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A Counterexample to the Fourteenth Problem of Hilbert in Dimension Three

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1. Introduction

Let K be a field, $K[X] = K[X_1, \dots, X_m]$ the polynomial ring in m variables over K, and $K(X)$ its field of fractions. Then, the fourteenth problem of Hilbert asks whether the K-subalgebra $L \cap K[X]$ of $K[X]$ is finitely generated whenever L is a subfield of $K(X)$ containing K. Zariski [23] showed that $L \cap K[X]$ is finitely generated if the transcendence degree of L over K is at most two. Consequently, the problem has an affirmative answer if $m \le 2$. On the other hand, a counterexample to the problem was first found by Nagata [17] in 1958 for $m \ge 32$ (see [8] for the progress on this problem).

Recently the author [12] gave a counterexample for $m = 4$, whereby the problem remained open only for $m = 3$. In fact, if $L \cap K[X]$ is not finitely generated, then $L \cap K[X][X_{m+1}, \dots, X_{m+r}]$ is also not finitely generated for each $r \ge 0$. In this paper we give the first counterexample to the problem for $m = 3$. Thus, the fourteenth problem of Hilbert is settled for all m at last.

Let γ and $\delta_{i,j}$ be positive integers for $i, j = 1, 2$, and let

$$\begin{aligned} \Pi_1 &= X_1^{\delta_{2,1}} X_2^{-\delta_{2,2}} - X_1^{-\delta_{1,1}} X_2^{\delta_{1,2}}, \\ \Pi_2 &= X_3^{\gamma} - X_1^{-\delta_{1,1}} X_2^{\delta_{1,2}}, \\ \Pi_3 &= 2 X_1^{\delta_{2,1} - \delta_{1,1}} X_2^{\delta_{1,2} - \delta_{2,2}} - X_1^{-2\delta_{1,1}} X_2^{2\delta_{1,2}}. \end{aligned} \tag{1.1}$$

Then we have the following result.

Theorem 1.1. *Assume that the characteristic of K is zero. If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}} + \frac{\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} < \frac{1}{2}, \tag{1.2}$$

then $K(\Pi_1, \Pi_2, \Pi_3) \cap K[X_1, X_2, X_3]$ is not finitely generated over K.

We remark that $m = 3$ is an exceptional dimension for the fourteenth problem of Hilbert with many partial positive answers as follows. As already mentioned, the answer to the fourteenth problem of Hilbert is affirmative when $m \le 2$ by

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Zariski's result. Zariski's result also implies that no counterexample to Hilbert's original fourteenth problem can be found even if $m = 3$ (cf. [8]). Here, *Hilbert's original fourteenth problem* is a special case of his fourteenth problem as follows. Let G be a group of linear automorphisms of $K[X]$. Then, is the invariant subring $K[X]^G$ of $K[X]$ for G finitely generated over K? Nagata [17] gave a counterexample to this problem (see also [1], [16], [21], and [22] for counterexamples to Hilbert's original fourteenth problem).

A K-linear map $D\colon A \to A$ of a commutative K-algebra A is called a *derivation* if $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. For a K-subalgebra B of A, we define

$$B^D = \{b \in B \mid D(b) = 0\},$$

which is a K-subalgebra of B. Then, the problem of finite generation of the kernel $K[X]^D$ of a derivation D on $K[X]$ is a part of the fourteenth problem of Hilbert and is well studied. In the case where the characteristic of K is zero, Zariski's result implies that $K[X]^D$ is always finitely generated if $m \leq 3$ [18]. Various sufficient conditions for finite generation of the kernels of derivations are found in [3], [9], [11], [13], and [15]. Derksen [4] showed that Nagata's counterexample is obtained as the kernel of a derivation (see also [19]). Moreover, several counterexamples to the fourteenth problem of Hilbert were constructed or described as the kernels of derivations (cf. [2; 5; 7; 10; 14; 19; 20]). However, Zariski's result implies that we can never obtain a counterexample in dimension three as the kernel of a derivation on $K[X_1, X_2, X_3]$.

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2. The Structure of $K(\Pi_1, \Pi_2, \Pi_3) \cap K[X_1, X_2, X_3]$

Let A be a finitely generated domain over a field K of characteristic zero, and let $K(A)$ be its field of fractions. Assume that D is a derivation on A. Then D extends naturally to a derivation on $K(A)$. We say that D is *locally nilpotent* if, for each $a \in A$, there exists $r \geq 0$ such that $D^r(a) = 0$.

First, we review some basic properties of the kernel of a locally nilpotent derivation. Lemmas 2.1 and 2.2 are well known (see e.g. [6, Chap. 1.3]).

LEMMA 2.1. *Let D be a locally nilpotent derivation on A.*

(i) *If $D(ab) = 0$ for $a, b \in A \setminus \{0\}$, then $D(a) = 0$ and $D(b) = 0$.*
(ii) *$K(A)^D$ is equal to the field of fractions of A^D.*
(iii) *If $D(a)$ is divisible by a, then $D(a) = 0$ for $a \in A$.*

We say that $s \in A$ is a *slice* of D if $D(s) = 1$. Assume that D has a slice $s \in A$. Then we may consider the *Dixmier map* $\sigma\colon A \to A$ defined by

$$\sigma(a) = \sum_{k=0}^{\infty} (-s)^k \frac{D^k(a)}{k!} \tag{2.1}$$

for each $a \in A$. Since D is locally nilpotent, $D^k(a) = 0$ for all $k \geq r$ for some r. Hence, the sum in (2.1) is well-defined.

LEMMA 2.2. *Let D be a locally nilpotent derivation on A with a slice $s \in A$. If A is generated by $S \subset A$ over K, then A^D is generated by $\{\sigma(a) \mid a \in S\}$ over K.*

Now let $K[Y] = K[Y_1, Y_2, Y_3, Y_4]$ be the polynomial ring in four variables over K, and let $K(Y)$ be its field of fractions. Consider the derivation

$$D = \frac{\partial}{\partial Y_1} + \frac{\partial}{\partial Y_2} + \frac{\partial}{\partial Y_3} + Y_1 \frac{\partial}{\partial Y_4} \tag{2.2}$$

on $K[Y]$, that is, the derivation defined by $D(Y_1) = D(Y_2) = D(Y_3) = 1$ and $D(Y_4) = Y_1$. We note that D is locally nilpotent. Let σ be the Dixmier map defined for the slice $s = Y_1$. Then, $K[Y]^D$ is generated by

$$\sigma(Y_2) = Y_2 - Y_1, \quad \sigma(Y_3) = Y_3 - Y_1, \quad \sigma(Y_4) = Y_4 - Y_1^2/2 \tag{2.3}$$

over K by Lemma 2.2, since $\sigma(Y_1) = 0$.

In what follows we assume that $m = 3$. For $f = \sum_{a \in \mathbf{Z}^3} \lambda_a X^a \in K[X]$, define the *support* $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) = \{a \in \mathbf{Z}^3 \mid \lambda_a \neq 0\},$$

where $\lambda_a \in K$ and X^a denotes $X_1^{a_1} X_2^{a_2} X_3^{a_3}$ for $a = (a_1, a_2, a_3) \in \mathbf{Z}^3$. We define the support of each element of $K[Y]$ similarly. For the definition of the support of f, one should allow f to be a Laurent polynomial, not just a polynomial.

We set

$$\delta_1 = (-\delta_{1,1}, \delta_{1,2}, 0), \;\; \delta_2 = (\delta_{2,1}, -\delta_{2,2}, 0), \;\; \delta_3 = (0, 0, 1), \;\; \delta_4 = \delta_1 + \delta_2. \tag{2.4}$$

It would be good to recall at this point that $\Pi_1 = X^{\delta_2} - X^{\delta_1}$, $\Pi_2 = X^{\delta_3} - X^{\delta_1}$, and $\Pi_3 = 2X^{\delta_4} - X^{2\delta_1}$. Let $K[X^\delta] = K[X^{\delta_1}, X^{\delta_2}, X^{\delta_3}, X^{\delta_4}]$ and $K[X^{\pm\delta}] = K[X^{\pm\delta_1}, X^{\pm\delta_2}, X^{\pm\delta_3}, X^{\pm\delta_4}]$, and let $K(X^\delta)$ be the field of fractions. Define the homomorphism $\Phi_0\colon K[Y] \to K(X^\delta)$ of K-algebras by $Y_i \mapsto X^{\delta_i}$ for each i. Then the kernel of Φ_0 is $\pi K[Y]$, where $\pi = Y_1 Y_2 - Y_4$. Let us denote by $K[Y]_{(\pi)}$ the localization of $K[Y]$ by the prime ideal $\pi K[Y]$. Then Φ_0 can be extended to the homomorphism $\Phi\colon K[Y]_{(\pi)} \to K(X^\delta)$. We remark that $K[Y]_{(\pi)}$ contains $K(Y)^D$. Actually, $K(Y)^D$ is equal to the field of fractions of $K[Y]^D$ by Lemma 2.1(ii), and π is not a factor of any element of $K[Y]^D \setminus \{0\}$ by Lemma 2.1(iii), since $D(\pi) = Y_2 \neq 0$.

Observe that $\Phi(\sigma(Y_2)) = \Pi_1$, $\Phi(\sigma(Y_3)) = \Pi_2$, and $\Phi(2\sigma(Y_4)) = \Pi_3$. We now set $K[\Pi] = K[\Pi_1, \Pi_2, \Pi_3]$ and $K(\Pi) = K(\Pi_1, \Pi_2, \Pi_3)$. Then it follows that $\Phi(K[Y]^D) = K[\Pi]$ and $\Phi(K(Y)^D) = K(\Pi)$ by Lemmas 2.2 and 2.1(ii).

PROPOSITION 2.3. *With notation as before, we have*

$$K(\Pi) \cap K[X] = K[\Pi] \cap K[X].$$

To prove this proposition, we need some lemmas.

LEMMA 2.4. $(K[Y] + \pi K[Y]_{(\pi)})^D = K[Y]^D$.

Proof. It suffices to show that $D(F+G) \neq 0$ for any $F \in K[Y]$ and any $G \in \pi K[Y]_{(\pi)} \setminus K[Y]$. Suppose to the contrary that $D(F+G) = 0$ for such F and G. Write $G = \pi G_1/G_2$, where $G_1, G_2 \in K[Y]$ with $\gcd(\pi G_1, G_2) = 1$. Then G_2^2 divides $D(\pi G_1)G_2 - \pi G_1 D(G_2)$, since $D(G) = -D(F)$ is in $K[Y]$. This implies that G_2 divides $D(G_2)$. Hence, $D(G_2) = 0$ by Lemma 2.1(iii). Therefore, $(F+G)G_2$ is in $K[Y]^D$, since $D(F+G) = 0$ and $(F+G)G_2 \in K[Y]$. On the other hand, G_2 divides $\pi G_1 - (F+G)G_2 = -G_2 F$. This contradicts Lemma 2.5 (to follow), since $G_2 \in K[Y]^D \setminus K$, $\gcd(G_1, G_2) = 1$, $\pi = Y_1Y_2 - Y_4$ is prime, and $D(\pi) = Y_2$ is not divisible by any element of $K[Y]^D \setminus K$. Consequently, $D(F+G) \neq 0$. □

The proofs of the following two lemmas have been simplified thanks to the referee's suggestion.

LEMMA 2.5. *Let D be a locally nilpotent derivation on $K[Y]$. Let $f, g \in K[Y]$ and $w \in K[Y]^D \setminus K$ be such that* $\gcd(g, w) = 1$, *f is a prime element of $K[Y]$, and $D(f)$ is not divisible by any element of $K[Y]^D \setminus K$. Then w does not divide $fg + v$ for any $v \in K[Y]^D$.*

Proof. Without loss of generality, we may assume that w is irreducible. Let $A = K[Y]/(w)$. The derivation D also induces a locally nilpotent derivation on A, which we also will denote by D. Suppose that w divides $fg + v$ for some $v \in K[Y]^D$. Then we have $\bar{f}\bar{g} \in A^D$, where $\bar{h}$ denotes the image of h in A for each $h \in K[Y]$. Since $\bar{g} \neq 0$ by assumption, we have $\bar{f} \in A^D$ by Lemma 2.1(i). It follows that $\overline{D(f)} = D(\bar{f}) = 0$, so w divides $D(f)$. This contradicts the choice of f. □

LEMMA 2.6. $K(\Pi) \cap K[X] \subset K[X^\delta]$.

Proof. We have $\sum_{i=1}^4 \mathbf{Z}\delta_i = \sum_{i=1}^3 \mathbf{Z}\delta_i$. If $\sum_{i=1}^3 \alpha_i\delta_i$ is in $(\mathbf{Z}_{\geq 0})^3$ for $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{Z}$, then $\alpha_1, \alpha_2, \alpha_3$ are nonnegative. Hence $\left(\sum_{i=1}^4 \mathbf{Z}\delta_i\right) \cap (\mathbf{Z}_{\geq 0})^3$ is contained in $\sum_{i=1}^4 \mathbf{Z}_{\geq 0}\delta_i$, which implies $K[X^{\pm\delta}] \cap K[X] = K[X^\delta] \cap K[X]$. So, we show that $K(\Pi) \cap K[X] \subset K[X^{\pm\delta}]$. Clearly, $K(\Pi) \cap K[X] \subset K(X^\delta) \cap K[X^{\pm 1}]$, where $K[X^{\pm 1}] = K[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. Hence, it suffices to show that $K(X^\delta) \cap K[X^{\pm 1}] = K[X^{\pm\delta}]$.

Without loss of generality, we may assume that K is algebraically closed. By (1.2), the rank of $\sum_{i=1}^4 \mathbf{Z}\delta_i$ is three. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a $\mathbf{Z}$-basis of $\mathbf{Z}^3$ such that $\sum_{i=1}^4 \mathbf{Z}\delta_i = \sum_{i=1}^3 t_i\mathbf{v}_i$ for some integers $t_1, t_2, t_3 > 0$. Then, $K[X^{\pm 1}] = K[X^{\pm\mathbf{v}_1}, X^{\pm\mathbf{v}_2}, X^{\pm\mathbf{v}_3}]$. We define an action of the group $G = \prod_{i=1}^3(\mathbf{Z}/t_i\mathbf{Z})$ on $K[X^{\pm 1}]$ by $(u_1, u_2, u_3)X^{\mathbf{v}_i} = \zeta_i^{u_i}X^{\mathbf{v}_i}$ for each i and $(u_1, u_2, u_3) \in G$, where ζ_i is a primitive (t_i)th root of unity. Then

$$K(X^{\delta}) \cap K[X^{\pm 1}] = K(X)^G \cap K[X^{\pm 1}] = K[X^{\pm 1}]^G = K[X^{\pm \delta}].$$

Hence the lemma is proved. □

Proof of Proposition 2.3. It is clear that $K(\Pi) \cap K[X] \supset K[\Pi] \cap K[X]$. We must prove the reverse inclusion. By Lemma 2.4, we have

$$\begin{aligned} &\Phi^{-1}(K(\Pi) \cap K[X^{\delta}]) \\ &\quad = \Phi^{-1}(K(\Pi)) \cap \Phi^{-1}(K[X^{\delta}]) \\ &\quad = (K(Y)^D + \pi K[Y]_{(\pi)}) \cap (K[Y] + \pi K[Y]_{(\pi)}) \\ &\quad = (K[Y] + \pi K[Y]_{(\pi)})^D + \pi K[Y]_{(\pi)} \\ &\quad = K[Y]^D + \pi K[Y]_{(\pi)}. \end{aligned} \tag{2.5}$$

Since $K(\Pi) \cap K[X^{\delta}]$ is contained in the image of Φ, it follows that $\Phi(\Phi^{-1}(K(\Pi) \cap K[X^{\delta}])) = K(\Pi) \cap K[X^{\delta}]$. On the other hand, $\Phi(K[Y]^D + \pi K[Y]_{(\pi)}) = K[\Pi]$. Hence $K(\Pi) \cap K[X^{\delta}] = K[\Pi]$ by (2.5). Since $K(\Pi) \cap K[X] \subset K[X^{\delta}]$ by Lemma 2.6, we have $K(\Pi) \cap K[X] \subset K[\Pi]$. Thus $K(\Pi) \cap K[X] \subset K[\Pi] \cap K[X]$ and so Proposition 2.3 is proved. □

Now set $\omega_1 = (-\delta_{1,1}, \delta_{2,1}, 0, \delta_{2,1} - \delta_{1,1})$ and $\omega_2 = (\delta_{1,2}, -\delta_{2,2}, 0, \delta_{1,2} - \delta_{2,2})$. Then

$$\Phi(Y^b) = X_1^{\omega_1 \cdot b} X_2^{\omega_2 \cdot b} X_3^{\gamma b_3} \tag{2.6}$$

for $b = (b_1, b_2, b_3, b_4) \in \mathbf{Z}^4$. Here, $\omega_i \cdot b$ is the inner product of ω_i and b for $i = 1, 2$.

LEMMA 2.7. *If (a_1, a_2, a_3) is in* $\operatorname{supp}(f)$ *for some $f \in K(\Pi) \cap K[X]$ with $a_3 > 0$, then $a_1 + a_2 > 0$.*

Proof. Suppose to the contrary that $(0, 0, a_3)$ is in $\operatorname{supp}(f)$ with $a_3 > 0$ for some $f \in K(\Pi) \cap K[X]$. Since $K(\Pi) \cap K[X] = K[\Pi] \cap K[X]$ by Proposition 2.3, there exists a polynomial $g \in K[Y]^D$ such that $f = \Phi(g)$. Then there exists an element $b = (b_1, b_2, b_3, b_4)$ of $\operatorname{supp}(g)$ such that $\Phi(Y^b) = X_3^{a_3}$. By (2.6) we have

$$\begin{aligned} 0 &= \omega_1 \cdot b = -(b_1 + b_2 + 2b_4)\delta_{1,1} + (b_2 + b_4)(\delta_{1,1} + \delta_{2,1}), \\ 0 &= \omega_2 \cdot b = -(b_1 + b_2 + 2b_4)\delta_{2,2} + (b_1 + b_4)(\delta_{2,2} + \delta_{1,2}), \end{aligned}$$

and $b_3 = a_3/\gamma$. This implies $b_1 = b_2 = b_4 = 0$ by (1.2), because $b_i \geq 0$ for each i. Moreover, b_3 is positive. Since $D(Y_3^{b_3}) = b_3 Y_3^{b_3 - 1}$ and $D(g) = 0$, there exists an element c of $\operatorname{supp}(g) \setminus \{b\}$ such that $Y_3^{b_3 - 1}$ appears in $D(Y^c)$. Then, c must be $(1, 0, b_3 - 1, 0)$ or $(0, 1, b_3 - 1, 0)$. Since $\Phi(Y_1 Y_3^{b_3 - 1}) = X^{\delta_1} X_3^{(b_3 - 1)\gamma}$ and $\Phi(Y_2 Y_3^{b_3 - 1}) = X^{\delta_2} X_3^{(b_3 - 1)\gamma}$ are not in $K[X]$, the monomial $\Phi(Y^c)$ does not appear in $\Phi(g)$. Hence there exists an element c' of $\operatorname{supp}(g) \setminus \{c\}$ such that $\Phi(Y^{c'}) = \Phi(Y^c)$. It follows that $Y^c - Y^{c'}$ is in $\pi K[Y]$. However, c is not contained in the support of any element of $\pi K[Y]$, since $\pi = Y_1 Y_2 - Y_4$. This is a contradiction. Therefore, $(0, 0, a_3)$ is not contained in $\operatorname{supp}(f)$. □

Let $\mathcal{C}$ be the set of $a \in (\mathbf{R}_{\geq 0})^4$ such that $\omega_i \cdot a \geq 0$ for $i = 1, 2$. Then $\mathcal{C}$ is a convex polyhedral cone in $\mathbf{R}^4$. We remark that, if $\operatorname{supp}(g) \subset \mathcal{C}$, then $\Phi(g) \in K[X]$ for $g \in K[Y]$ by (2.6).

The following is the key lemma, which will be proved in Section 3.

LEMMA 2.8. *There exist integers $p_1, p_2 > 0$ such that, for each integer $l > 0$, we may find $F_l \in K[Y]^D$ with $\operatorname{supp}(F_l) \subset \mathcal{C}$ of the form*

$$F_l = (Y_1^2 - 2Y_4)^{p_1}(Y_2^2 - 2Y_1Y_2 + 2Y_4)^{p_2}Y_3^l + (\textit{terms of lower degree in } Y_3). \tag{2.7}$$

Now, we prove Theorem 1.1 as a consequence of Lemma 2.8. Suppose to the contrary that $K(\Pi) \cap K[X]$ is generated by a finite number of elements $g_1, \dots, g_r$. By Lemma 2.7, there exists $\mu > 0$ such that $N(a) < \mu$ for every $a \in \bigcup_{i=1}^r \operatorname{supp}(g_i) \setminus \{0\}$, where $N(a) = a_3/(a_1 + a_2)$ for $a = (a_1, a_2, a_3)$. Since $g_1, \dots, g_r$ generate $K(\Pi) \cap K[X]$, the set S of the union of the supports of elements of $K(\Pi) \cap K[X]$ is contained in the subsemigroup of $\mathbf{Z}^4$ generated by $\bigcup_{i=1}^r \operatorname{supp}(g_i)$. On the other hand, $N(a+b) < \mu$ for any a, b with $N(a), N(b) < \mu$. Hence we have $N(a) < \mu$ for all $a \in S \setminus \{0\}$. Take $p_1, p_2 > 0$ as in Lemma 2.8. Then, for each $l > 0$, there exists a polynomial $F_l \in K[Y]^D$ as in (2.7) with $\operatorname{supp}(F_l) \subset \mathcal{C}$ by Lemma 2.8. Since

$$\Phi(F_l) = (X^{2\delta_1} - 2X^{\delta_1+\delta_2})^{p_1}X^{2p_2\delta_2}X_3^{l\gamma} + (\text{terms of lower degree in } X_3)$$

is contained in $K(\Pi) \cap K[X]$, it follows that the vector $a_l = 2p_1\delta_1 + 2p_2\delta_2 + (0, 0, l\gamma)$ is contained in S for any $l > 0$. This is a contradiction, since $N(a_l) > \mu$ for sufficiently large l. Thus, $K(\Pi) \cap K[X]$ is not finitely generated. This proves Theorem 1.1 under the assumption that Lemma 2.8 is true.

3. Proof of Lemma 2.8

For each integer $q > 0$, we set $\varepsilon(q) = 0$ if q is even and $\varepsilon(q) = 1$ otherwise, and we set $\eta(q) = \lfloor q/2 \rfloor = (q - \varepsilon(q))/2$. Then, define

$$\begin{aligned} f_{q,p} &= (-\sigma(Y_2))^{\varepsilon(q)}(-2\sigma(Y_4))^{\eta(q)-p}(\sigma(Y_2)^2 + 2\sigma(Y_4))^p \\ &= (Y_1 - Y_2)^{\varepsilon(q)}(Y_1^2 - 2Y_4)^{\eta(q)-p}(Y_2^2 - 2Y_1Y_2 + 2Y_4)^p \end{aligned} \tag{3.1}$$

for $q > 0$ and $0 \leq p \leq \eta(q)$.

Let q_0 be an integer such that

$$q_0\left(\frac{1}{2} - \frac{\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}} - \frac{\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}}\right) \geq \frac{3}{2}. \tag{3.2}$$

Then, for an integer $q \geq q_0$, we have

$$\eta(q) > \frac{q}{2} - \left(\frac{q\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} + \frac{3}{2}\right) \geq \frac{q\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}}. \tag{3.3}$$

LEMMA 3.1. *Let $q \geq q_0$ be an integer, and let p be the minimal integer such that*

$$p > \frac{q\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}}. \tag{3.4}$$

Then $\mathrm{supp}(f_{q,p}) \subset \mathcal{C}$, *and* $\omega_2 \cdot a \geq 0$ *for each* $a \in \mathrm{supp}(f_{q,p'})$ *for* $0 \leq p' \leq p$.

Proof. By (3.3) we have $p \leq \eta(q)$. Hence, $f_{q,p'}$ is defined for $0 \leq p' \leq p$. Note that each monomial appearing in $f_{q,p'}$ is written as

$$(Y_1^{\varepsilon(q)-\alpha} Y_2^{\alpha})(Y_1^{2\beta} Y_4^{\eta(q)-p'-\beta})(Y_2^{2\gamma_1}(Y_1Y_2)^{\gamma_2} Y_4^{\gamma_3}) \\ = Y_1^{\varepsilon(q)-\alpha+2\beta+\gamma_2} Y_2^{\alpha+2\gamma_1+\gamma_2} Y_4^{\eta(q)-p'-\beta+\gamma_3}$$

for some $0 \leq \alpha \leq \varepsilon(q)$, $0 \leq \beta \leq \eta(q) - p'$, and $\gamma_1, \gamma_2, \gamma_3 \geq 0$ with $\gamma_1 + \gamma_2 + \gamma_3 = p'$. We set

$$b_{q,p'} = (\varepsilon(q) - \alpha + 2\beta + \gamma_2)\mathbf{e}_1 + (\alpha + 2\gamma_1 + \gamma_2)\mathbf{e}_2 + (\eta(q) - p' - \beta + \gamma_3)\mathbf{e}_4$$

for q, p' and $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3$ as before. Here, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are the coordinate unit vectors of $\mathbf{R}^4$. Then we have

$$\begin{aligned} \omega_1 \cdot b_{q,p} &= -(\varepsilon(q) - \alpha + 2\beta + \gamma_2)\delta_{1,1} + (\alpha + 2\gamma_1 + \gamma_2)\delta_{2,1} \\ &\quad + (\eta(q) - p - \beta + \gamma_3)(\delta_{2,1} - \delta_{1,1}) \\ &= -(2\eta(q) + \varepsilon(q))\delta_{1,1} + (\eta(q) + \alpha - \beta + \gamma_1)(\delta_{1,1} + \delta_{2,1}) \\ &\geq -q\delta_{1,1} + p(\delta_{1,1} + \delta_{2,1}) = (\delta_{1,1} + \delta_{2,1})\left(p - \frac{q\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}}\right) > 0, \end{aligned}$$

where the first inequality is obtained by substituting $\alpha = 0$, $\beta = \eta(q) - p$, and $\gamma_1 = 0$; the second inequality follows from (3.4). A similar formula holds for $\omega_2 \cdot b_{q,p'}$. For $0 \leq p' \leq p$, we have

$$\begin{aligned} \omega_2 \cdot b_{q,p'} &= (\varepsilon(q) - \alpha + 2\beta + \gamma_2)\delta_{1,2} - (\alpha + 2\gamma_1 + \gamma_2)\delta_{2,2} \\ &\quad + (\eta(q) - p' - \beta + \gamma_3)(\delta_{1,2} - \delta_{2,2}) \\ &= (\eta(q) + \varepsilon(q) - \alpha + \beta - \gamma_1)(\delta_{2,2} + \delta_{1,2}) - (2\eta(q) + \varepsilon(q))\delta_{2,2} \\ &\geq (\eta(q) - p')(\delta_{2,2} + \delta_{1,2}) - q\delta_{2,2} \\ &= (\delta_{2,2} + \delta_{1,2})\left(\frac{q}{2} - p' - \frac{q\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} - \frac{\varepsilon(q)}{2}\right) \\ &> (\delta_{2,2} + \delta_{1,2})\left(\frac{q}{2} - \frac{q\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}} - \frac{q\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} - \left(1 + \frac{\varepsilon(q)}{2}\right)\right) \geq 0, \end{aligned}$$

where the first inequality is obtained by substituting $\alpha = \varepsilon(q)$, $\beta = 0$, and $\gamma_1 = p'$; the second inequality follows from $p' \leq p < q\delta_{1,1}/(\delta_{1,1} + \delta_{2,1}) + 1$; and the third inequality follows from (3.2). Thus, the assertion of the lemma is true. □

Let $q \geq q_0$ be an even number, and let p be the minimal integer such that $p > q\delta_{1,1}/(\delta_{1,1} + \delta_{2,1})$. We set $p_1 = \eta(q) - p$, $p_2 = p$, and

$$f_0 = (Y_1^2 - 2Y_4)^{p_1}(Y_2^2 - 2Y_1Y_2 + 2Y_4)^{p_2}.$$

Then f_0 is in $K[Y]^D$, and $\mathrm{supp}(f_0)$ is contained in $\mathcal{C}$ by Lemma 3.1 because $f_0 = f_{q,p}$.

We define a $\mathbf{Z}$-grading on $K[Y]$ by setting $\deg(Y_1) = \deg(Y_2) = \deg(Y_3) = 1$ and $\deg(Y_4) = 2$. Note that f_0 is a $\mathbf{Z}$-homogeneous element of $\mathbf{Z}$-degree q.

Let l be any positive integer, and let $\mathcal{S}$ be the set of $\mathbf{Z}$-homogeneous elements $F \in K[Y]^D$ of $\mathbf{Z}$-degree $l + q$ having the form $F = f_0Y_3^l +$ (terms of lower degree in Y_3) such that $\omega_2 \cdot a \geq 0$ for each $a \in \mathrm{supp}(F)$. Since $\omega_2 \cdot (i\mathbf{e}_1 + j\mathbf{e}_3) \geq 0$ for $i, j \geq 0$, it follows that $\omega_2 \cdot a \geq 0$ for $a \in \mathrm{supp}((Y_3 - Y_1)^l)$. Hence $f_0(Y_3 - Y_1)^l$ is in $\mathcal{S}$ and so $\mathcal{S}$ is not empty. To complete the proof of Lemma 2.8, it suffices to show that there exists a polynomial $F \in \mathcal{S}$ such that $\omega_1 \cdot a \geq 0$ for all $a \in \mathrm{supp}(F)$. Suppose the contrary. Then $O(F) = (d, e) \in (\mathbf{Z}_{\geq 0})^2$ is defined for each $F \in \mathcal{S}$ as follows. Let $f_i \in K[Y_1, Y_2, Y_4]$ be the coefficient of Y_3^{l-i} in F for each i. Then, define e to be the minimal integer such that $\omega_1 \cdot a < 0$ for some $a \in \mathrm{supp}(f_eY_3^{l-e})$. Note that each monomial in Y_1, Y_2, and Y_4 of $\mathbf{Z}$-degree $q + e$ is written as $Y^{a(i,d)}$ for some d and i, where

$$\begin{aligned} a(i,d) &= (q + e - 2d)\mathbf{e}_1 + d\mathbf{e}_4 + i(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4) \\ &= i\mathbf{e}_2 + (d - i)\mathbf{e}_4 + (q + e - 2d + i)\mathbf{e}_1. \end{aligned}$$

We define d to be the minimal integer such that $a(i,d) \in \mathrm{supp}(f_e)$ for some i. Clearly, the cardinality of the set of $O(F')$ for $F' \in \mathcal{S}$ is finite. Let $\preceq$ be the total order on $(\mathbf{Z}_{\geq 0})^2$ defined by $a \preceq b$ if the last nonzero component of $b - a$ is positive. Then, take $F \in \mathcal{S}$ such that $O(F') \preceq O(F)$ for any $F' \in \mathcal{S}$. Let $f_i \in K[Y_1, Y_2, Y_4]$ be the coefficient of Y_3^{l-i} in F for each i and write $f_e = \sum_{a \in (\mathbf{Z}_{\geq 0})^4} \lambda_a Y^a$, where $O(F) = (d, e)$. We remark that e is positive, since the second component of $O(f_0(Y_3 - Y_1)^l)$ is positive. Define $h = \sum_{i=0}^{d} \lambda_{a(i)} Y^{a(i)}$, where $a(i) = a(i,d)$ for each i. Then we have

$$\omega_1 \cdot a(i) = \omega_1 \cdot ((q+e-2d)\mathbf{e}_1 + d\mathbf{e}_4) = -(q+e)\delta_{1,1} + d(\delta_{1,1} + \delta_{2,1}) < 0 \quad (3.5)$$

for all i. Actually, (3.5) holds for i with $\lambda_{a(i)} \neq 0$ by the definition of e and the minimality of d. Since the left-hand side of the inequality in (3.5) does not depend on i, the inequality holds for all i.

LEMMA 3.2. *There exists $\kappa \in K \setminus \{0\}$ such that $h = \kappa Y_1^{q+e-2d}(-Y_1Y_2 + Y_4)^d$.*

Proof. It suffices to show that $i\lambda_{a(i)} + (d - i + 1)\lambda_{a(i-1)} = 0$ for $i = 1, \ldots, d$. Suppose that $i\lambda_{a(i)} + (d - i + 1)\lambda_{a(i-1)} \neq 0$ for some i. Then the monomial $Y^{a(i)}Y_2^{-1}Y_3^{l-e}$ appears in $D(\lambda_{a(i-1)}Y^{a(i-1)}Y_3^{l-e} + \lambda_{a(i)}Y^{a(i)}Y_3^{l-e})$, since

$$\begin{aligned} (\partial/\partial Y_2)Y^{a(i)}Y_3^{l-e} &= iY^{a(i)}Y_2^{-1}Y_3^{l-e}, \\ (Y_1\partial/\partial Y_4)Y^{a(i-1)}Y_3^{l-e} &= (d - i + 1)Y^{a(i-1)}Y_1Y_4^{-1}Y_3^{l-e} \\ &= (d - i + 1)Y^{a(i)}Y_2^{-1}Y_3^{l-e}, \end{aligned}$$

and the other monomials appearing in $D(Y^{a(i-1)}Y_3^{l-e})$ and $D(Y^{a(i)}Y_3^{l-e})$ are not equal to $Y^{a(i)}Y_2^{-1}Y_3^{l-e}$. Since f_e is the coefficient of Y_3^{l-e} in F, the monomial

$Y^{a(j)}Y_3^{l-e}$ appears in F with coefficient $\lambda_{a(j)}$ for each j. Hence there exists an element c of $\mathrm{supp}(F)$ such that $a(i) - \mathbf{e}_2 + (l-e)\mathbf{e}_3$ is in $\mathrm{supp}(D(Y^c))$ and $c - (l-e)\mathbf{e}_3$ does not equal $a(i)$ or $a(i-1)$, since $D(F) = 0$. Then, c must be equal to $c(1)$ or $c(3)$, where $c(j) = a(i) - \mathbf{e}_2 + \mathbf{e}_j + (l-e)\mathbf{e}_3$ for $j = 1, 3$. Since $c(1) - (l-e)\mathbf{e}_3 = a(i) + 2\mathbf{e}_1 - \mathbf{e}_4 - (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4)$, we have $c(1) \notin \mathrm{supp}(F)$ by the minimality of d. Hence, $c(3)$ is in $\mathrm{supp}(F)$. Then $c(3)$ belongs to $\mathrm{supp}(f_{e-1}Y_3^{l-e+1})$ by definition. However, $\omega_1 \cdot c(3) = \omega_1 \cdot a(i) - \delta_{2,1} < \omega_1 \cdot a(i) < 0$ by (3.5), which contradicts the minimality of e. Thus, $i\lambda_{a(i)} + (d-i+1)\lambda_{a(i-1)} = 0$ for all $i = 1, \ldots, d$. □

Now let $G = F - \kappa f_{q+e,d}(Y_3 - Y_1)^{l-e}$. Since $d < (q+e)\delta_{1,1}/(\delta_{1,1} + \delta_{2,1})$ by (3.5), it follows that $\omega_2 \cdot a \geq 0$ for each $a \in \mathrm{supp}(f_{q+e,d})$ by Lemma 3.1. Therefore, $\omega_2 \cdot a \geq 0$ for each $a \in \mathrm{supp}(G)$. Because e is positive, G has the form $f_0 Y_3^l +$ (terms of lower degree in Y_3). Hence, G is in $\mathcal{S}$. Moreover, $O(F) \preceq O(G)$ by definition. Note that we may write

$$f_{q+e,d} = Y_1^{q+e-2d}(-Y_1Y_2 + Y_4)^d + \sum_{i,d'} \mu_{a(i,d')} Y^{a(i,d')}$$

for some $\mu_{a(i,d')} \in K$, where the sum is taken over i and d' with $d' < d$; consequently, any monomials appearing in hY_3^{l-e} do not appear in G, by Lemma 3.2. This implies that $O(F) \neq O(G)$, which contradicts the choice of F. Hence there exists a polynomial $F \in \mathcal{S}$ such that $\omega_1 \cdot a \geq 0$ for each $a \in \mathrm{supp}(F)$. We have thus proved Lemma 2.8, thereby completing the proof of Theorem 1.1.

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