j^* = (1-s, R_2(1-t)e^{i(\phi_j+\psi)}).$$

Finally, for a small constant $\nu > 0$ we denote by ℓ_j^* ($j = 0, \dots, \mathcal{N} - 1$) the complex line obtained from the complex tangent to $R\partial\mathbb{B}^2$ at p_j^* via turning by a fixed angle:

$$\ell_j^* = \{p_j^* + w_j \cdot \zeta : \zeta \in \mathbb{C}\}, \quad w_j = ((1-t)R_2, -(1-s+\nu)e^{i(\phi_j+\psi)}). \quad (7)$$

Let g_j be annihilating functions of ℓ_j^* with gradient of norm 1.

LEMMA 2. (a) *Let $s \in (0, 1)$ and let $t > 0$ be sufficiently small. There exist positive constants $\varepsilon' = \varepsilon'(s, t)$ and $B' = B'(s, t)$ such that, if $\varepsilon \in (0, \varepsilon')$ and $B > B'$, then for the f_j defined previously for chosen parameters $s, t, \mathcal{N}, \varepsilon$ (with B related to \mathcal{N} and ε as before), the sets $\{|f_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint.*

(b) *If, in addition, ν is small enough (depending on s), there exist constants $\varepsilon' = \varepsilon'(s, t, \nu)$ and $B' = B'(s, t, \nu)$ such that, if $\varepsilon \in (0, \varepsilon')$ and $B > B'$, then for the g_j defined for the parameters $s, t, \mathcal{N}, \nu, \varepsilon$ and an arbitrary parameter ψ , the sets $\{|g_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint.*

LEMMA 3. *With suitable constants ε' and B' that are smaller (resp., greater) than the constants of Lemma 2(a) and 2(b) and with parameters $s, t, \mathcal{N}, \nu, \varepsilon$ as in Lemma 2, with $B > B'$ one can choose the constant ψ in such a way that the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ of Lemma 2 are disjoint for all j and k .*

Proof of Lemma 2. The argument is roughly that, for small $t > 0$, the intersection $\ell_j \cap \overline{\mathbb{B}^2}$ is a disc of small diameter. If two such discs intersect, their centers p_j must be close. But then the corresponding complex tangencies to $R\partial\mathbb{B}^2$ are “almost parallel”, so they cannot intersect at points close to the p_j .

More precisely, the set $\{|f_j| \leq 2\varepsilon\}$ is the union of complex lines $\mathcal{L}_j = \{\tilde{p}_j + v_j \cdot \zeta : \zeta \in \mathbb{C}\}$, with the same direction v_j as ℓ_j through points $\tilde{p}_j \in \ell^s$. The \tilde{p}_j have distance from p_j not exceeding $A\varepsilon$ for a constant A that depends on the angle between ℓ^s and ℓ_j and hence—by unitary equivalence—on s and t only. Therefore,

$$\tilde{p}_j = (1-s, R_2(1-t)e^{i\phi_j} + \alpha_j) \quad \text{with } |\alpha_j| \leq A\varepsilon. \quad (8)$$

Let $j \neq k$. For the intersection point $\mathcal{L}_j \cap \mathcal{L}_k \neq \emptyset$, we have

$$\tilde{p}_j + v_j \zeta = \tilde{p}_k + v_k \zeta' \quad \text{for some } \zeta, \zeta' \in \mathbb{C}.$$

From (6) and (8) we obtain that $\zeta = \zeta'$ and

$$\zeta = \frac{R_2(1-t)}{(1-s)} + \frac{\alpha_j - \alpha_k}{(1-s)(e^{i\phi_j} - e^{i\phi_k})} \quad \text{with } |\alpha_j| \leq A\varepsilon, |\alpha_k| \leq A\varepsilon. \quad (9)$$

For small $t > 0$ and $\varepsilon < \varepsilon'(t, s)$, the absolute value $|\zeta|$ of this number can be estimated from below by a positive constant depending only on s , provided that $B \geq B'(s, t)$ and that $B'(s, t)$ is chosen so that

$$|e^{i\phi_j} - e^{i\phi_k}|^{-1} \cdot 2A\varepsilon \leq \frac{2A\varepsilon}{B'(s, t) \cdot \varepsilon} < \frac{1}{2}R_2(1 - t).$$

On the other hand, $\ell_j \cap \overline{\mathbb{B}^2}$ is a closed disc of radius $\sqrt{1 - R^2}$, which for fixed s is small if $t > 0$ is small. Hence for $\varepsilon < \varepsilon'(t, s)$ the diameter of the intersection $\mathcal{L}_j \cap \overline{\mathbb{B}^2}$ is small (\mathcal{L}_j , as before, contained in $\{|f_j| \leq 2\varepsilon\}$). For those t and ε , the intersection $\mathcal{L}_j \cap \overline{\mathbb{B}^2}$ cannot contain the two points \tilde{p}_j and $\tilde{p}_j + v_j \zeta \in \mathcal{L}_j \cap \mathcal{L}_k$, since their distance is bounded from below by a constant depending only on s . Since $\tilde{p}_j \in \mathcal{L}_j \cap \overline{\mathbb{B}^2}$, the point $\tilde{p}_j + v_j \zeta$ is not in $\overline{\mathbb{B}^2}$. Part (a) is proved.

To prove assertion (b), increase A if necessary and replace the number $(1 - s)$ in (9) by $(1 - s + \nu)$. Use that for ν small, $\nu < \nu(s)$, and $\varepsilon < \varepsilon'(s, t, \nu)$, the complex lines \mathcal{L}_j^* parallel to ℓ_j^* and of distance not exceeding 2ε from ℓ_j^* still intersect $\overline{\mathbb{B}^2}$ along a disc of small diameter (not exceeding $\text{const} \cdot (\nu + \sqrt{t})$). The remaining arguments are the same as for part (a). □

REMARK 4. Note that the unitary transformation $(z_1, z_2) \rightarrow (z_1, z_2 e^{-i\phi_j})$ maps the torus $T_j(\varepsilon) = T_{f_j}(\varepsilon)$ to the torus $T_0(\varepsilon) = T_{f_0}(\varepsilon)$ and $T_j^*(\varepsilon) = T_{g_j}(\varepsilon)$ to $T_0^*(\varepsilon) = T_{g_0}(\varepsilon)$. Moreover, the tori $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ with

$$s_1 = 1 - R = (1 + R)^{-1}(2t - t^2)(2s - s^2) \tag{10}$$

and the $T_j^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$ for some number $s_2 > s_1$, which can be estimated from above by $\text{const} \cdot (t + \nu^2)$.

Proof of Lemma 3. We want to choose ψ so that $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint for all j and k . Note that the norm $|P_{j,k}|$ of the intersection point $P_{jk} = \ell_j \cap \ell_k^*$ depends only on $m = k - j$. The idea is as follows. When $|P_{0,m}|$ is close to 1, the points $P_{0,m}$ form approximately an arithmetic progression with step $\text{const} \cdot B\varepsilon$ on a real line in the complex line ℓ_0 . Changing the parameter ψ leads approximately to translating the approximate arithmetic progression by the parameter ψ inside the real line. If B is large enough, this enables us to choose ψ in such a way that the intersection points $P_{0,m}$ are not in $\partial\mathbb{B}^2$; moreover, a neighborhood of them of size comparable to ε (containing the intersection $\{|f_0| \leq 2\varepsilon\} \cap \{|g_m| \leq 2\varepsilon\}$) does not meet $\partial\mathbb{B}^2$.

Here is the precise argument. Let $s, t, \mathcal{N}, \nu, \varepsilon$ be chosen according to Lemma 2, with constants ε' and B' as specified hereafter and smaller (resp., greater) than the constants in parts (a) and (b) of that lemma. We will change the parameter ψ . The points p_k^* , the complex lines ℓ_k^* , the intersection points $P_{jk} = \ell_j \cap \ell_k^*$, and the tori $T_k^*(\varepsilon)$ will all depend on ψ , but we will indicate the dependence on ψ only when we want to draw special attention to this fact.

From (6) and (7) it follows that the intersection point $P_{j,k} = P_{j,k}(\psi)$ of the complex lines ℓ_j and ℓ_k^* is determined by

$$P_{j,k}(\psi) = p_j + v_j \cdot \zeta(\psi) = p_k^*(\psi) + w_k(\psi) \cdot \zeta'(\psi)$$

for some $\zeta(\psi), \zeta'(\psi) \in \mathbb{C}$. From the same formulas we now obtain that $\zeta(\psi) = \zeta'(\psi) \stackrel{\text{def}}{=} \zeta_{j,k}(\psi)$ and

$$\zeta_{j,k}(\psi) = \frac{R_2 \cdot (1-t)}{1-s} \frac{1 - e^{i(\phi_k - \phi_j + \psi)}}{1 - Qe^{i(\phi_k - \phi_j + \psi)}}, \tag{11}$$

where

$$Q = (1-s+v)(1-s)^{-1} > 1.$$

Here, as before, we put $\phi_j = 2\pi j/\mathcal{N}$ and assume that the natural number \mathcal{N} is large. Put

$$F(\phi) = \left| \frac{1 - e^{i\phi}}{1 - Qe^{i\phi}} \right| = \left| \frac{2 \sin(\phi/2)}{1 - Qe^{i\phi}} \right|, \tag{12}$$

so that

$$|\zeta_{j,k}(\psi)| = \frac{R_2(1-t)}{1-s} F(\phi_k - \phi_j + \psi). \tag{13}$$

Note that the function $(|1 - Qe^{i\phi}|)^{-1}$ is of class $C^\infty(\mathbb{R})$, so for small $\phi \neq 0$ we have

$$F'(\phi) = \frac{\cos(\phi/2) \cdot \text{sgn } \phi}{|1 - Qe^{i\phi}|} + \left| 2 \sin \frac{\phi}{2} \right| \cdot \left(\frac{1}{|1 - Qe^{i\phi}|} \right)'$$

Hence

$$0 < C_Q \leq |F'(\phi)| \leq 2C_Q \quad \text{for } \phi \neq 0 \text{ and } |\phi| \leq \Phi_Q \tag{14}$$

for positive constants Φ_Q and C_Q depending only on Q . (Φ_Q may be chosen comparable to v and C_Q comparable to v^{-1} .) Since $p_j \in R\partial\mathbb{B}^2$ and since v_j is the direction of the complex tangent line to $R\partial\mathbb{B}^2$, it follows that

$$|P_{j,k}|^2 = |p_j + v_j \cdot \zeta_{j,k}|^2 = |p_j|^2 + |v_j|^2 |\zeta_{j,k}|^2.$$

By (6), we have $|v_j|^2 = |p_j|^2 = R^2$. Hence

$$|P_{j,k}(\psi)|^2 = R^2 \left(1 + \frac{R_2^2(1-t)^2}{(1-s)^2} F^2(\phi_k - \phi_j + \psi) \right), \tag{15}$$

where

$$R^2 = 1 - (2t - t^2)(2s - s^2). \tag{16}$$

The set $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$ is contained in the (closed) $A'\varepsilon$ -neighborhood of $P_{j,k} = P_{j,k}(\psi)$. The constant A' depends only on v, s , and t . For the square of the norm of points in $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$, this implies that these numbers are contained in the (open) $4A'\varepsilon$ -neighborhood of $|P_{j,k}(\psi)|^2$ provided $|P_{j,k}(\psi)| \leq 3/2$ and $A'\varepsilon < 1$.

We want to choose ψ so that the $4A'\varepsilon$ -neighborhoods of the $|P_{j,k}(\psi)|^2$ do not contain 1. Put $m = k - j$. Then $\phi_k - \phi_j + \psi = \phi_m + \psi = -\phi_{-m} + \psi$ and $|P_{j,k}(\psi)|^2 = |P_{0,m}(\psi)|^2 = |P_{0,-m}(-\psi)|^2$. Hence all possible values of (15) are obtained when $j = 0$ and $k = m$ runs over integers.

If for some ψ and m_0 we have

$$||P_{0,m_0}(\psi)|^2 - 1| < 4A'\varepsilon, \tag{17}$$

then by (15) and (16) it follows that $F^2(\phi_{m_0} + \psi)t^{-1}$ can be estimated from above and below by positive constants depending only on s , provided ε is small compared with t . Hence by (12) in this case $\phi_{m_0} + \psi$ is comparable to either $\nu \cdot \sqrt{t}$ or $-\nu \cdot \sqrt{t}$.

Suppose now that for some m_0 the $4A'\varepsilon$ -neighborhood of $|P_{0,m_0}(0)|^2$ contains 1. (Otherwise we are done.) Note that by symmetry this is true also for $-m_0$. If t is small then, by the preceding arguments, $|\phi_{m_0}|$ and $|\phi_{-m_0}|$ are much smaller than Φ_Q . By (14) and (15) we see that, for fixed ψ close to 0 and for integer numbers m close to m_0 (close to $-m_0$, respectively), the possible values of the right-hand side of (15) have distance from each other at least equal to $c(s, t, \nu) \cdot R_2(1-t)|\phi_{m+1} - \phi_m|$ for a positive constant $c(s, t, \nu)$ depending on s, t , and ν . Since $|\phi_{m+1} - \phi_m| = 2\pi/\mathcal{N} > 2 \sin(\pi/\mathcal{N})$ and since $R_2(1-t)2 \sin(\pi/\mathcal{N}) = B\varepsilon$, the difference of the values of (15) for fixed ψ close to 0 and for m close to m_0 ($-m_0$, resp.) is at least $c(s, t, \nu) \cdot B \cdot \varepsilon$. Take the constant B' large enough so that the latter constant is at least $40A'\varepsilon$.

Using again (14) and (15), take ψ so that

$$10A'\varepsilon < ||P_{0,m_0}(\psi)|^2 - |P_{0,m_0}(0)|^2| < 30A'\varepsilon$$

and the same estimate holds for m_0 replaced by $-m_0$. Note that ψ is comparable to ε with multiplicative constants depending on s, t , and ν , so ψ is small if ε is small. The $4A'\varepsilon$ -neighborhood of $|P_{0,m_0}(\psi)|^2$ does not contain 1, and the same is true for m_0 replaced by $-m_0$. The preceding arguments give that, for all m close to m_0 ,

$$||P_{0,m}(\psi)|^2 - 1| > 4A'\varepsilon;$$

hence, since the function F is strictly monotonic on the positive half-axis (resp., on the negative half-axis), this holds for all m .

With this choice of ψ , we obtain that for all j and k the set $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$ does not meet $\partial\mathbb{B}^2$ if ε is small enough. Hence the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint. □

Proof of Proposition 2. For $s \in (0, 1)$ as in the statement, we first choose t and ν small enough that the numbers s_1 and s_2 are less than s' (see (10) and Remark 4) and $R(t) > q$ (see (16)). By choosing t and ν small we may also achieve that, with any choice of the natural number \mathcal{N} , the parameter ψ , and a small enough positive number ε , the sets $\{|f_j| \leq 2\varepsilon\}$ and $\{|g_k| \leq 2\varepsilon\}$ are contained in $\{|z_1 - (1-s)| \leq \delta\}$. Choose by Lemmas 2 and 3 the relation of the numbers \mathcal{N}, ε , and ψ so that the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are pairwise disjoint. They are unitarily equivalent to $T^{s_1}(\varepsilon)$ and $T^{s_2}(\varepsilon)$, respectively.

It remains to prove (5). Apply the Main Lemma to the pair ℓ_j and $\ell_j^* = \ell_j^*(\psi)$ for any j . Since the pairs obtained for different j are unitarily equivalent to ℓ_0 and $\ell_0^*(\psi)$ (see Remark 4) and since $\ell_0^*(\psi)$ is close to $\ell_0^*(0)$ if ψ is small, there exist numbers $a > 0$ and $r' \in (0, 1)$ depending only on s, t, ν such that (1) holds with f

replaced by f_j and g replaced by g_j . By Corollary 1, for $r \in [r', 1]$ the polynomial hull of ${}^rT_j(\varepsilon) \cup {}^rT_j^*(\varepsilon)$ contains

$$\{|f_j g_j| \leq a\varepsilon\} \cap r\overline{\mathbb{B}^2} \supset \{|f_j| \leq \sqrt{a\varepsilon}\} \cap \{|g_j| \leq \sqrt{a\varepsilon}\} \cap r\overline{\mathbb{B}^2}.$$

So $f_j(p_j) = 0$ and, since $g_j(p_j^*) = 0$, we obtain $|g_j(p_j)| = |g_j(p_j) - g_j(p_j^*)| \leq |p_j - p_j^*| = O(\varepsilon)$. This implies that, for small ε , the latter set contains the bidisc $p_j + b\sqrt{a\varepsilon} (\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ of radius $b \cdot \sqrt{a\varepsilon}$ around the point p_j for some constant b depending only on s, t , and ν . Recall that the p_j are equidistributed on the circle $C_t = \{1 - s\} \times \{|z| = R_2(1 - t)\}$ with distance between nearest points equal to $B\varepsilon$. If $b \cdot \sqrt{a\varepsilon} > B\varepsilon$ (i.e., if $\varepsilon < (b^2/B^2)a$), then the polynomial hull of $\bigcup_j (T_j(\varepsilon) \cup T_j^*(\varepsilon))$ contains $\{z_1\} \times \{|z| = R_2(1 - t)\}$ for all z_1 with $|z_1 - (1 - s)| \leq B\varepsilon$. Since $(1 - s)^2 + (R_2(1 - t))^2 = R(t)^2 > q^2$ we obtain that, for $\varepsilon < \varepsilon'$ and with a suitable choice of the constants r' and $\varepsilon'(s, s_1, s_2, \delta)$, the inclusion (5_r) holds for $r \in [r', 1]$. The weaker inclusion (5) follows. \square

The proof of Proposition 2 does not yield good estimates for the ratio of the constants ε and δ . The point is that the Main Lemma gives useful effects essentially only for the pair ℓ_j and ℓ_j^* of complex lines—not for arbitrary pairs ℓ_j and ℓ_k^* , when the intersection point may be close to $\partial\mathbb{B}^2$ and the constant a of the Main Lemma is not bounded away from zero. We shall state and prove the stronger Proposition 4, which can be directly used for inductive construction of the E_N .

The first step toward this goal is Proposition 3, which is in the spirit of Proposition 1. For its proof we use again two families of complex lines. The first family, ℓ_k^* , consists of certain complex lines parallel to the z_2 -axis. For some small $\sigma > 0$ and a suitable constant α , their z_1 -coordinates form an α -net of the $\frac{4}{3}\sigma$ -neighborhood of $1 - s$ ($s \in (0, 1)$). Note that the intersections of these lines with $\partial\mathbb{B}^2$ are circles of diameters comparable to \sqrt{s} . The second family, ℓ_j , consists of complex tangent lines to a smaller sphere through equidistributed points on a circle C_t contained in $\ell^s = \{z \in \mathbb{C}^2 : z_1 = 1 - s\}$. Here t cannot be chosen arbitrarily small; it will take a value that is determined by s and σ . As in Proposition 1, the Main Lemma will be applied to all pairs (ℓ_j, ℓ_k^*) .

PROPOSITION 3. *For positive numbers σ and α we will denote by ζ_k all points of the disc $\{\zeta \in \mathbb{C} : |\zeta - s| \leq \frac{4}{3}\sigma\}$ that are contained in the lattice $s + \alpha\mathbb{Z} + i\alpha\mathbb{Z}$. Associate to these points functions $g_k(z) = z_1 - (1 - \zeta_k)$.*

Let $s \in (0, 1)$. Then there exist positive constants $\sigma' = \sigma'(s)$ and $\alpha = \alpha(s, \sigma')$ such that the following holds: For each $\sigma \in (0, \sigma')$ there exist positive constants $\varepsilon' = \varepsilon'(s, \sigma)$ and $s^ = s^*(s, \sigma) < s$ ($s^*(s, \sigma) \rightarrow 0$ for $\sigma \rightarrow 0$) such that, for any $\varepsilon \in (0, \varepsilon')$, the following is true.*

One can find finitely many complex affine functions f_j such that the tori $T_j(\varepsilon) = \{z \in \partial\mathbb{B}^2 : |f_j(z)| \leq \varepsilon\}$ are unitarily equivalent to $T^{s^}(\varepsilon)$. Moreover, the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon) = \{z \in \partial\mathbb{B}^2 : |g_k(z)| \leq 2\varepsilon\}$ (for g_k as defined previously) are all pairwise disjoint and contained in $\{|z_1 - (1 - s)| \leq 2\sigma\}$. Finally, for the polynomial hull of the union of the tori, we have*

$$\{|z_1 - (1 - s)| \leq \sigma\} \cap (1 - \rho)\overline{\mathbb{B}^2} \subset \left(\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon) \right)^\wedge \tag{18}$$

for a constant $\rho = \rho(s, \sigma)$ such that $\rho(s, \sigma) \rightarrow 0$ for s fixed and $\sigma \rightarrow 0$.

The following sharper statement holds: There exists a constant $r' = r'(s, \sigma) > 0$ such that, for $r \in [r', 1]$,

$$\{|z_1 - (1 - s)| \leq \sigma\} \cap (1 - \rho)\overline{\mathbb{B}^2} \subset \left(\bigcup_j {}^r T_j(\varepsilon) \cup \bigcup_k {}^r T_k^*(\varepsilon) \right)^\wedge. \tag{18_r}$$

Proof. Let s and σ be as in the statement of the proposition. The complex lines ℓ_j will be obtained as in our plan for the proof of Proposition 2 for some parameters t and \mathcal{N} : they will be complex tangent to the sphere $R(t)\partial\mathbb{B}^2$ (see (16)) through equidistributed points $p_j \in C_t$ with distance $|p_{j+1} - p_j|$ depending on \mathcal{N} .

Determine now the parameter t (so far, \mathcal{N} is arbitrary). Let t be maximal so that $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| \leq \frac{5}{3}\sigma\}$ and hence, for this parameter t , $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| = \frac{5}{3}\sigma\}$ (see (6)). This number t does not depend on \mathcal{N} or j , and t tends to zero for $\sigma \rightarrow 0$. Hence, we also have $s^*(s, \sigma) \rightarrow 0$ for $\sigma \rightarrow 0$. Moreover, if σ' is small then there is a uniform estimate from below of the angle of intersection of $\ell^{s'}$ and ℓ_j for $s' \in (s - 2\sigma, s + 2\sigma)$, with $\sigma \in (0, \sigma')$ and t related to σ as just described. Indeed, $\ell^{s'}$ is transversal to the complex tangent space of $\partial\mathbb{B}^2$ with a uniform estimate of the angle for the mentioned s' . Therefore, if σ' is small and $\sigma < \sigma'$ then t is small; hence $R(t)$ is close to 1 and so the ℓ_j (as complex tangents to $R(t)\partial\mathbb{B}^2$) are transversal to the $\ell^{s'}$ with uniform estimate of the angle.

The positive number α will be specified later. Let ζ_k, g_k , and $\ell_k^* = \{g_k = 0\}$ be as in the statement of the proposition. The ℓ_k^* are parallel to ℓ^s , and $\ell_k^* \cap \partial\mathbb{B}^2$ is contained in $\{|z_1 - (1 - s)| \leq \frac{4}{3}\sigma\}$ while $\ell_j \cap \partial\mathbb{B}^2$ is contained in $\{|z_1 - (1 - s)| = \frac{5}{3}\sigma\}$. The foregoing observations imply two facts as follows.

- (i) If σ' and hence t is small, if ε is small depending on s' and σ , and if moreover the parameters \mathcal{N} and ε satisfy the conditions of Lemma 2(a), then all tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint and contained in $\{|z_1 - (1 - s)| \leq 2\sigma\}$.
- (ii) The Main Lemma can be applied to all pairs (ℓ_j, ℓ_k^*) with uniform constants a and r' .

Furthermore, using the notation $\ell^\xi = \{z \in \mathbb{C}^2 : z_1 = 1 - \xi\}$ with $\xi \in \mathbb{C}$ and $|\xi - s| \leq \frac{4}{3}\sigma$, we obtain that for each positive number η the set $\ell^\xi \cap \{|f_j| \leq \eta\}$ is a closed disc in ℓ^ξ of radius at least $D \cdot \eta$ around the intersection point $\ell^\xi \cap \ell_j$, where the constant D depends only on s and σ' .

We now prove the assertion on the polynomial hull of the union of the tori for suitably chosen α . Note first that, for $|\xi - s| \leq \frac{4}{3}\sigma$, the intersection points $\ell^\xi \cap \ell_j$ ($j = 0, \dots, \mathcal{N} - 1$) are equidistributed on the circle $\{1 - \xi\} \times \{|z| = R'_\xi\} \subset \mathbb{B}^2$ for a number R'_ξ that is close to $\sqrt{1 - |1 - \xi|^2}$ if $\sigma < \sigma'$ is small. This is because the unitary transformation $(z_1, z_2) \rightarrow (z_1, e^{i\phi_j} z_2)$ maps the pair (ℓ^ξ, ℓ_0) to the pair (ℓ^ξ, ℓ_j) .

Corollary 1 implies that, for $r \in [r', 1]$ and all k and j , the polynomial hull of ${}^r T_k^*(\varepsilon) \cup {}^r T_j(\varepsilon)$ contains the set

$$\{|z_1 - (1 - \zeta_k)| \cdot |f_j(z)| \leq a\varepsilon\} \cap r\overline{\mathbb{B}^2}. \tag{19}$$

Let ε be as small as hitherto required (i.e., $\varepsilon < \varepsilon'(s, \sigma)$) and let $B > B'$, with B' the constant of Lemma 2(a). Choose \mathcal{N} so that the distance between nearest points $|p_{j+1} - p_j|$ is between $B\varepsilon$ and $(B + 1)\varepsilon$. If σ is small, then for $|\xi - s| \leq \frac{4}{3}\sigma$ the distance between the nearest of intersection points $\ell^\xi \cap \ell_j$ ($j = 0, \dots, \mathcal{N} - 1$) does not exceed $2(B + 1)\varepsilon$. Take a constant C such that $C \cdot D > 2(B + 1)$. The set in (19) contains the set

$$\{z \in r\overline{\mathbb{B}^2} : |z_1 - (1 - \zeta_k)| \leq a/C, |f_j(z)| \leq C\varepsilon\}.$$

Hence, for any fixed ξ with $|\xi - \zeta_k| \leq a/C$ and for r as before, the polynomial hull of ${}^r T_k^*(\varepsilon) \cup \bigcup_j {}^r T_j(\varepsilon)$ contains the circle $\{1 - \xi\} \times \{|z| = R'_\xi\}$. Take any $\alpha < a/C$. Then, for $r \in [r', 1]$,

$$\left(\bigcup_k {}^r T_k^*(\varepsilon) \cup \bigcup_j {}^r T_j(\varepsilon) \right)^\wedge \supset \bigcup_{|\xi - s| \leq \sigma} \{1 - \xi\} \times \{|z| \leq R'_\xi\}.$$

The right-hand side contains $\{|z_1 - (1 - s)| \leq \sigma\} \cap (1 - \rho)\overline{\mathbb{B}^2}$ for suitable $\rho = \rho(s, \sigma)$, with $\rho(s, \sigma) \rightarrow 0$ for $\sigma \rightarrow 0$. □

Now we are ready to state and prove the main proposition.

PROPOSITION 4. *Let $s \in (0, 1)$. There is a positive constant $\sigma' = \sigma'(s)$ such that, for any $\sigma \in (0, \sigma')$, there exist finitely many numbers $s_m \in (0, 1)$ (their number depends on s and σ , and each of them tends to zero for $\sigma \rightarrow 0$) and a positive number $\varepsilon' = \varepsilon'(s, \sigma, s_m)$ such that the following holds.*

For any $\varepsilon \in (0, \varepsilon')$ there exist finitely many complex affine functions f_n and related tori $T_n(2\varepsilon)$, where each torus is unitarily equivalent to $T^{s_m}(2\varepsilon)$ for some m , with $T_n(2\varepsilon)$ pairwise disjoint and contained in $\{|z_1 - (1 - s)| \leq 2\sigma\}$ and such that

$$(1 - \rho)\overline{\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \sigma\} \subset \left(\bigcup_n T_n(\varepsilon) \right)^\wedge \tag{20}$$

for a constant $\rho = \rho(s, \sigma)$ such that, for fixed s , we have $\rho(s, \sigma) \rightarrow 0$ if $\sigma \rightarrow 0$. Moreover, there exists an $r' = r'(s, \sigma, s_m)$ such that, for $r \in [r', 1]$,

$$(1 - \rho)\overline{\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \sigma\} \subset \left(\bigcup_n {}^r T_n(\varepsilon) \right)^\wedge. \tag{20_r}$$

Proof. Let σ', r' and α be as in Proposition 3, let $\sigma < \sigma'$, and let $\varepsilon'(s, \sigma)$ be the constant from the statement of that proposition. Recall that Proposition 3 gives two families of functions f_j and g_k . If σ is small, then the tori related to the f_j have small diameter but the tori related to the g_k have large diameter. Our aim is to apply Proposition 2 to each g_k .

Let $s(g_k)$ be the number for which $|g_k|$ is unitarily equivalent to $|z_1 - (1 - s(g_k))|$. For each k , apply Proposition 2 with $s = s(g_k)$, $\delta = \alpha/3$, and $q = r'$. For each k we obtain two numbers $s_{k,1}, s_{k,2} \in (0, 1)$ (we can make them as small as we wish) and a bound for ε below which we can find finitely many nonintersecting tori $T(2\varepsilon)$, unitarily equivalent to $T^{s_{k,1}}(2\varepsilon)$ or $T^{s_{k,2}}(2\varepsilon)$ and contained in $\{|z_1 - (1 - s(g_k))| \leq \alpha/3\}$, with the following property:

$$\overline{r'\mathbb{B}^2} \cap \{|z_1 - (1 - s(g_k))| \leq \varepsilon\} \subset \left(\bigcup^r T(\varepsilon)\right)^\wedge \tag{21_r}$$

for all $r \in [\tilde{r}_k, 1]$ with a suitable \tilde{r}_k .

Assume that ε is less than all the mentioned bounds and also that $\varepsilon < \varepsilon'(s, \sigma)$. Consider the second family of tori $T_j(2\varepsilon)$ from Proposition 3. The $T_j(2\varepsilon)$ are related to complex affine functions f_j and to complex lines $\ell_j = \{f_j = 0\}$.

Labeling our collection of all tori (i.e., the tori obtained for each k by Proposition 2 and the tori $T_j(2\varepsilon)$ from Proposition 3) yields the family $T_n(2\varepsilon)$. The $T_n(2\varepsilon)$ are pairwise disjoint. Indeed, by the choice of δ the tori obtained by Proposition 2 for different k do not intersect. Since $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| = \frac{5}{3}\sigma\}$ and $\{g_k = 0\} \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| \leq \frac{4}{3}\sigma\}$, the tori obtained by Proposition 2 do not meet the $T_j(2\varepsilon)$ from Proposition 3 if ε is small. Together with (21_r) applied for each k with $r \in [\tilde{r}_k, 1]$, the inclusion (18_{r'}) from Proposition 3 implies (20_r), with r larger than the maximum of all \tilde{r}_k and also larger than r' from Proposition 3. Proposition 4 is proved. □

Proposition 4 enables us to construct the sets E_N inductively.

Proof of the Theorem.

Step 1. Lemma 1 and Proposition 1 give us, for any sufficiently small $\varepsilon > 0$, a finite collection of tori. More precisely, we obtain a number r_1 (see Remark 2) and a finite collection of numbers $s_m^{(1)}$ such that, for each small $\varepsilon > 0$ and for each m , there exists a finite number of tori unitarily equivalent to $T^{s_m^{(1)}}(\varepsilon)$ with the following properties. Denote all the tori by $T^{(1)}$, skipping labeling indices but indicating that they are obtained at step 1. The $T^{(1)}(2\varepsilon)$ are pairwise disjoint, and

$$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup^{r_1} T^{(1)}(\varepsilon)\right)^\wedge. \tag{22}$$

The number ε will be chosen at the second step of the induction. (Note that the number of tori as well as the choice of the complex lines that are the symmetry axes of the tori also depend on ε .)

Step 2. For any $s_m^{(1)}$ of step 1 we apply Proposition 4. We obtain a bound $\sigma'(s_m^{(1)})$. Choose now the number ε of step 1 (and hence the tori of that step). It has to satisfy the following requirements. First, it must be so small that it fits for step 1. Next, for each m we require $\varepsilon < \sigma'(s_m^{(1)})$. Further, for chosen $s = s_m^{(1)}$ and $\sigma = \varepsilon$, Proposition 4 allows us to find new s -parameters that tend to zero for $\sigma = \varepsilon \rightarrow 0$.

We denote the collection (over all m) of all new s -parameters at the second step by $s^{(2)}$ (again we skip labeling indices) and require $\sigma = \varepsilon$ to be so small that each $s^{(2)}$ is less than $1/2^2$. Finally, Proposition 4 asserts for each m and ε the existence of a number $\rho(s_m^{(1)}, \varepsilon)$ (see (20)). We wish to estimate $1 - \rho(s_m^{(1)}, \varepsilon) > r_1$ for each m . Choose an $\varepsilon > 0$ satisfying all these conditions and denote it by ε_1 .

Denote $E_1 = \bigcup T^{(1)}(2\varepsilon_1)$, so E_1 is the union of pairwise disjoint closed tori of diameter determined by the $s^{(1)}$ and ε_1 . Then (22) implies that

$$\overline{\beta\mathbb{B}^2} \subset \hat{E}_1.$$

Indeed, $T^{(1)}(2\varepsilon_1) \supset T^{(1)}(\varepsilon_1)$ and $(T^{(1)}(\varepsilon_1))^\wedge \supset r_1 T^{(1)}(\varepsilon_1)$ for any of the tori $T^{(1)}(\varepsilon_1)$.

Having constructed the set E_1 , we now describe the construction of E_2 modulo the choice of the parameter ε_2 .

Denote the functions corresponding to the $T^{(1)}(\varepsilon_1)$ by $f^{(1)}$ (omitting indices as before). By applying Proposition 4 to each of the $T^{(1)}(\varepsilon_1)$ we obtain the aforementioned collection of numbers $s^{(2)} < 1/2^2$; we also obtain a number r_2 (see (20,_r)) such that the following holds: For each sufficiently small $\varepsilon > 0$ there exist finitely many pairwise disjoint tori $T^{(2)}(2\varepsilon)$ contained in E_1 , each of them unitarily equivalent to a torus $T^{s_m^{(2)}}(2\varepsilon)$ for some of the numbers $s_m^{(2)}$ of the collection $s^{(2)}$, such that (since $1 - \rho(s_m^{(1)}, \varepsilon) > r_1$ for each m)

$$r_1 \overline{\beta\mathbb{B}^2} \cap \bigcup \{|f^{(1)}| \leq \varepsilon_1\} \subset \left(\bigcup r_2 T^{(2)}(\varepsilon) \right)^\wedge;$$

hence, by (22),

$$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup r_2 T^{(2)}(\varepsilon) \right)^\wedge.$$

This describes the construction of the set E_2 modulo the choice of the parameter ε_2 .

Step N . This is the general step of the induction. Let $N > 2$. Suppose that the sets $E_1 \supset \dots \supset E_{N-2}$ are constructed with $\hat{E}_{N-2} \supset \overline{\beta\mathbb{B}^2}$ and that the construction of the set E_{N-1} is described modulo the choice of the parameter ε_{N-1} . We want to choose the parameter ε_{N-1} and describe the construction of E_N modulo the choice of the ε_N .

More precisely, we suppose the following has been done at step $N - 1$. There was found a finite collection of numbers $s^{(N-1)}$, all less than $1/2^{N-1}$, and a number $r_{N-1} \in (0, 1)$ such that for each sufficiently small $\varepsilon > 0$ there exist finitely many pairwise disjoint tori $T^{(N-1)}(2\varepsilon)$ contained in E_{N-2} , each of them unitarily equivalent to a torus $T^{s_m^{(N-1)}}(2\varepsilon)$ for some number $s_m^{(N-1)}$ of the collection $s^{(N-1)}$, with the following relation for the polynomial hull:

$$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup r_{N-1} T^{(N-1)}(\varepsilon) \right)^\wedge. \quad (23)$$

We want to choose a suitable number ε_{N-1} for ε . Here are the requirements for the number ε . First, it must be small enough for step $N - 1$ to go through. Further,

apply Proposition 4 using any of the numbers $s^{(N-1)}$ in place of s . We thus obtain an upper bound for σ . Let σ' be the least of all such upper bounds over numbers $s^{(N-1)}$ and require that $\varepsilon < \sigma'$. Now, for s equal to any of the $s^{(N-1)}$ and for $\sigma = \varepsilon$, Proposition 4 allows us to choose new s -parameters that tend to zero together with $\sigma = \varepsilon$. Require ε to be so small that all the new s -parameters denoted by $s^{(N)}$ are less than 2^{-N} . Finally, the proposition asserts the existence of a number $\rho(s_m^{(N-1)}, \varepsilon)$ for each number $s_m^{(N-1)}$ of the $s^{(N-1)}$ (see (20)). We wish to estimate $1 - \rho(s_m^{(N-1)}, \varepsilon) > r_{N-1}$ for each m . Choose a number $\varepsilon > 0$ satisfying all these conditions and denote it by ε_{N-1} . Put

$$E_{N-1} = \bigcup T^{(N-1)}(2\varepsilon_{N-1}).$$

Thus the set E_{N-1} is constructed (recall that also the number of the tori and the choice of their symmetry axes depend on ε_{N-1}) and, by (23),

$$\overline{\beta\mathbb{B}^2} \subset \hat{E}_{N-1}.$$

Use now Proposition 4 to define E_N modulo the choice of ε_N . Denote the functions corresponding to the tori $T^{(N-1)}(\varepsilon_{N-1})$ of the previous generation by $f^{(N-1)}$. We already mentioned the numbers $s^{(N)}$ obtained by Proposition 4. This proposition (applied with s being any of the $s^{(N-1)}$ and with $\sigma = \varepsilon_{N-1}$) also gives, for each sufficiently small $\varepsilon > 0$, finitely many pairwise disjoint tori $T^{(N)}(2\varepsilon)$ contained in E_{N-1} (each of them unitarily equivalent to $T^{s_m^{(N)}}(2\varepsilon)$, with $s_m^{(N)}$ being some of the numbers $s^{(N)}$) as well as a number $r_N \in (0, 1)$ with the following relation for the polynomial hull:

$$\overline{r_{N-1}\mathbb{B}^2} \cap \bigcup \{|f^{(N-1)}| \leq \varepsilon_{N-1}\} \subset \left(\bigcup r_N T^{(N)}(\varepsilon) \right)^\wedge. \tag{24}$$

Here we have used that $1 - \rho(s_m^{(N-1)}, \varepsilon) > r_{N-1}$ for each m . Hence, using (23) with the ε replaced by ε_{N-1} yields

$$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup r_N T^{(N)}(\varepsilon) \right)^\wedge$$

for sufficiently small positive ε of step N and for the tori $T^{(N)}(\varepsilon)$ whose existence is obtained at step N .

The induction is now complete and hence the theorem is proved. □

After this paper was written, L. Stout informed me that he had constructed a Cantor set in \mathbb{C}^n with nontrivial polynomial hull in case $n \geq 4$. He uses purely topological results and the existence of suitable plurisubharmonic Morse functions, a method that does not work in dimensions 2 and 3.

REMARK 5. Not every wild Cantor set in the sphere has nontrivial polynomial hull.

Indeed, this can be seen even by varying the previous construction. Let the general step N of the present construction look as follows. Suppose we obtained sets

$E_1^* \supset \cdots \supset E_{N-1}^*$, so that the E_k^* are disjoint unions of closed solid tori (the same kind as before) of width $2\varepsilon_k^*$. Suppose the polynomial hull of E_k^* is contained in the δ_k -neighborhood of $(\bigcup \ell^{(k)}) \cap \overline{\mathbb{B}^2}$ for some sufficiently small positive number δ_k . Here $\ell^{(k)}$ denotes the collection of complex lines that are the symmetry axes of the tori in E_k^* . At step N , choose disjoint tori $T^{*(N)} \subset E_{N-1}^*$ by first doing the construction from the proof of Proposition 4 and then fixing the symmetry axes of the tori and taking ε_N^* so small that the polynomial hull of $E_N^* = \bigcup T^{*(N)}(2\varepsilon_N^*)$ is contained in the δ_N -neighborhood of $(\bigcup \ell^{(N)}) \cap \overline{\mathbb{B}^2}$. This is possible because $(\bigcup \ell^{(k)}) \cap \overline{\mathbb{B}^2}$ is the polynomial hull of $(\bigcup \ell^{(k)}) \cap \partial\mathbb{B}^2$. If $\delta_N \rightarrow 0$ for $N \rightarrow \infty$ fast enough, then (since $(\bigcup \ell^{(N)}) \cap \overline{\mathbb{B}^2}$ is close to the sphere for large N) the accumulation points of the mentioned δ_N -neighborhoods are contained in the sphere. Hence the set $E^* = \bigcap E_N^*$ is polynomially convex. Using Bing's theorem, one can prove that this set is a wild Cantor set.

REMARK 6. The set E constructed in the theorem is rationally convex.

In fact, each pair of points of E can be separated by a 2-torus T in $\partial\mathbb{B}^2 \setminus E$. Indeed, for the 2-torus T we can take the boundary of a certain solid torus $T_m^{(N)}(2\varepsilon)$, where $T_m^{(N)}(2\varepsilon_N)$ is one of the tori contributing to E_N for some N and where ε is slightly bigger than ε_N . The 2-torus T is contained in the cylinder $\{z \in \mathbb{C}^2 : |f_m^{(N)}(z)| = 2\varepsilon\}$, which is the union of complex lines and does not meet E . Hence, the cylinder does not meet the rational hull of E . Now the same argument works as for polynomial convexity of tame Cantor sets in the sphere.

We do not know whether Cantor sets in the sphere are always rationally convex.

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Department of Mathematics
Uppsala University
Box 480
S-75106 Uppsala
Sweden
joericke@math.uu.se