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Automorphisms of Affine Surfaces with $\mathbb{A}^1$-Fibrations

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1. Introduction

Let X be a normal affine surface defined over the complex field $\mathbf{C}$, which has at worst quotient singularities. We call X simply a *log affine surface*. If further $H_i(X;\mathbb{Q}) = (0)$ for $i > 0$ then X is called a *log $\mathbb{Q}$-homology plane* (if X is smooth then X is simply called a $\mathbb{Q}$-homology plane). Let G_a denote the complex numbers with addition as an algebraic group. In this paper we are mainly interested in log affine surfaces X that have an $\mathbb{A}^1$-fibration. Of particular interest are surfaces that admit a regular action of G_a. Such actions up to conjugacy correspond in a bijective manner to $\mathbb{A}^1$-fibrations on X with base a smooth affine curve. Algebraically, these actions correspond bijectively to locally nilpotent derivations of the coordinate ring $\Gamma(X)$ of X. The set of all elements of $\Gamma(X)$ that are killed under all the locally nilpotent derivations of $\Gamma(X)$ is called the *Makar-Limanov invariant* of X and denoted by $\mathrm{ML}(X)$.

If a smooth affine surface has two independent G_a actions then its Makar-Limanov invariant is trivial. Gizatullin [9] and Bertin [2] gave a necessary and sufficient condition for this to happen. More recently, Bandman and Makar-Limanov [1] proved that a smooth affine surface X with trivial canonical bundle and $\mathrm{ML}(X) = \mathbf{C}$ is an affine surface in $\mathbb{A}^3$ defined by $\{xy = p(z)\}$, where $p(z)$ is a polynomial with distinct roots. Masuda and Miyanishi [12] applied this to determine the structure of a $\mathbb{Q}$-homology plane with trivial ML-invariant. They proved that such a surface is a quotient of the Bandman–Makar-Limanov hypersurface by the action of a finite cyclic group (see result (3) in the listing that follows).

In this paper we extend the last result to the case of log $\mathbb{Q}$-homology planes in Section 2. Similar and related results in Section 2 and Section 3 have been obtained independently by Daigle and Russell [4] and Dubouloz [5]. An automorphism of a smooth affine surface sends fibers of one $\mathbb{A}^1$-fibration with affine base to the fibers of another $\mathbb{A}^1$-fibration. If these two fibrations are different then the Makar-Limanov invariant of the surface is trivial. If a smooth affine surface has an $\mathbb{A}^1$-fibration whose base is not an affine curve, then this fibration does not correspond to a G_a action. In this case the geometry of the fibration enters into the picture. In Section 4 we give a sufficient condition for uniqueness of an $\mathbb{A}^1$-fibration on a smooth affine surface. This involves the number of multiple fibers

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of a given $\mathbb{A}^1$-fibration and the type of the base of the fibration. The proof of this result is rather involved. However, as a consequence we are able to prove a result about the automorphism group of a smooth affine surface with an $\mathbb{A}^1$-fibration. We also prove two results that deal with uniqueness of a $\mathbf{C}^*$-fibration on a normal quasi-homogeneous surface. In the last section we give typical examples to illustrate the various results proved in this paper. Throughout, we denote by $\mathbb{A}^n$ the affine n-space.

We now summarize our main results.

(1) *Let X be a log affine surface. Then* $\mathrm{ML}(X) = \mathbf{C}$ *if and only if the divisor at infinity for X in a suitable minimal normal compactification of X is a linear chain of rational curves* (Theorem 3.1).

(2) *If X is a log $\mathbb{Q}$-homology plane, then* $\mathrm{ML}(X) = \mathbf{C}$ *if and only if* $\pi_{1,\infty}(X)$ *is finite cyclic* (Lemma 2.5, Theorem 2.9).

(3) *If X is as in* (2) *then the quasi-universal cover of X is isomorphic to the surface $xy = z^a - 1$ in $\mathbb{A}^3$. Here $a = 1$ is allowed, so the quasi-universal cover is isomorphic to* $\mathbb{A}^2$ (cf. Theorem 2.8; for the definition of the quasi-universal cover, see Section 2).

(4) *Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\psi: X \to B$. Assume that one of the following conditions is satisfied*:
  - (i) *B is nonrational*;
  - (ii) *B is rational with at least two places at infinity*;
  - (iii) *$B \cong \mathbb{A}^1$, and ψ has at least two multiple fibers*;
  - (iv) *$B \cong \mathbf{P}^1$, every fiber of ψ is irreducible, and ψ has at least three multiple fibers.*

  Then ψ is the unique $\mathbb{A}^1$-fibration on X (Theorem 4.1).

(5) *Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\psi: X \to \mathbf{P}^1$ with at least three multiple fibers. If every fiber of ψ is irreducible, then* $\mathrm{Aut}(X)$ *is finite* (Theorem 4.2).

(6) *Let X be a normal affine surface with a good $\mathbf{C}^*$-action such that at least three orbits exist with nontrivial isotropy subgroups. Then any curve contained in the smooth locus of X that is isomorphic to $\mathbf{C}^*$ is one of the orbits of the $\mathbf{C}^*$-action* (Theorem 4.6).

Remark. We conjecture that (4)(iv) and (5) are true without the condition of irreducibility of every fiber of ψ.

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2. Case of Log $\mathbb{Q}$-Homology Planes

We will deal only with complex algebraic varieties. For a normal projective surface W with only quotient singularities (abbreviated as a *log projective surface*), a curve C on W is an (n)-curve if C is a smooth, irreducible, rational curve contained in $W - \mathrm{Sing}\, W$ and with $(C^2) = n$. For a (possibly reducible) curve C on a log projective surface W, by a “component” of C we mean an irreducible component of C. Let Z be a normal quasi-projective surface. By a $\mathbf{P}^1$-*fibration* (resp., an $\mathbb{A}^1$-fibration) on Z we mean a morphism $Z \to B$ onto a smooth algebraic curve whose general fiber is isomorphic to $\mathbf{P}^1$ (resp., $\mathbb{A}^1$). A $\mathbf{C}^*$-fibration on a normal quasi-projective surface is defined similarly. We say that a normal affine surface Z is *quasi-homogeneous* or, equivalently, that it has *a good* $\mathbf{C}^*$*-action* if there is an algebraic action of the algebraic group of $\mathbf{C}^*$ on Z such that Z contains a unique point, say ν, that is in the closure of every orbit. The point ν is called the *vertex* of Z. Corresponding to such an action is a quotient map $X - \{\nu\} \to \Delta$, where Δ is a smooth projective curve.

For any variety Z we denote the set of smooth points of Z by Z^0. For a normal algebraic surface X such that $\pi_1(X^0)$ is finite, the normalization of X in the function field of the universal covering Y^0 of X^0 is called the *quasi-universal covering* of X (cf. [20]). Let X be a complex affine surface with at worst quotient singularities. By a *minimal normal compactification* of X we mean a projective completion V of X such that V is smooth outside X and $D := V - X$ is a simple normal crossing divisor such that any (-1)-curve in D meets at least three other components of D. Suppose that the divisor D is a tree of rational curves. In [21], Mumford gives a presentation of the fundamental group of the boundary of a nice tubular neighborhood of D in V in terms of the intersection matrix of the components of D. Following Ramanujam [23], we call this the “fundamental group at infinity of X” and denote it by $\pi_{1,\infty}(X)$.

Recall that a log $\mathbb{Q}$-homology plane Y is a log affine surface such that $H_i(Y;\mathbb{Q}) = (0)$ for $i > 0$. It is well known that the divisor at infinity of Y in a minimal normal compactification is a (connected) tree of rational curves (see [19]). Hence we can use the foregoing presentation of $\pi_{1,\infty}(Y)$. If D is a linear tree of smooth rational curves then it follows easily from Mumford’s presentation that $\pi_{1,\infty}(Y)$ is a finite cyclic group with a generator corresponding to an end component of D. This observation will be quite useful later. We will use repeatedly the following general properties of a singular fiber of a $\mathbf{P}^1$-fibration proved by Gizatullin [8].

LEMMA 2.1. *Let $p: V \to B$ be a $\mathbf{P}^1$-fibration on a smooth projective surface V with base a smooth curve B. Let G be a singular fiber of p. Then the following assertions hold.*

(1) *G is a tree of smooth rational curves.*
(2) *G contains a (-1)-curve, and any (-1)-curve in G meets at most two other components of G. If a (-1)-curve E occurs with multiplicity 1 in the scheme-theoretic fiber G, then G contains another (-1)-curve.*
(3) *By successively contracting (-1)-curves in G and their images, we can reduce G to a regular fiber.*

We now recall a result about singular fibers of an $\mathbb{A}^1$-fibration on a normal affine surface (cf. [16]).

LEMMA 2.2. *Let Z be a normal affine surface with an $\mathbb{A}^1$-fibration $f: Z \to B$, where B is a smooth curve. Then we have the following assertions.*

(1) *Z has at most cyclic quotient singularities.*
(2) *Every fiber of f is a disjoint union of curves isomorphic to $\mathbb{A}^1$.*
(3) *A component of a fiber of f contains at most one singular point of Z. If a component of a fiber occurs with multiplicity* 1 *in the scheme-theoretic fiber, then no singular point of Z lies on this component.*

The next result clarifies the relation between G_a-actions and $\mathbb{A}^1$-fibrations on a normal affine surface (cf. 14, Chap. I, Lemma 1.5; 16]).

LEMMA 2.3. *Let X be a normal affine surface. Then the quotient morphsim under any nontrivial algebraic action of the additive group G_a on X gives rise to an $\mathbb{A}^1$-fibration $\rho: X \to B$ with a smooth affine curve B. Conversely, given an $\mathbb{A}^1$-fibration $\rho: X \to B$ with a smooth affine curve B, there is a nontrivial G_a-action on X such that ρ is the associated $\mathbb{A}^1$-fibration.*

We use repeatedly the following result of Bundagaard and Nielsen [3] and Fox [7], which is the solution of Fenchel's conjecture.

LEMMA 2.4. *Let C be a smooth projective curve of genus g and let $P_1, \dots, P_s$ be points of C. Let $m_1, \dots, m_s$ be integers larger than* 1. *Then there exists a finite Galois covering $p: \tilde{C} \to C$ that ramifies over the points P_i with respective ramification indices $m_1, \dots, m_s$ unless either* (i) $g = 0$ *and* $s = 1$ *or* (ii) $g = 0$, $s = 2$, *and* $m_1 \neq m_2$.

The next result is quite important for the proofs of the main results (1), (2), and (3) stated in the Introduction.

LEMMA 2.5. *Let X be a log affine surface such that D, the divisor at infinity for X in a minimal normal compactification V of X, is a linear tree of rational curves such that the intersection form on the irreducible components of D has nonzero determinant. Then X has an $\mathbb{A}^1$ fibration $f: X \to \mathbb{A}^1$ with at most one multiple fiber mF_1 with $m > 1$. Further, $\pi_1(X^0)$ is isomorphic to $\mathbb{Z}/(m)$.*

Proof. Since X is affine, the intersection form on the components of D has at least one positive eigenvalue. By assumption, 0 is not an eigenvalue of this form. It is easy to see that, by a suitable sequence of blow-ups with centers in D and contractions of (-1)-curves in the proper transforms of D, we can transform D into a linear tree of rational curves $D_1, D_2, \dots, D_r$ such that $D_1^2 = D_2^2 = 0$ (cf. [11, Lemma 5]). Now $K_V \cdot D_1 = -2$. It follows that V is a rational surface and $|D_1|$ gives a $\mathbf{P}^1$-fibration $\varphi: V \to \mathbf{P}^1$ such that D_1 is a full fiber, D_2 is a cross-section, and $D_3, \dots, D_r$ are contained in a fiber of φ. Hence $f := \varphi|_X$ is an $\mathbb{A}^1$-fibration on X with base $\mathbb{A}^1$. By Mumford's result quoted earlier, $\pi_{1,\infty}(X)$ is finite cyclic. By a Lefschetz theorem for open surfaces (see [22, Cor. 2.3]), there is a surjection $\pi_{1,\infty}(X) \to \pi_1(X^0)$. This implies that $\pi_1(X^0)$ is finite cyclic.

Suppose $m_1F_1, m_2F_2, \dots, m_rF_r$ are all the multiple fibers of f. By Lemma 2.4, there is a finite Galois covering $\Delta \to \mathbb{A}^1$ such that the ramification index over the point $f(F_i)$ is m_i for $i = 1, 2, \dots, r$. Then the normalization of the fiber product

$Z := \overline{X \times_{\mathbb{A}^1} \Delta}$ contains a Zariski-open subset that is a finite unramified covering of X^0 with a noncyclic covering transformation group. This contradicts the fact that $\pi_1(X^0)$ is finite cyclic, so f has at most one multiple fiber mF_1. If such a multiple fiber exists then we consider again the surface Z just described. There is an induced $\mathbb{A}^1$-fibration on Z with base $\mathbb{A}^1$ (now $\Delta \cong \mathbb{A}^1$) without multiple fibers. By Lemma 4.3 (to follow), there is a short exact sequence

$$\pi_1(\mathbb{A}^1) \to \pi_1(Z^0) \to \pi_1(\mathbb{A}^1) \to (0).$$

This shows that Z^0 is simply connected and $\pi_1(X^0) \cong \mathbb{Z}/(m)$. □

Lemma 2.5 applies in particular to a log $\mathbb{Q}$-homology plane with $\mathrm{ML}(X) = \mathbf{C}$, since we will show (in Lemma 2.6) that the divisor at infinity for X in a minimal normal compactification of X is a linear tree of rational curves.

LEMMA 2.6. *Let X be a log $\mathbb{Q}$-homology plane and let $\rho\colon X \to B$ be an $\mathbb{A}^1$-fibration. Then the following assertions hold* (cf. [19]).

(1) *B is isomorphic to the affine line $\mathbb{A}^1$. Hence there is a smooth normal compactification V of X such that the $\mathbb{A}^1$-fibration ρ extends to a $\mathbf{P}^1$-fibration $p\colon V \to \bar{B} \cong \mathbf{P}^1$ and the fiber at infinity $F_\infty = p^{-1}(P_\infty)$ is a smooth fiber, where $\bar{B} - B = \{P_\infty\}$. The fibration p has a cross-section S lying outside X.*
(2) *Every fiber of ρ is irreducible, and its reduced form is isomorphic to $\mathbb{A}^1$.*

The hypothesis that $\mathrm{ML}(X) = \mathbf{C}$ for a log $\mathbb{Q}$-homology plane X implies more precise results, as follows.

LEMMA 2.7. *Let X be a log $\mathbb{Q}$-homology plane with* $\mathrm{ML}(X) = \mathbf{C}$. *Assume that $X \not\cong \mathbb{A}^2$. Then the following assertions hold.*

(1) *Every $\mathbb{A}^1$-fibration $\rho\colon X \to B$ has a unique multiple fiber mA with $m > 1$.*
(2) *There is a smooth normal compactification V of X such that $D := V - X$ is a linear chain of rational curves.*
(3) *The surface X has at most one singular point P such that $P \in A$.*

Proof. (1) Let $\rho\colon X \to B$ be an $\mathbb{A}^1$-fibration. Since $B \cong \mathbb{A}^1$, ρ is the quotient morphism with respect to a G_a-action σ. If there are no multiple fibers in ρ, then X is smooth and isomorphic to $\mathbb{A}^2$. Hence ρ has at least one multiple fiber. We use the argument in Lemma 2.5. If ρ has $r \geq 2$ multiple fibers then we consider (a) the finite Galois cover Δ of B ramified over $\rho(F_i)$ for $1 \leq i \leq r$ and (b) the point at infinity for B such that the ramification index at any point over $\rho(F_i)$ is m_i and at any point over ∞ is equal to 2. Then the normalized fiber product $Z = \overline{X \times_B \Delta}$ has an $\mathbb{A}^1$-fibration over Δ. It is easy to see that Δ is either nonrational or rational with at least two places at infinity. By assumption, X has a transverse $\mathbb{A}^1$-fibration; let G be a general fiber of this transverse fibration. Then the map $G \to X^0$ lifts to a map $G \to Z^0$. But then G dominates Δ, a contradiction. Hence ρ has exactly one multiple fiber mA. By the argument in part (3) of Lemma 2.2, X has at most one singular point and it is a cyclic quotient singular point.

(2) We consider a minimal normal compactification V of X such that the $\mathbb{A}^1$-fibration ρ extends to a $\mathbf{P}^1$-fibration $p\colon V \to \bar{B}$. We may assume that the fiber F_∞ lying over the point at infinity P_∞ of B is smooth. Let S be the cross-section of p contained in $D := V - X$. Let G be the part of the singular fiber F_0 of p lying over the point $P_0 := \rho(A)$. Let σ' be a G_a-action that is algebraically independent of the G_a-action σ, and let $\rho'\colon X \to B'$ be the associated $\mathbb{A}^1$-fibration. Let Λ' be the linear pencil spanned by the closures of the general fibers of ρ' on V. Now the arguments in [12, Lemma 2.4 & Thm. 2.5] apply (up to some minor modifications) to the pencil Λ', enabling us to conclude that G is a linear chain. □

The following results give a characterization of log $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariants.

THEOREM 2.8. *Let X be a log $\mathbb{Q}$-homology plane with* $\mathrm{ML}(X) = \mathbf{C}$. *Then $\pi_1(X^0) \cong \mathbb{Z}/(m)$ (cf. Proof of Lemma 2.5). The quasi-universal cover of X is isomorphic to either $\mathbb{A}^2$ or the surface $z^a - 1 = xy$ in $\mathbb{A}^3$.*

Proof. We consider the $\mathbb{A}^1$-fibration $\rho\colon X \to B$, which is associated to a G_a-action σ. Let mA be a unique multiple fiber of ρ. Then the quasi-universal covering Y of X is obtained as the normalization of $X \times_B \Delta$, where Δ is an m-tuple cyclic covering of B totally ramifying over the point $P_0 := \rho(A)$ and the point at infinity P_∞ of B (cf. Lemma 2.5). Let $f\colon Y \to X$ be the composite of the normalization morphism and the projection of $X \times_B \Delta$ to X. Since the induced $\mathbb{A}^1$-fibration on Y has only reduced fibers, it follows by Lemma 2.2(3) that Y is a smooth surface and that f is étale and finite over X^0. If all the fibers of the induced $\mathbb{A}^1$-fibration on Y are irreducible then Y is isomorphic to $\mathbb{A}^2$. Any G_a-action on X extends to Y, since f is étale over X^0. Hence $\mathrm{ML}(Y) = \mathbf{C}$. Now, by the result of Bandman and Makar-Limanov [1], Y is isomorphic to the surface $z^a - 1 = xy$ in $\mathbb{A}^3$. □

We now give another proof of the result of Bandman and Makar-Limanov just cited, generalized slightly to work for normal surfaces.

THEOREM 2.9. *Let X be a log affine surface with trivial Makar-Limanov invariant. Then X has a minimal normal compactification V such that $D := V - X$ is a linear chain of rational curves. In particular, $\pi_{1,\infty}(X)$ is a finite cyclic group.*

Proof. For the proof we will use some arguments from [12, Lemma 2.6 & Thm. 2.7]. If X is isomorphic to the affine plane $\mathbb{A}^2$, it is well known (see [23]) that the boundary divisor of any minimal normal compactification of $\mathbb{A}^2$ is a linear chain. Hence we may and shall assume that X is not isomorphic to $\mathbb{A}^2$. Let σ, σ' be two G_a-actions on X. By making use of one G_a-action σ on X, we consider an associated $\mathbb{A}^1$-fibration $\rho\colon X \to B$. We claim that $B \cong \mathbb{A}^1$. First of all, by Lemma 2.3, B is an affine curve. Since a general fiber of the $\mathbb{A}^1$-fibration corresponding to σ' dominates B, we conclude that $B \cong \mathbb{A}^1$.

For a suitable smooth compactification V of X, we can extend ρ to a $\mathbf{P}^1$-fibration $p\colon V \to \bar{B} \cong \mathbf{P}^1$ such that $D := V - X$ consists of a smooth fiber F_∞, a cross-section S, and a union G_1 of irreducible components contained in a degenerate fiber of p. By using Lemma 2.1 repeatedly we can assume that, for any component C of G_1, $(C^2) < -1$.

Let Λ' be the pencil of rational curves corrresponding to σ' and let T' be the closure of a general orbit of σ'. If $T' \cap F_\infty = \emptyset$ then the $\mathbb{A}^1$-fibrations corresponding to σ, σ' are the same. Hence T' meets F_∞. Suppose that Λ' has no base point on F_∞. Then we get another $\mathbf{P}^1$-fibration p' on V such that F_∞ is a cross-section for p'. We then claim that $X \cong \mathbb{A}^2$. Since a general fiber of p' is disjoint from S, it follows by the Hodge index theorem that $(S^2) \le 0$. If $(S^2) = 0$ then S is a member of the pencil Λ'. In this case, $D = F_\infty \cup S$ and we see that $X \cong \mathbb{A}^2$. Suppose $(S^2) < 0$. Then $S \cup G_1$ is contained in a fiber of p'. In fact, since the base of the $\mathbb{A}^1$-fibration on X corresponding to p' is also isomorphic to $\mathbb{A}^1$, the union $S \cup G_1$ is a full fiber of p'. It follows that $(S^2) = -1$ and, starting with the contraction of S, we can contract $S \cup G_1$ to a smooth rational curve with self-intersection 0. Then again we see that $X \cong \mathbb{A}^2$.

Now, we know that Λ' has a base point on F_∞. By performing elementary transformations at $F_\infty \cap S$ we can further assume that this base point is not the point $F_\infty \cap S$. By the Hodge index theorem, $(S^2) < 0$. Blowing up the base point of Λ' and its infinitely near points yields a surface V' that admits a $\mathbf{P}^1$-fibration p' such that the proper transform of F_∞, S, G_1, and some exceptional curves obtained by blow-ups form a single fiber—say, G' of p'. By Lemma 2.1 we can contract G' to a regular fiber. Since no irreducible component of G_1 is a (-1)-curve, the first (-1)-curve to be contracted is the proper transform of F_∞ or S. Again using Lemma 2.1, we deduce that $D = F_\infty \cup S \cup G_1$ is linear. In particular, $\pi_{1,\infty}(X)$ is a finite cyclic group. □

Next we show that the converse to Theorem 2.8 holds when X is a log $\mathbb{Q}$-homology plane. We shall prove the following result.

THEOREM 2.10. *Let X be a log $\mathbb{Q}$-homology plane. Suppose that $\pi_{1,\infty}(X)$, the fundamental group at infinity, is a finite cyclic group. Then X has a minimal normal compactification V such that $D := V - X$ is a linear chain of rational curves. Furthermore,* ML(X) *is trivial.*

We first recall the following result from [24].

LEMMA 2.11. *Let X be a smooth affine surface. Assume that the fundamental group at infinity of X is finite cyclic. Then X has a minimal normal compactification V such that*: (a) *$D := V - X$ is a tree of rational curves*; (b) *D contains components D_1, D_2 with the self-intersections of D_1, D_2 both zero*; *and* (c) *after removing D_1, D_2 from D we get a connected linear chain of rational curves that has a negative definite intersection form. Moreover, V is rational.*

With the notation of Theorem 2.10, D then supports a divisor with strictly positive self-intersection, since X is affine. Because V is rational, the linear system $|D_1|$ gives a $\mathbf{P}^1$-fibration p on V such that D_2 is a cross-section of p and all the other components of D are contained in a singular fiber G of p. By Lemma 2.5 we can see that p has no singular fiber other than G. The part of D contained in G is a linear chain. This observation will be used in what follows.

Proof of Theorem 2.10. If X is isomorphic to $\mathbb{A}^2$, then all the assertions hold. Hence we assume that X is not isomorphic to $\mathbb{A}^2$; in particular, $\mathrm{Pic}(X^0) \neq 0$. The restriction of p to X is an $\mathbb{A}^1$-fibration $\rho\colon X \to B \cong \mathbb{A}^1$. Since X is a $\mathbb{Q}$-homology plane, we see easily that every fiber of ρ is irreducible. For example, $G = G_1 \cup A_1$ where $G_1 = D \cap G$ and $A := A_1 - D \cong \mathbb{A}^1$. By Lemma 2.1 we can assume that no component of G_1 is a (-1)-curve and hence A_1 is a unique (-1)-curve in G after the desingularization of a possible singular point on A, and the multiplicity m of A_1 in G exceeds unity.

Let $P_0 := p(G)$. By assumption, G_1 is a linear chain. Let D_3 be the component of G_1 that meets D_2. We claim that D_3 meets at most one other component of G_1. Suppose that D_3 meets two components of D, say D_4 and D_5. Then $\overline{G_1 - D_3}$ has exactly two connected components, say Δ_1 and Δ_2. Starting with A_1 we can successively contract (-1)-curves in G, the exceptional curves arising from the desingularization of a possible singular point on A, and their images in order to reduce G to a (0)-curve.

Suppose that at some stage the image of D_3, say D_3', becomes a (-1)-curve and that D_3' still meets two other components of the image G' of G. Then the multiplicity of D_3' in G' is at least 2. This is a contradiction, since D_3 meets the cross-section D_2. Hence, if D_3' is a (-1)-curve then it meets only one other component of G'. Further, all the other components of G' have self-intersection < -1. Since G' is still a linear chain, it clearly follows that G' cannot be contracted to a (0)-curve. Now we see that D is a linear chain of rational curves. This proves the first part of Theorem 2.10; the second part follows from Theorem 3.1. □

3. The General Case

Our objective in this section is to prove the following result, which will completely explain the relation between $\mathrm{ML}(X)$ and $\pi_{1,\infty}(X)$.

Theorem 3.1. *Let X be a log affine surface. Then* $\mathrm{ML}(X)$ *is trivial if and only if X has a minimal normal compactification V such that the dual graph of $D := V - X$ is a linear chain of rational curves and $\pi_{1,\infty}(X)$ is a finite group.*

Proof. The "only if" part follows from Theorem 2.9; here we show the "if" part. Again, for simplicity we will assume that X is smooth. The proof for the log affine case is almost similar.

By assumption, X has a minimal normal compactification V such that $D := V - X$ is a linear chain of smooth rational curves. We can also assume that $D = D_1 + D_2 + \cdots + D_r$ such that $D_{r-1}^2 = 0 = D_r^2$. We call $D_{r-1} + D_r$ an *appendix* of D. Then the linear system $|D_r|$ gives a $\mathbf{P}^1$-fibration p on V such that D_r is a full fiber and D_{r-1} is a cross-section. Restricting p to X yields an $\mathbb{A}^1$-fibration $\rho\colon X \to B$ with $B \cong \mathbb{A}^1$. By Lemma 2.3, X admits a G_a-action such that a general fiber of ρ is an orbit for this action. If $r = 2$ then $X \cong \mathbb{A}^2$ and obviously

$\mathrm{ML}(X) = \mathbf{C}$. So, we assume that $r > 2$. Then $D_{r-2}^2 \le 0$ by the Hodge index theorem. If $D_{r-2}^2 = 0$ then $r = 3$ and $\pi_{1,\infty}(X) \cong \mathbb{Z}$, a contradiction. Therefore, $D_{r-2}^2 < 0$ and hence $D_i^2 < 0$ for $1 \le i \le r-2$. Observe that D_1 is contained in a singular fiber of p and is thus disjoint from a general fiber of p.

The idea of the proof is to shift the appendix to the beginning of the linear chain so that D_1 becomes a (0)-curve and then use $|D_1|$ to construct another G_a-action on X. The proof of [11, Lemma 5] shows that, by blowing up points in D and blowing down (-1)-curves that are proper transforms of irreducible components of D, we reach a minimal normal compactification of X, say W, such that the proper transform of D_1 in W becomes a (0)-curve. We will indicate a few steps in this process.

Let $(D_{r-2}^2) = -a \le -2$. Blow up $D_{r-1} \cap D_r$ to obtain a surface V' and let E be the exceptional curve obtained by this blow-up. Then $(D_{r-1}'^2) = -1 = (E^2) = (D_r'^2)$, where the prime denotes proper transform. Blow down D_{r-1}' to obtain the surface V_1. On V_1 the proper transform of D_{r-2} has self-intersection $-a+1$. The self-intersections of the images of E and D_r' on V_1 (say, E_1 and $D_{r,1}$) are 0 and -1, respectively. We see that the pencil in V_1 corresponding to $|D_r|$ has a base point on the proper transform $D_{r-2,1}$ of D_{r-2}. Next blow up $E_1 \cap D_{r,1}$ and blow down the proper transform of E_1 to obtain the surface V_2. The self-intersection of $D_{r-2,2}$ on V_2 is $-a+2$, and the self-intersection of the proper transform of $D_{r,1}$ is -2. The pencil on V_2 has a base point on $D_{r-2,2}$. Continue this process to obtain the surface V_a such that the self-intersection of the proper transform $D_{r-2,a}$ of D_{r-2} on V_a is 0. The pencil has a base point on $D_{r-2,a}$. Observe that the proper transform of D on V is still linear, $(E_a^2) = 0$, and that E_a meets $D_{r,a}$. Now we start blowing up $D_{r-2,a} \cap E_a$, and so forth.

Finally, we reach a surface V_b on which the proper transform of D is a linear chain, the proper transform $D_{1,b}$ of D_1 is a (0)-curve, and the curve next to it is also a (0)-curve. The pencil corresponding to $|D_r|$ on V_b has a base point on $D_{1,b}$. Using $|D_{1,b}|$, we get another $\mathbb{A}^1$-fibration on X that is transverse to the original $\mathbb{A}^1$-fibration. By Lemma 2.3, $\mathrm{ML}(X) = \mathbf{C}$. This completes the proof of Theorem 3.1. □

Combining Theorems 2.8, 2.9, 2.10, and 3.1 completes our proof of the main results (1), (2), and (3) stated in the Introduction.

4. Uniqueness of $\mathbb{A}^1$-Fibrations and Aut(X)

In this section we give a sufficient condition for a smooth affine surface to have a unique $\mathbb{A}^1$-fibration.

Theorem 4.1. *Let $\psi\colon X \to B$ be an $\mathbb{A}^1$-fibration on a smooth affine surface X with base B a smooth curve such that every fiber of ψ is irreducible. Assume further that B is isomorphic to $\mathbb{A}^1$ or $\mathbf{P}^1$ and that ψ has at least two (resp., three) multiple fibers if $B \cong \mathbb{A}^1$ (resp., if $B \cong \mathbf{P}^1$). Then X has no other $\mathbb{A}^1$-fibrations whose general fibers are transverse to ψ.*

A consequence of this result is the following theorem.

Theorem 4.2. *Let $\psi: X \to B$ be an $\mathbb{A}^1$-fibration such that $B \cong \mathbf{P}^1$, all the fibers of ψ are irreducible, and ψ has at least three multiple fibers. Then* $\mathrm{Aut}(X)$ *is finite. If, further, the multiplicities of the multiple fibers are pairwise coprime, then* $\mathrm{Aut}(X)$ *is trivial.*

Remarks. (1) The assumption in Theorem 4.1 that B is isomorphic to $\mathbb{A}^1$ or $\mathbf{P}^1$ is quite harmless. For if X has another $\mathbb{A}^1$-fibration with general fiber transverse to a general fiber of ψ, then (by Lüroth's theorem) B will be isomorphic to $\mathbb{A}^1$ or $\mathbf{P}^1$.

(2) It is most probable that Theorem 4.1 is valid with*out* assuming the irreducibility of all the fibers of ψ. Similarly, Theorem 4.2 should be true without the assumption of irreducibility of all the fibers of ψ.

(3) An example in Section 5 (see paragraph 4) shows that the hypothesis of having at least three multiple fibers in Theorem 4.2 is necessary.

We shall first recall the following result (see e.g. [27, Sec. 1]). Let Y be a smooth quasi-projective surface, let B be a smooth quasi-projective curve, and let $f: Y \to B$ be a fibration in the sense that all fibers have pure dimension 1 and all but a finite number of them are smooth and connected. Let $F_0 = \sum_{i=1}^{n} \mu_i C_i$ be its fiber, where the C_i are irreducible components and the μ_i are multiplicities of the C_i in F_0. Let $\mu = \gcd(\mu_1, \dots, \mu_n)$, which we call the *multiplicity* of F_0. If $\mu > 1$, we call F_0 a *multiple* fiber and write $F_0 = \mu F_0'$, where $F_0' = \sum_{i=1}^{n} (\mu_i/\mu) C_i$.

Lemma 4.3. *With the preceding notation, let F be a general fiber of f, let $m_1 F_1, \dots, m_s F_s$ exhaust all multiple fibers of f, and let $P_i = f(F_i)$. Set $B' = B - \{P_1, \dots, P_s\}$. Then there exists a short exact sequence*

$$\pi_1(F) \to \pi_1(Y) \to \Gamma \to (1),$$

where Γ is the quotient of $\pi_1(B')$ by the normal subgroup generated by $e_1^{m_1}, \dots, e_s^{m_s}$ with the e_i corresponding to a small loop in B around the point P_i.

This lemma shows that even a reducible fiber without reduced components behaves like a smooth fiber in $\pi_1(Y)$ if the multiplicity of the fiber is 1.

Proof of Theorem 4.1

Let X be as in the statement of Theorem 4.1. Let $X \subset V$ be a smooth projective compactification such that (a) $D := V - X$ is a simple normal crossing divisor and (b) ψ extends to a $\mathbf{P}^1$-fibration $\Psi: V \to \bar{B}$, where $\bar{B}$ is a smooth projective compactification of B. An irreducible component of D will be called a *boundary* component. There is a unique component S of D that is a cross-section of Ψ such that the point at infinity for a general fiber of ψ lies on S. Using Lemma 2.1, we can contract (-1)-curves in any singular fiber of Ψ that are contained in D and can assume that D does not contain any (-1)-curve that is contained in a fiber of Ψ. Hence, if $B \cong \mathbf{P}^1$ then every boundary fiber component of Ψ has self-intersection number ≤ -2, and if $B \cong \mathbb{A}^1$ then the same holds—except for the full fiber of

Ψ, which is contained in D. In this latter case we can assume that this fiber is a $\mathbf{P}^1$ with self-intersection 0. Let mF'_0 be a multiple fiber of ψ with $m > 1$. Then $\Psi^{-1}(P) - \overline{F'_0}$ is nonempty, connected, and meets the section S ($\overline{F'_0}$ is the closure of F'_0 in V). This is because X is affine.

Now assume that X has another $\mathbb{A}^1$-fibration $g\colon X \to B'$ whose general fiber, say G', is horizontal with respect to ψ (or, equivalently, transverse to ψ). We will arrive at a contradiction. Again by Lüroth's theorem, B' is isomorphic to $\mathbb{A}^1$ of $\mathbf{P}^1$ (since a general fiber of ψ dominates B').

Step 1. First we will deal with the case where $B \cong \mathbb{A}^1$. Then the assertion follows from a more general result.

LEMMA 4.4. *Let $\psi\colon X \to B$ be an $\mathbb{A}^1$-fibration from a smooth affine surface X onto the affine line B. Suppose that $m_1F_1, \dots, m_rF_r$ exhaust all multiple fibers of ψ, where $m_i \geq 2$, $r \geq 2$, and the F_i might be reducible. Then there are no curves G' such that G' is isomorphic to the affine line and transverse to the fibration ψ.*

Proof. Let $P_i = \psi(F_i)$ for $1 \leq i \leq r$, and let P_∞ be the point at infinity of B when B is embedded into $\bar{B} = \mathbf{P}^1$. Now apply Lemma 2.5 to $\bar{B}, P_1, \dots, P_r, P_\infty$ and integers $m_1, \dots, m_r, m_\infty$ in order to find a finite Galois covering $\bar{\tau}\colon \bar{\Delta} \to \bar{B}$, where m_∞ is a positive integer to be chosen arbitrarily. By the Riemann–Hurwitz theorem, it is easy to show that either $\bar{\Delta}$ has genus > 0 or $\bar{\tau}^{-1}(P_\infty)$ has at least two points. More precisely, $\bar{\Delta}$ has positive genus if either (i) $r \geq 3$ or (ii) $r = 2$ and $m_\infty \geq 6$, except for the case $r = m_1 = m_2 = 2$ in which $\bar{\Delta}$ has two places above the point P_∞.

Let $\Delta = \bar{\tau}^{-1}(B)$ and $\tau = \bar{\tau}|_\Delta$. Then $\tau\colon \Delta \to B$ is a finite Galois covering. Let Y be the normalization of the fiber product $X \times_B \Delta$. Then Y is an étale covering of X and so Δ either is nonrational or is a rational curve with at least two places at infinity.

Suppose that there exists a curve G' such that G' is isomorphic to $\mathbb{A}^1$ and transverse to ψ. Then the inverse image of G' in Y is a disjoint union of curves isomorphic to $\mathbb{A}^1$ each of which dominates Δ. This is a contradiction. □

We shall make use later of the following well-known result.

LEMMA 4.5. *Let $\psi\colon X \to B$ be an $\mathbb{A}^1$-fibration from a smooth affine surface X with a smooth curve B. Then the following assertions hold.*

(1) *Let n_P be the number of irreducible components of the fiber $\psi^{-1}(P)$ for $P \in B$, and let N be the number of places of B lying at infinity. Then the Picard number of X is equal to*

$$\rho(X) = 1 + \sum_{P \in B}(n_p - 1) - \varepsilon,$$

where $\varepsilon = 0$ or 1 according as $N = 0$ or $N \geq 1$.

(2) *Let $\chi(Y)$ denote the topological Euler–Poincare characteristic of a topological manifold Y. Let F be a general fiber of ψ and let $F_1, \dots, F_s$ exhaust all*

the singular fibers, which are (by definition) the fibers not isomorphic to $\mathbb{A}^1$ in the scheme-theoretic sense. We have the following formula of Suzuki [26] *and Zaidenberg* [28]:

$$\chi(X) = \chi(F)\chi(B) + \sum_{i=1}^{s}(\chi(F_i) - \chi(F)).$$

Proof. For the proof of the first assertion, consider a smooth projective compactification $X \subset V$ such that $\psi\colon X \to B$ extends to a $\mathbf{P}^1$-fibration $\Psi\colon V \to \bar{B}$. We assume that the fibers contained in $V \setminus X$ are irreducible if they exist at all. Now V is obtained from a relatively minimal $\mathbf{P}^1$-fibration by iterating blow-ups with centers on the fibers. Then the result is standard. □

Hereafter in the proof of Theorem 4.1, *we assume that $B \cong \mathbf{P}^1$ and that ψ has irreducible multiple fibers $m_1F_1, \dots, m_rF_r$ with $r \geq 3$.*

Step 2. We make the following claim.

CLAIM.
(1) $B' \cong \mathbf{P}^1$.
(2) *Let G' be a general fiber of g. Then G' meets each F_i for $1 \leq i \leq r$.*
(3) *Suppose $m_1 \leq m_2 \leq \cdots \leq m_r$. Then $r = 3$ and $(m_1, m_2, m_3) = (2, 2, n)$,* $(2, 3, 3)$, $(2, 3, 4)$, *or* $(2, 3, 5)$. *Namely, it is one of the Platonic triplets.*
(4) *All fibers of g' are irreducible, and there are three multiple fibers of g' whose multiplicities form one of the Platonic triplets.*

For the proof of (1), let F be a general fiber of ψ and let

$$\Gamma = \langle e_1, e_2, \dots, e_r \mid e_1e_2\cdots e_r = e_1^{m_1} = e_2^{m_2} = \cdots = e_r^{m_r} = 1\rangle$$

be the group given by generators and relations, which is the group given in Lemma 4.3 for $B \cong \mathbf{P}^1$. Hence we obtain an isomorphism $\pi_1(X) \cong \Gamma$ because $\pi_1(F) = (1)$. By the assumption that $r \geq 3$, it follows that $\pi_1(X)$ is not a finite cyclic group. Furthermore, we know by Lemma 4.5 that the Picard group $\mathrm{Pic}(X)$ has rank 1 and that the topological Euler–Poincaré characteristic $\chi(X)$ is 2. Now we show that $B' \cong \mathbf{P}^1$. Suppose to the contrary that $B' \cong \mathbb{A}^1$. Since $\mathrm{Pic}(X)$ has rank 1, it follows that the fibration g has one reducible fiber $\mu_1C_1 + \mu_2C_2$ and that all other fibers are irreducible. Lemma 4.3 implies that g has at least two multiple fibers because $\pi_1(X)$ is not a finite cyclic group. We then have a contradiction by Lemma 4.4, since a general fiber F is transverse to g. Hence, $B' \cong \mathbf{P}^1$.

We now show that G' meets each F_i. If G' does *not* meet some F_i then we consider $X' := X - F_i$. Then X' is a smooth affine surface that has an induced $\mathbb{A}^1$-fibration from ψ with at least two multiple fibers with base $\mathbb{A}^1$ and another $\mathbb{A}^1$-fibration induced from g. This is impossible (by Lemma 4.4), so we know that G' meets each F_i.

We show that $r = 3$ and that (m_1, m_2, m_3) is one of the Platonic triplets. In fact, if either $r \geq 4$ or (m_1, m_2, m_3) is not a Platonic triplet then we use the argument in the proof of Lemma 4.4. The curve Δ in this case is nonrational, whereas the inverse image of G' in Y is a disjoint union of the affine lines. Hence we obtain a contradiction.

The last assertion is easy to see. Since the Picard number of X is 1 and since $B' \cong \mathbf{P}^1$, it follows that all fibers of g' are irreducible. Then Lemma 4.3 implies that g' has at least three multiple fibers because $\pi_1(X)$ is not a finite cyclic group. If one notes that a general fiber of ψ is transverse to the fibration g', then the same argument as in the previous assertion (3) implies that the multiplicities of the singular fibers of g' form one of the Platonic triplets.

Step 3. Taking the closures of the fibers of g yields a pencil of rational curves Λ with at most one base point on V. Note that the base point lies on D if it exists.

CLAIM.

(1) Λ *has no base point. In particular, V has another* $\mathbf{P}^1$*-fibration* (*say,* $\tilde{g}$) *whose general fiber is transverse to* Ψ.

(2) *S is also a cross-section for* $\tilde{g}$.

For the proof of (1), suppose that Q is a base point of Λ. Then Q lies either on S or on a boundary fiber component of Ψ. Let W be obtained from V by a shortest succession of blow-ups at Q (and its infinitely near points), so that W has a $\mathbf{P}^1$-fibration $\tilde{g}$ that extends g. The last (-1)-curve E obtained by blow-ups is a cross-section of $\tilde{g}$. We note that every irreducible component of $W - X$, except for E and possibly the proper transform S' of S, has self-intersection number ≤ -2.

The proper transform S' is contained in a fiber (say, G) of $\tilde{g}$ and either (a) meets at least three other components of G or (b) meets two components of G and also meets E. This follows from the assumption that there are at least three multiple fibers of ψ. In either case, by Lemma 2.1(2) we can see that S' is not a (-1)-curve. On the other hand, the proper transforms of at least two singular fibers (which remain untouched under the blow-ups $W \to V$) of Ψ corresponding to the multiple fibers of ψ, say $\tilde{F}_1$ and $\tilde{F}_2$, have the property that $(\mathrm{Supp}\,\tilde{F}_1 - F_1) \cup (\mathrm{Supp}\,\tilde{F}_2 - F_2)$ is contained in G. All the components of this last union are components of D, and none is a (-1)-curve (by our initial assumption). Every fiber of g is also irreducible, as remarked in Step 2. It follows that the closure of every singular fiber of g in W is the unique (-1)-curve in the corresponding fiber of the $\mathbf{P}^1$-fibration $\tilde{g}$ on W.

The curve S' is connected to E by a connected union of irreducible components G (possibly, $S' \cap E \neq \emptyset$). If we successively contract (-1)-curves in G using Lemma 2.1, we reach a stage when the image of S' becomes a (-1)-curve and either (a) meets at least three other irreducible components of the image of G or (b) meets two irreducible components of the image of G and meets the image of E. In case (a) we have a contradiction to Lemma 2.1(2). Suppose that case (b) occurs. Because the image of S' meets two irreducible components of the image of G, the multiplicity of S' in G is 2. But then its image cannot meet the image of E'. This proves part (1) of the claim.

For the proof of part (2) we observe that, if S is not a cross-section of $\tilde{g}$, then it is contained in a fiber of $\tilde{g}$. In this case we argue exactly as in part (1) and arrive at a contradiction.

Step 4. Now changing the notation, let G be a general fiber of Φ_Λ. Then $G^2 = 0$. The idea of the proof is to blow down V to a minimal model and then calculate the arithmetic genus of the image of G in the minimal model in two different ways in order to arrive at a contradiction.

The only singular fibers of Ψ are the fibers $\tilde{F}_i$ containing F_i for $i = 1, 2, 3$, and the multiplicities form a Platonic triplet. We have already seen that the closure $\bar{F}_i$ of F_i is the only (-1)-curve in $\tilde{F}_i$. Starting with $\bar{F}_i$, we successively contract (-1)-curves for all i and arrive at a $\mathbf{P}^1$-bundle V_0 over $\bar{B}$. The component of $\tilde{F}_i$ meeting S occurs with multiplicity 1 in $\tilde{F}_i$. Hence, by Lemma 2.1, in the process of these contractions there will always be a (-1)-curve that differs from the image of this curve. Let G_0, S_0 be (respectively) the images of G, S in V_0, and let F be a general fiber of Ψ.

Let $n := G \cdot F$. We denote the general fiber of the $\mathbf{P}^1$-bundle $V_0 \to \bar{B}$ again by F. Write $G_0 \sim aF + bS_0$, and denote S_0^2 by $-c$. Since $G_0 \cdot F = n$, we have $b = n$. From $G_0 \cdot S_0 = 1$, we obtain $1 = a + nS_0^2 = a - cn$. Hence $G_0 \sim (1+cn)F + nS_0$. This gives $G_0^2 = 2n + cn^2$. Now let K be the canonical divisor of V_0. Then $K \sim -2S_0 - (c+2)F$ and so $K \cdot G_0 = (1+cn)(-2) + n(-2+c) = -2n - cn - 2$. Therefore, $p_a(G_0) = cn(n-1)/2$. Now we calculate $p_a(G_0)$ in a different way.

Clearly $G \cdot \bar{F}_i = n/m_i$ for each i. Thus, contraction of $\bar{F}_i$ produces a singular point of multiplicity n/m_i on the image of G. Let

$$e_{i1} = n/m_i \le e_{i2} \le \cdots \le e_{ir_i}$$

be the multiplicities of the images of G after the succession of contractions of (-1)-curves. Because G is rational,

$$p_a(G_0) = \sum_{i=1}^{3} \sum_{j=1}^{r_i} e_{ij} \frac{e_{ij} - 1}{2}.$$

Since $G^2 = 0$, we get $G_0^2 = \sum_{i,j} e_{ij}^2$. Suppose $cn(n-1)/2 = \sum_{i,j} e_{ij}(e_{ij} - 1)/2$. Then $cn^2 - cn = G_0^2 - \sum e_{ij}$ and hence $\sum e_{ij} = 2n + cn$. From $\sum e_{ij}^2 = 2n + cn^2$ we have

$$\sum \left(\frac{e_{ij}}{n}\right)^2 = \frac{2}{n} + c. \tag{4.1}$$

Similarly, from $\sum e_{ij} = 2n + cn$ we obtain

$$\sum \frac{e_{ij}}{n} = 2 + c. \tag{4.2}$$

Subtracting (4.1) from (4.2) then yields

$$\sum \frac{e_{ij}}{n} - \left(\frac{e_{ij}}{n}\right)^2 = 2 - \frac{2}{n}. \tag{$*$}$$

We will now use the observation made in Step 2 that (m_1, m_2, m_3) is a Platonic triplet. Then $m_1 = 2$. First we concentrate on the fiber $\tilde{F}_1$. Since $\bar{F}_1$ is the only (-1)-curve in $\tilde{F}_1$, it follows that the self-intersection of any other component of $\tilde{F}_1$

is ≤ -2. Write $\tilde{F}_1 = 2\bar{F}_1 + \Delta$ as the scheme-theoretic fiber. Then $K \cdot \tilde{F}_1 = -2 = -2 + K \cdot \Delta$. From this and the fact that Supp Δ is connected (since X is affine) we infer that every component of Supp Δ is a (-2)-curve. By [29, Lemma 1.5], the dual graph of $\tilde{F}_1$ has exactly one branch point and $\bar{F}_1$ is a tip of one of the branches at the branch point. Hence we see that the multiplicity sequence on the image of G during contractions of curves in $\tilde{F}_1$ is $n/2, n/2, \ldots, n/2$ (at least three blow-downs). Hence the contribution to the sum $\sum(e_{ij}/n) - (e_{ij}/n)^2$ from this fiber is at least 3/4.

Now consider the other two fibers. If $m_2 = m_3 = 2$ then by the same observation as before we see that the LHS of $(*)$ is greater than 2 whereas the RHS is less than 2. Suppose that $m_2 = 2$ and $m_3 > 2$. We will show in what follows that, in this case, the contribution from $\tilde{F}_3$ is at least $2/3$. Hence in all the cases we get a contradiction to $(*)$.

Step 5. Finally we consider the case when $m_2 > 2$ and $m_3 > 2$. For simplicity we consider only the case of m_3 and write $m_3 = m$. Since G meets F_3 transversally in n/m distinct points, we have n/m smooth subarcs of G meeting F_3 in distinct points. We consider the images of these arcs in V_0, say $G_{0,1}, G_{0,2}, \ldots, G_{0,n/m}$. Since G is a general fiber of g, the multiplicity sequences for all these unibranch curves are the same. In particular, it follows that e_{3j} is divisible by n/m. Now $G_{0,j} \cdot F = m$ for each j. Let n_1 be the multiplicity of $G_{0,j}$ of the singular point lying on the fiber L on V_0 that is the image of $\tilde{F}_3$. We consider the reverse process to obtain F_3.

If $n_1 = 1$, then the proper transform L' of L in V is a (-1)-component lying in the boundary $V \setminus X$. Because such a component does not exist on V (by our assumption), we have $n_1 > 1$. Then the Euclidean transformation with respect to the pair (m, n_1) (see [14]) will be the first process that we must perform in order to produce the singular fiber $\tilde{F}_3$. This process produces a linear chain of the components. Then we have to blow up a point on the (-1)-component of the linear chain (not the end components of the linear chain) as well as additional points to obtain the multiple fiber F_3 on X. This last process produces the side tree.

Let $n_1 > n_2 > \cdots > n_s$ be the multiplicities of $G_{0,1}$ in the Euclidean transformation. Then $n_s \geq 1$. It follows that the distinct multiplicities occurring in the resolution of singularities for G_0 contain $n_1 \cdot n/m, n_2 \cdot n/m, \ldots, n_s \cdot n/m$. Hence the contribution to the LHS of $(*)$ from $\tilde{F}_3$ is at least

$$a_1(n_1/m - n_1^2/m^2) + a_2(n_2/m - n_2^2/m^2) + \cdots + a_s(n_s/m - n_s^2/m^2),$$

where the integers $a_1, a_2, \ldots, a_s$ are defined as follows:

$$\begin{aligned}
m &= a_1 n_1 + n_2, & n_2 &< n_1,\\
n_1 &= a_2 n_2 + n_3, & n_3 &< n_2,\\
&\ \ \vdots & &\ \ \vdots\\
n_{s-2} &= a_{s-1} n_{s-1} + n_s, & n_s &< n_{s-1},\\
n_{s-1} &= a_s n_s. & &
\end{aligned}$$

From $G_{0j} \cdot F = m$ we see that the arc G_{0j} on V_0 has a parameterization of the form

$$z_1 = t^{n_1}, \qquad z_2 = t^m + \text{higher-degree terms.}$$

Hence, in the resolution by blow-ups, the multiplicity sequence for G_{0j} contains $n_1^{a_1}, \dots, n_{s-1}^{a_{s-1}}, n_s^{a_s}$ (where n^a signifies that n is repeated a times). We thus have

$$\begin{aligned} 1 &= a_1 \cdot n_1/m + n_2/m, \\ n_1/m &= a_2 \cdot n_2/m + n_3/m, \\ &\vdots \\ n_{s-2}/m &= a_{s-1} \cdot n_{s-1}/m + n_s/m, \\ n_{s-1}/m &= a_s \cdot n_s/m. \end{aligned}$$

Adding up both the left- and right-hand sides yields

$$1 + \frac{n_1}{m} - \frac{n_s}{m} = \sum_{i=1}^{s} a_i \cdot \frac{n_i}{m}. \tag{4.3}$$

Again multiplying respectively by $n_1/m, n_2/m, \dots, n_s/m$, we obtain

$$\begin{aligned} n_1/m &= a_1(n_1/m)^2 + n_1 n_2/m^2, \\ n_1 n_2/m^2 &= a_2(n_2/m)^2 + n_2 n_3/m^2, \\ &\vdots \\ n_{s-1} n_s/m &= a_s(n_s/m)^2. \end{aligned}$$

Hence it follows that

$$\frac{n_1}{m} = \sum_{1}^{s} a_i \left(\frac{n_i}{m}\right)^2. \tag{4.4}$$

From (4.3) and (4.4) we can derive

$$\sum a_i \left\{ \frac{n_i}{m} - \left(\frac{n_i}{m}\right)^2 \right\} = 1 - \frac{n_s}{m}.$$

Here we note that $n_s | m$ and $n_s < m$. If m is a prime number then $n_s = 1$; if $n_s \neq 1$, then the first blow-up to produce the side tree of $\tilde{F}_3$ will give a contribution $(n_s/m) - (n_s/m)^2$. Hence the contribution is at least $1 - 1/m$ if $n_s = 1$ and $1 - (n_s/m)^2$ if $n_s \neq 1$. That is, the contribution is at least $2/3, 3/4, 4/5$ as $m = 3, 4, 5$ (respectively) and $3/4$ if $m > 5$. Therefore, the contributions to the left side of $(*)$ from $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$ are at least 2. This is a contradiction to the relation $(*)$ in Step 2 and so completes the proof of Theorem 4.1.

Proof of Theorem 4.2

Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\pi: X \to B$. By Theorem 4.1, X has no $\mathbb{A}^1$-fibration whose fibers are transverse to the fibers of ψ.

Hence any automorphism of X permutes the fibers of ψ. Let $G := \mathrm{Aut}(X)$, and let $m_1F_1, m_2F_2, \ldots, m_rF_r$ be all the multiple fibers of ψ and $P_i := \psi(F_i)$. By hypothesis, $r \geq 3$. Hence there exists a subgroup H of finite index in G such that every fiber of ψ is stable under every element of H. If the multiplicities are pairwise coprime, then clearly every element of G keeps every fiber of ψ stable and hence the induced action of G on B is trivial. Now we can assume that G itself keeps every fiber stable and acts trivially on B.

Step 1. First we will show that G is finite.

By Lemma 2.4, there exists a finite Galois covering $\tau: \Delta \to B$ such that the ramification index at any point over P_i is m_i for every i. Then the normalization of the fiber product $Y := \overline{X \times_B \Delta}$ is an étale covering of X. There is an induced $\mathbb{A}^1$-fibration ψ' on Y whose fibers are all reduced; the group G also acts on Y, permuting the fibers. By taking a subgroup of finite index of G, we can assume that every element of G keeps stable every component of every fiber of ψ'. Let Z be obtained by omitting all components but one from every reducible fiber of ψ'. Then Z is a smooth affine surface with an $\mathbb{A}^1$-bundle $\tilde{\pi}: Z \to \Delta$ and G acts on Z by automorphisms, keeping every fiber stable. There is a smooth compactification $W \subset T$ such that $T - Z$ is a cross-section $\tilde{S}$ of $\tilde{\psi}$. The action of G extends to T, keeping $\tilde{S}$ pointwise fixed. Assuming that the action of G on T is nontrivial, we will show that such a surface T does not exist; this will prove that G is finite. Observe that W is affine.

Case 1. Suppose that $T = \mathbf{P}^1 \times \mathbf{P}^1$ such that G keeps each $\{x\} \times \mathbf{P}^1$ stable and keeps $\tilde{S}$ pointwise fixed. Then the action of G is independent of the point x in the first factor $\mathbf{P}^1$. This implies that the fixed point locus of G cannot contain an ample irreducible curve (in this case, $\tilde{S}$). (This argument was shown to us by A. Fujiki.)

Case 2. Suppose next that T is a rational surface that is not isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Then T contains a unique irreducible curve Γ with $\Gamma^2 < 0$ and Γ is a cross-section of $\tilde{\psi}$ that is also pointwise fixed by G. Hence most fibers of $\tilde{\psi}$ have at least two fixed points. Let $Q_1, Q_2, \ldots, Q_s$ be the points in $\tilde{S} \cap \Gamma$. Each Q_i is fixed by G. By performing elementary transformations at these points repeatedly, we can separate the proper transforms of $\tilde{S}$ and Γ and still have a G-action along fibers of a $\mathbf{P}^1$-bundle $T' \to \Delta$ while keeping the proper transforms of $\tilde{S}$ and Γ pointwise fixed. Then it is easy to see that the G-action extends to an action of the multiplicative group $\mathbf{C}^*$ on T' and that the process of obtaining T from T' is $\mathbf{C}^*$-equivariant. Hence the G-action on T extends to an action of $\mathbf{C}^*$ on T. At Q_i, the fixed point locus of this action is not smooth; this is a contradiction, since $\mathbf{C}^*$ is reductive.

Case 3. Now assume that Δ has genus $g > 0$, that $\tilde{S}$ is an ample cross-section of $\tilde{\psi}: T \to \Delta$, and that G acts on T keeping every fiber stable and keeping $\tilde{S}$ pointwise fixed. Let U_1, U_2 be Zariski-open subsets of Δ such that the $\mathbb{A}^1$-bundle is trivial over both U_1, U_2 and $\Delta = U_1 \cup U_2$. We can use the section $\tilde{S}$ to choose a point at infinity on every fiber; W is obtained from $U_1 \times \mathbb{A}^1$ and $U_2 \times \mathbb{A}^1$ by patching. Let z, w be fiber coordinates on $U_1 \times \mathbb{A}^1$ and $U_2 \times \mathbb{A}^1$, respectively. On $U_1 \cap U_2$ we have $w = a(u)z + b(u)$, where a, b are regular functions on $U_1 \cap U_2$ and a is nowhere zero. Next we use an automorphism σ in G.

Now suppose that $\sigma(u,z) = (u, \alpha_1(u)z + \beta_1(u))$ on $U_1 \times \mathbb{A}^1$ and $\sigma(u,w) = (u, \alpha_2(u)w + \beta_2(u))$ on $U_2 \times \mathbb{A}^1$, where $\alpha_i(u)$ are units on U_i, et cetera. Hence $a(\alpha_1 z + \beta_1) + b = \alpha_2(az + b) + \beta_2$ on $(U_1 \cap U_2) \times \mathbb{A}^1$. This gives $a\alpha_1 = a\alpha_2$ and $a\beta_1 + b = b\alpha_2 + \beta_2$ on $U_1 \cap U_2$. Then $\alpha_1 = \alpha_2$ on $U_1 \cap U_2$, whence the functions a_i on U_i patch together to give an invertible regular function on Δ. It follows that $\alpha_1 = \alpha_2$ is a nonzero constant α. *We claim that* $\alpha = 1$. In fact, if this is not true, then σ has another fixed point on every fiber and our argument for Case 2 works in this case also to give a contradiction.

Assume that $\alpha = 1$. Now we have $a\beta_1 = \beta_2$ on $U_1 \cap U_2$. The conormal bundle of $\tilde{S}$ in T is $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of $\tilde{S}$ in T. On $U_1 \times \mathbf{P}^1$ the ideal sheaf is generated by $1/z = z'$ and on $U_2 \times \mathbf{P}^1$ by $1/w = w'$. Since $w = az + b$, it follows that $w' = z'/(a + bz') = z'/a \pmod{\mathcal{I}^2}$. Thus, on $\tilde{S}$ the equation $a\beta_1 = \beta_2$ gives a cross-section $\beta_1 z' = \beta_2 w'$ of $\mathcal{I}/\mathcal{I}^2$. But $\tilde{S}^2 > 0$ since W is affine. Hence there is no such nonzero cross-section of $\mathcal{I}/\mathcal{I}^2$ and thus no such automorphism can exist.

Step 2. Now assume that $m_1, m_2, \dots, m_r$ are pairwise coprime. We will show that $\mathrm{Aut}(X)$ is trivial; for this purpose, it suffices to show that there is no nontrivial finite automorphism of X. Suppose that σ is such an automorphism. By Sumihiro's result [25], we can find a smooth projective compactification $X \subset V$ such that (a) V has a $\mathbf{P}^1$-fibration $\tilde{\psi} : V \to \mathbf{P}^1$ that extends ψ and (b) the action of σ extends to V. In the fiber $\tilde{F}_i$ of $\tilde{\psi}$ containing F_i, we may assume that the closure $\bar{F}_i$ is the only (-1)-curve and hence is stable under σ. Since $\bar{F}_i$ is a tip of $\tilde{F}_i$ and since $\tilde{S}$ is stable under σ, we can see that there exists a component (say, G_i) of $\tilde{F}_i$ that meets at least three other components of $\tilde{F}_i$ and that is pointwise fixed by σ. Let z_1, z_2 be suitable local coordinates at a general point p_i of G_i such that G_i is $\{z_1 = 0\}$ and ψ is given by $(z_1, z_2) \to z_1^{m_i}$. The action of σ on the base B is trivial. Hence we can diagonalize the action of σ at p_i as $\sigma(z_1, z_2) = (\zeta z_1, z_2)$, where $\zeta^{m_i} = 1$. It follows that σ^{m_i} is trivial in a neighborhood of p_i and thus trivial everywhere on X. If $m_1, m_2, \dots, m_r$ are pairwise coprime then σ is the identity. This completes the proof of Theorem 4.2.

Further Results

Our next result deals with $\mathbf{C}^*$-fibrations on (affine) quasi-homogeneous surfaces.

THEOREM 4.6. *Let X be a normal quasi-homogeneous surface with vertex v. Suppose there exist at least three orbits with nontrivial isotropy subgroups. Then any curve C in X^0 that is isomorphic to $\mathbf{C}^*$ is one of the orbits of the good $\mathbf{C}^*$-action on X.*

Proof. Let the $\mathbf{C}^*$-action be denoted by σ_λ for $\lambda \in \mathbf{C}^*$. For a general λ, the translate $\sigma_\lambda(C)$ meets a general orbit transversally if C is not an orbit. There exists a normal projective compactification V of X such that the $\mathbf{C}^*$-action extends to V and $V - X$ contains an irreducible curve B that is pointwise fixed by this action. There is a natural map $X^0 \to B$ whose fibers are the orbits.

Suppose that C is not an orbit. Then C dominates B and so B is a rational curve. Now C and $\sigma_\lambda(C)$ define a pencil of rational curves on V. There are at most two base points for this pencil that are contained in $B \cup \{\nu\}$. Resolving the base locus yields a $\mathbf{P}^1$-fibration on a blow-up of V whose restriction to X^0 is a $\mathbf{C}^*$-fibration $\pi\colon X^0 \to \Delta$. Every fiber of π contains a reduced irreducible component. The only possible member of the pencil that does not intersect X^0 is B. Hence $\Delta \cong \mathbb{A}^1$ or $\mathbf{P}^1$.

By Lemma 2.4, we have an exact sequence

$$\pi_1(\mathbf{C}^*) \to \pi_1(X^0) \to (1).$$

This implies that $\pi_1(X^0)$ is cyclic. Since σ_λ has at least three orbits with nontrivial isotropy subgroups, the map $X^0 \to B$ has at least three multiple fibers. By the argument in the proof of Lemma 2.5 and using Lemma 2.4, we can construct a noncyclic étale finite covering of X^0. This is a contradiction, proving that C is an orbit of σ_λ. □

An easy consequence of Theorem 4.6 is the following result.

THEOREM 4.7. *With X as in Theorem 4.6, there is a short exact sequence*

$$(1) \to G_m \to \mathrm{Aut}(X) \to \Gamma \to (1),$$

where Γ is a finite group.

Our next result is similar in spirit to Theorem 4.6.

THEOREM 4.8. *Let (X, ν) be a quasi-homogeneous surface with the corresponding quotient map $\psi\colon X^0 \to B$. If ψ either has at least four multiple fibers or has three multiple fibers whose multiplicities do not form a Platonic triplet, then the image of any nonconstant morphism $f\colon \mathbf{C}^* \to X^0$ is a fiber of ψ.*

Proof. We will give only a brief sketch of the proof, since most of the arguments have already been made. Suppose that the result is false. Then $B \cong \mathbf{P}^1$ because B is rational. Let $P_1, P_2, P_3, \ldots$ be the points in B corresponding to the orbits with nontrivial isotropies. By Lemma 2.4, we can construct a Galois ramified covering $\Delta \to B$ that is *correctly* ramified over the points P_i. Then Δ is nonrational. The normalization Y' of the fiber product $\overline{X^0 \times_B \Delta}$ is an étale finite covering of X^0, and there is a $\mathbf{C}^*$-action on Y' such that the map $Y' \to X^0$ is equivariant. Now $\pi_1(Y') \to \pi_1(X^0)$ has finite index. From this we see that there is a suitable morphism $\mathbf{C}^* \to Y'$ whose image is not contained in a fiber of the quotient map $Y' \to \Delta$. This is a contradiction, since Δ is nonrational. □

Theorem 4.8 has the following consequence.

THEOREM 4.9. *For X as in Theorem 4.8, any self-map $X \to X$ permutes the orbits of the good C^*-action.*

5. Examples

1. Let $W = \mathbf{P}^1 \times \mathbf{P}^1$ and let G_1, G_2 be two fibers of one of the natural $\mathbf{P}^1$-fibrations on W. Let S be a cross-section of this fibration. By blowing up points of G_1 suitably we obtain a surface V such that the inverse image of G_1 in V is the linear chain $D_1 + D_2 + D_3 + D_4 + D_5$, where D_1, D_5 are (-1)-curves, D_2, D_3, D_4 are (-2)-curves, the proper transforms S', G_2' of S, G_2 on V are (0)-curves, and S' meets D_3. It is easy to see that $X := V - (D_2 + D_3 + D_4 + S' + G_2')$ is an affine surface. The curve $B := D_2 + D_3 + D_4 + S' + G_2'$ is the divisor at infinity for X. By Lemma 2.5, $\pi_{1,\infty}(X)$ is finite cyclic (of order 4). We claim that $\mathrm{ML}(X) \neq \mathbf{C}$, for if $\mathrm{ML}(X) = \mathbf{C}$ then (by Theorem 2.9) X has a minimal normal compactification Z such that $D := Z - X$ is a linear chain of rational curves. But then Z is obtained from V by blow-ups and blow-downs of (-1)-curves with points in B and hence D is the proper transform of B. However, we can see that this is not possible and so $\mathrm{ML}(X) \neq \mathbf{C}$.

2. As an application of Theorem 3.1, we will prove the following result related to the Jacobian problem.

Proposition. *Let $\varphi\colon X_1 \to X_2$ be an étale endomorphism of the affine plane, where X_1 and X_2 are isomorphic to $\mathbb{A}^2$. Let $\tilde{X}_2$ be the normalization of X_2 in the function field of X_1. Then X_1 is a Zariski open subset of $\tilde{X}_2$ and* $\mathrm{ML}(\tilde{X}_2) \neq \mathbf{C}$, *provided there are at least three singular points on $\tilde{X}_2$.*

Proof. It is known by [16; 17] that X_1 is a Zariski open set of $\tilde{X}_2$, that $\tilde{X}_2$ is a log affine surface with at most cyclic quotient singularities, and that $\tilde{X}_2 - X_1$ is a disjoint union of irreducible components isomorphic to the affine line. Note that any $\mathbb{A}^1$-fibration on X_1 extends to an $\mathbb{A}^1$-fibration $\rho\colon \tilde{X}_2 \to B$ for $B \cong \mathbb{A}^1$ or $\mathbf{P}^1$ and that the restriction $\rho|_{X_1}$ consists only of reduced irreducible fibers because $X_1 \cong \mathbb{A}^2$. Let V be a minimal normal compactification of $\tilde{X}_2$ such that ρ extends to a $\mathbf{P}^1$-fibration $p\colon \tilde{X}_2 \to \bar{B}$, with a cross-section S contained in the boundary D at infinity. Suppose that $\tilde{X}_2 \neq X_1$. Write $\tilde{X}_2 - X_1 = \coprod_{i=1}^n C_i$, where the C_i are irreducible. We argue separately in the two cases $B \cong \mathbb{A}^1$ and $B \cong \mathbf{P}^1$.

First assume that $B \cong \mathbb{A}^1$. Then the closure $\bar{C}_i$ of C_i in V is an irreducible component of a fiber $p^{-1}(P)$, where $P \in B$. Let $A_i := (\rho|_{X_1})^{-1}(P)$ and let $\bar{A}_i$ be its closure on V. Since $\bar{A}_i$ has multiplicity 1 in the fiber $p^{-1}(P)$, it follows that $p^{-1}(P)$ contains components other than $\bar{A}_i$ and $\bar{C}_i$ (otherwise, $\bar{A}_i$ and $\bar{C}_i$ must meet on the cross-section S, which is impossible). Let D_i be the component of $p^{-1}(P)$ that meets the cross-section S. Then $D_i \neq \bar{A}_i, \bar{C}_i$, for otherwise $\bar{A}_i$ or $\bar{C}_i$ has more than one puncture. Suppose that $\#\{\rho(C_i); 1 \leq i \leq n\} \geq 2$. Then the divisor D is not a linear chain because the fiber F_∞ of $p^{-1}(P)$ lying over the point P_∞ at infinity of B is contained in the boundary divisor D. Suppose that $\#\{\rho(C_i); 1 \leq i \leq n\} = 1$. Namely, we assume that all the components C_i are contained in one and the same fiber $p^{-1}(P)$. If there are two or more singular points then they lie on some of the C_i and the $\bar{C}_i$ are connected to the component D_i, which meets S. If there is a singular point on C_i then the multiplicity of $\bar{C}_i$ in the fiber $p^{-1}(P)$ is at least 2. Hence there exists a nonempty subtree in D that connects $\bar{C}_i$ and D_i. If there are two or more singular points, then the divisor D is not a linear chain.

Next assume that $B \cong \mathbf{P}^1$. Then some component, say C_j, is contained in the fiber $F_\infty = p^{-1}(P_\infty)$. If F_∞ is reducible, it must be that F_∞ contains a component of D. We then argue as in the case $B \cong \mathbb{A}^1$ to conclude that the existence of two or more singular points on $\tilde{X}_2$ implies that D is not a linear chain. So, assume that $F_\infty = C_j$ is irreducible and hence $(C_j^2) = 0$. Then the existence of three or more singular points on $\tilde{X}_2$ implies that D is not a linear chain. Hence $\mathrm{ML}(\tilde{X}_2) \neq \mathbf{C}$ by Theorem 3.1. □

3. Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\pi : X \to B$, where $B \cong \mathbb{A}^1$. Then X has a G_a-action such that every fiber of π is an orbit for this action. We assume that π has at least two multiple fibers. If Y is obtained from X by removing a finite number of regular fibers, then clearly $\mathrm{Aut}(Y)$ contains G_a. Meanwhile, X has no other $\mathbb{A}^1$-fibrations whose general fibers are transverse to π (by Theorem 4.1). Similar examples can be given when base of the fibration is a curve of positive genus.

4. Let V be a Hirzebruch surface Σ_n with $n \gg 0$. Choose a cross-section S of the $\mathbf{P}^1$-bundle π on Σ_n with $S^2 = n$. By blowing up two points of S and its infinitely near points successively, we can create two singular fibers $\tilde{G}_1, \tilde{G}_2$ on the blow-up $\tilde{V}$ of V such that $C_i^2 = D_i^2 = -2$ for $1 \leq i \leq 3$, $C_4^2 = D_4^2 = -1$, $(C_3 \cdot C_4) = (D_3 \cdot D_4) = 1$, and C_1, D_1 are the proper transforms of the fibers of V. The surface $X := \tilde{V} - (S' \cup C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3)$ is affine, where S' is the proper transform of S in $\tilde{V}$. The divisor at infinity for X is a linear chain of $\mathbf{P}^1$s. Hence X admits two nonconjugate actions of the additive group G_a. Observe that there is an $\mathbb{A}^1$-fibration on X with exactly two multiple fibers (of multiplicity 2 each) over $\mathbf{P}^1$. Therefore, the hypothesis of Theorem 4.2—that π has at least three multiple fibers—is necessary to conclude the assertion.

5. We calculate the Makar-Limanov invariant of $X := \mathbf{P}^2 - C$, where C is a curve defined by $X_0 X_1^{m-1} = X_2^m$ with $m > 2$. We will show that X has a unique G_a-action up to conjugacy that is associated to the pencil generated by C and mL, where L is the line $X_1 = 0$.

Using blow-ups to resolve the base locus of this pencil yields an $\mathbb{A}^1$-fibration on X with base $\mathbb{A}^1$. Hence X has a nontrivial G_a-action. By suitable further blow-downs we can find a minimal normal compactification V of X such that $D := V - X$ is a nonlinear tree of rational curves.

We can easily show that X is a $\mathbb{Q}$-homology plane and that the $\mathbb{A}^1$-fibration has a unique multiple fiber of multiplicity m. By Theorem 3.1, $\mathrm{ML}(X) \neq \mathbf{C}$; in fact, $\mathrm{ML}(X) = \mathbf{C}[x]$, which is a polynomial ring in one variable. The surface X also has a $\mathbf{C}^*$-action given by $\sigma_\lambda([X_0, X_1, X_2]) = [X_0, \lambda^m X_1, \lambda^{m-1} X_2]$. This action of $\mathbf{C}^*$ on $\mathbf{P}^2$ keeps C stable and hence induces an action on X. The action of $\mathbf{C}^*$ on $\mathbf{P}^2$ has only finitely many fixed points. We claim that a general orbit of the $\mathbf{C}^*$-action on X is transverse to the fibers of the $\mathbb{A}^1$-fibration just described; otherwise, the fibers of the $\mathbb{A}^1$-fibration will be stable under the $\mathbf{C}^*$-action. Then on every fiber there will be at least one fixed point for the $\mathbf{C}^*$-action. This is not possible.

6. The surface X of paragraph 5 is one of the affine pseudo-planes, which—together with their universal coverings—have various interesting properties (cf. [13]). Using a different example, we shall offer just one remark about the $\mathbb{A}^1$-fibrations on such surfaces. Let Y be a smooth affine surface $x^r y = z^d - 1$, where $r \geq 2$ and $d \geq 2$. The quotient surface X of Y under a $(\mathbb{Z}/d\mathbb{Z})$-action defined by $\zeta \cdot (x, y, z) = (\zeta x, \zeta^{-r} y, \zeta z)$ is an affine pseudo-plane, where ζ is a primitive dth root of unity. In fact, the quotient morphism $Y \to X$ is a universal covering map of X.

It is known that any G_a-action on X lifts up to a G_a-action on Y that commutes with the $(\mathbb{Z}/d\mathbb{Z})$-action and vice versa (cf. [12]). Now $(x, y, z) \mapsto x$ gives rise to an $\mathbb{A}^1$-fibration on Y over the base $\mathbb{A}^1$, so Y has a nontrivial G_a-action that commutes with the $(\mathbb{Z}/d\mathbb{Z})$-action. But one can show that this is a unique $\mathbb{A}^1$-fibration on Y over an affine base curve. Meanwhile, there are at least $2d$ distinct $\mathbb{A}^1$-fibrations on Y over $\mathbf{P}^1$. In fact, a mapping $(x, y, z) \in X \mapsto [x^r : z - \zeta^i]$ (or $[y : z - \zeta^i]$) yields an $\mathbb{A}^1$-fibration over $\mathbf{P}^1$ for $0 \leq i < d$. This example shows that $\mathrm{ML}(Y) \neq \mathbf{C}$ while Y has at least two independent $\mathbb{A}^1$-fibrations.

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