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Nonisotropic Hölder Estimates on Convex Domains of Finite Type

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1. Introduction

In [DFo], Diederich and Fornæss constructed a smooth support function for convex domains of finite type that satisfies the correct estimates to be used in the construction of several integral kernels. (See e.g. [DFFo; DM1; DM2; F; H].) In [DFFo] this support function was used to construct solution operators for the Cauchy–Riemann equation that satisfy optimal (isotropic) Hölder estimates, as follows.

Theorem 1.1 [DFFo]. *Let $D \subset\subset \mathbb{C}^n$ be a linearly convex domain with $\mathcal{C}^\infty$-smooth boundary of finite type m. We denote by $C^0_{(0,q)}(\bar{D})$ the Banach space of $(0,q)$-forms with continuous coefficients on $\bar{D}$ and by $\Lambda^{1/m}_{(0,q)}(D)$ the Banach space of $(0,q)$-forms whose coefficients are uniformly Hölder continuous of order $1/m$ on D. Then there are bounded linear operators*

$$T_q : C^0_{(0,q+1)}(\bar{D}) \to \Lambda^{1/m}_{(0,q)}(D)$$

such that $\bar{\partial} T_q f = f$ for all $f \in C^0_{(0,q+1)}(\bar{D})$ with $\bar{\partial} f = 0$.

In fact, a simple modification of the standard example shows that in general the solution of a Cauchy–Riemann equation with bounded left-hand side cannot be better than $(1/m)$-Hölder continuous. However, a closer look at the example shows that it is in the *normal* direction that the Hölder exponent cannot be better than $1/m$. On the other hand, Krantz [K] has shown that $(1/2)$-Hölder continuous solutions of the Cauchy–Riemann equation in strongly pseudoconvex domains are almost 1-Hölder continuous in the complex tangent directions, and a similar result is known for finite-type domains in $\mathbb{C}^2$ (see [ChK]). In our case the situation is even more difficult because we have several complex tangential directions and, in contrast to the strongly pseudoconvex case, these directions cannot be handled in an isotropic way. We expect nevertheless to derive a solution that is $(1/m)$-Hölder continuous in the normal direction and satisfies better estimates in the complex tangential directions.

It turns out that such estimates are best expressed in terms of a certain pseudometric that is associated to the given domain D. So let $D = \{\rho < 0\} \subset \mathbb{C}^n$ be a bounded convex domain with $\mathcal{C}^\infty$-boundary of finite type m. We further assume

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that ρ is given in such a way that the domains $D_\zeta := \{z : \rho(z) < \rho(\zeta)\}$ are convex and of finite type m for all ζ in a certain neighborhood U of ∂D. Such a function could be given by $\rho(z) := \inf\{t > 0 : p + (z - p)/t \in D\} - 1$ for any $p \in D$. For $\zeta \in U$, an arbitrary direction vector v, and $\varepsilon < \varepsilon_0$, we define the complex directional level distances by

$$\tau(\zeta, v, \varepsilon) := \max\{c : |\rho(\zeta + \lambda v) - \rho(\zeta)| < \varepsilon \text{ for all } \lambda \in \mathbb{C},\ |\lambda| < c\}.$$

For a fixed point ζ and a fixed radius ε, we define the ε-extremal basis $(v_1, \dots, v_n)$ centered at ζ as in [Mc] or [H]. If it is important to mention the dependence on ζ and ε of the coordinates with respect to this basis, we denote their components by $z_{k,\zeta,\varepsilon}$. Let v_k be a unit vector in the $z_{k,\zeta,\varepsilon}$-direction and write $\tau_k(\zeta, \varepsilon) := \tau(\zeta, v_k, \varepsilon)$. We can now define the polydiscs

$$AP_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : |z_{k,\zeta,\varepsilon}| \le A\tau_k(\zeta, \varepsilon) \text{ for } k = 1, \dots, n\}.$$

(Note that the factor A in front means blowing up the polydisc around its center and not just multiplying each point by A.)

Using these polydiscs, we define the pseudodistance

$$d(\zeta, z) := \inf\{\varepsilon : z \in P_\varepsilon(\zeta)\}.$$

Now we can state the main result of this paper.

THEOREM 1.2. *Let D, T_q, and f be as in Theorem 1.1. Then, for every $\varepsilon > 0$, there exists a constant C such that the solution $u := T_q f$ of the Cauchy–Riemann equation $\bar\partial u = f$ satisfies the following nonisotropic Hölder estimate*:

$$|T_q f(z_0) - T_q f(z_1)| \le C\|f\|_\infty \max\{d(z_0, z_1)^{1/m}, |z_0 - z_1|^{1-\varepsilon}\}.$$

If we choose $0 < \varepsilon < 1/m$ and use the fact that $|z_0 - z_1|^m \lesssim d(z_0, z_1) \lesssim |z_0 - z_1|$ then we see that, for small values of $|z_0 - z_1|$, the term $d(z_0, z_1)^{1/m}$ will be larger than $|z_0 - z_1|^{1-\varepsilon}$ except when $d(z_0, z_1) \approx |z_0 - z_1|^m$. But this can happen only if z_0 is of type m and if the direction to z_1 is a direction where the order of contact with the tangent space is also maximal. In all other cases we obtain the estimate by $d(z_0, z_1)^{1/m}$.

In fact, it might be possible to avoid the term $|z_0 - z_1|^{1-\varepsilon}$ altogether. That could be done by constructing a special nonisotropic Hölder space out of one-dimensional Hölder spaces of different order, where the Hölder space of order 1 receives some special treatment. But this construction would be quite difficult (owing to the complexity of the geometry of convex domains of finite type), so we decided to avoid it and state the result as shown here.

Finally, we observe that $d(z_0, z_1) \lesssim |z_0 - z_1|$ implies that the maximum in Theorem 1.2 can be estimated by $C'|z_0 - z_1|^{1/m}$. Hence Theorem 1.2 also implies Theorem 1.1.

This paper is organized as follows. In Section 2 we recall the definition of the solution operator from [DFFo]; we split it into several parts, which will be estimated differently. We also state the results of these estimates as Lemma 2.1 and use them to prove Theorem 1.2. Section 3 contains the proof of Lemma 2.1, which is done in a series of smaller statements. In Section 4 we finally give an example to further clarify the result and to show that (in a certain sense) it is the best possible.

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2. Solution Operator

In this paper we use exactly the same integral operator as in [DFFo]. For the convenience of the reader we recall some details of its definition.

We write $l_\zeta(z) = \Phi(\zeta)(z-\zeta)$, where $\Phi(\zeta)$ is a unitary matrix depending smoothly on $\zeta \in \partial D$ such that the unit outer normal vector to ∂D will be turned into $(1, 0, \ldots, 0)$. The following definitions are as in [DFo]:

$$r_\zeta(w) := \rho(l_\zeta^{-1}(w)), \qquad a_\alpha(\zeta) := \frac{1}{\alpha!}\frac{\partial^{|\alpha|} r_\zeta}{\partial w^\alpha}(0);$$

$$S_\zeta(w) := 3w_1 + Kw_1^2 - c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j \\ \alpha_1=0}} a_\alpha(\zeta) w^\alpha \tag{1}$$

for K and M suitably large, c suitably small, and $\sigma_j = \operatorname{Re} i^j$. Now put

$$S(z,\zeta) := S_\zeta(l_\zeta(z)).$$

Next we want to define a decomposition of $S(z,\zeta)$ such that

$$S(z,\zeta) = \langle Q(z,\zeta), z-\zeta\rangle = \sum_{j=1}^{n} Q_j(z,\zeta)(z_j - \zeta_j).$$

For this we simply define

$$Q_\zeta^1(w) := 3 + Kw_1 \tag{2}$$

and (for $k > 1$)

$$Q_\zeta^k(w) := -c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j \\ \alpha_1=0,\alpha_k>0}} \frac{\alpha_k}{j} a_\alpha(\zeta)\frac{w^\alpha}{w_k} \tag{3}$$

and then set

$$Q(z,\zeta) := \Phi^T(\zeta) Q_\zeta(l_\zeta(z)).$$

Now we define Cauchy–Fantappiè integral operators R_q based on the support function S and its Hefer decomposition $Q(z,\zeta)$. We define the Cauchy–Fantappiè form

$$W(z,\zeta) := \sum_i \frac{Q_i(z,\zeta)}{S(z,\zeta)} d\zeta_i.$$

Let

$$B = \frac{b}{|\zeta - z|^2} = \sum_i \frac{\bar{\zeta}_i - \bar{z}_i}{|\zeta - z|^2} d\zeta_i$$

be the usual Martinelli–Bochner form and let K_q be the well-known Martinelli–Bochner operator. Further define

$$\begin{aligned} R_q f &:= \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta \in \partial D} f \wedge W \wedge B \wedge (\bar{\partial}_\zeta W)^k \wedge (\bar{\partial}_\zeta B)^{n-q-k-2} \wedge (\bar{\partial}_z B)^q \\ &= \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta \in \partial D} f \wedge \frac{Q \wedge b \wedge (\bar{\partial}_\zeta Q)^k \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)}}. \end{aligned} \tag{4}$$

In the last line we used the convention of denoting the (1, 0)-form $\sum_i Q_i(z, \zeta) d\zeta_i$ again by Q. Because S is a support function and hence $S(\zeta, z) \neq 0$ on $\partial D \times D$, the standard arguments (see e.g. [DFoW; LMi; R]) show that the operators $T_q = R_q + K_q$ are solution operators, which means that $\bar{\partial} T_q f = f$ for all $\bar{\partial}$-closed $(0, q+1)$-forms on D.

The Martinelli–Bochner operator is known to satisfy isotropic α-Hölder estimates for all $\alpha < 1$. In particular this implies our nonisotropic estimates, so it only remains to estimate $R_q f$. In order to do so we consider z-derivatives of this form. Let δ_γ be the z-derivative in the γ-direction. It is easy to see that $\delta_\gamma \bar{\partial}_\zeta b = \delta_\gamma \bar{\partial}_z b = 0$. Thus $\delta_\gamma R_q f$ can be written as a sum of integrals of the form

$$\int_{\partial D} f \wedge \frac{\delta_\gamma Q \wedge b \wedge (\bar{\partial}_\zeta Q)^k \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge \delta_\gamma b \wedge (\bar{\partial}_\zeta Q)^k \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge \delta_\gamma(\bar{\partial}_\zeta Q) \wedge (\bar{\partial}_\zeta Q)^{k-1} \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge (\bar{\partial}_\zeta Q)^k \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+2} |\zeta - z|^{2(n-k-1)}} \delta_\gamma S,$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge (\bar{\partial}_\zeta Q)^k \wedge (\bar{\partial}_\zeta b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)+1}} \delta_\gamma |\zeta - z|,$$

where the third integral appears only for $k > 0$.

Now we want to estimate these integrals. First we observe that f is bounded and thus $|f|$ can be estimated by $\|f\|_\infty$. Next we observe that $\delta_\gamma b \lesssim 1$ and $\delta_\gamma |\zeta - z| \lesssim 1$. Hence the second and the fifth integral can both be estimated by $\|f\|_\infty I_a$ with

$$I_a := \int_{\partial D} \frac{|(Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta)_t|}{|S|^{k+1} |\zeta - z|^{2(n-k-1)}} d\sigma_{2n-1}.$$

Here β is a differential form that contains all the remaining $d\zeta_j$ and $d\bar{\zeta}_j$ such that $Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta$ is of bidegree $(n, n-1)$ in ζ, and $(\omega)_t$ denotes the tangential part of the form ω, which is the only part of the form that contributes to the integral over ∂D.

For the other three integrals, we need to consider

$$I_b := \int_{\partial D} \frac{|(\delta_\gamma Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta)_t|}{|S|^{k+1}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1},$$

$$I_c := \int_{\partial D} \frac{|(Q \wedge \delta_\gamma(\bar{\partial}_\zeta Q) \wedge (\bar{\partial}_\zeta Q)^{k-1} \wedge \beta)_t|}{|S|^{k+1}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1},$$

$$I_d := \int_{\partial D} \frac{|(Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta)_t||\delta_\gamma S|}{|S|^{k+2}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1}.$$

We can now formulate the integral estimates that are essential for the proof of our main theorem.

LEMMA 2.1. *If* $\rho = |\rho(z)|$ *and* $0 < \sigma \le 1$, *then*

$$|I_a| \lesssim \rho^{\sigma(1/m-1)}.$$

The remaining integrals satisfy the estimate

$$|I_b|, |I_c|, |I_d| \lesssim \frac{\rho^{1/m}}{\tau(z, \gamma, \rho)} + |\log \rho|.$$

Note that, since $\rho \lesssim \tau(z, \gamma, \rho) \lesssim \rho^{1/m}$, the lemma also implies

$$|I_a|, |I_b|, |I_c|, |I_d| \lesssim \frac{\rho^{1/m}}{\tau(z, \gamma, \rho)}$$

except if $\tau(z, \gamma, \rho) \approx \rho^{1/m}$, in which case

$$|I_a|, |I_b|, |I_c|, |I_d| \lesssim \frac{\rho^{(1-\varepsilon)/m}}{\tau(z, \gamma, \rho)}$$

for $\varepsilon = \sigma(m-1) > 0$.

The proof of this lemma will be given in Section 3. Now we come to the proof of our main theorem.

Proof of Theorem 1.2. For simplicity we write $u(z) = R_q f(z)$. Let $A = d(z_0, z_1)$, let $\gamma = (z_1 - z_0)/|z_1 - z_0|$, and let ν be the inward normal direction at $\zeta_0 = \pi(z_0)$. Consider the additional points $\tilde{z}_0 = z_0 + A\nu$ and $\tilde{z}_1 = z_1 + A\nu$. Now we can estimate

$$\begin{aligned} |u(z_0) - u(z_1)| &\le |u(z_0) - u(\tilde{z}_0)| + |u(\tilde{z}_0) - u(\tilde{z}_1)| + |u(\tilde{z}_1) - u(z_1)| \\ &\le \int_{z_0}^{\tilde{z}_0} |\delta_\nu u(t)|\, dt + \int_{\tilde{z}_0}^{\tilde{z}_1} |\delta_\gamma u(t)|\, dt + \int_{z_1}^{\tilde{z}_1} |\delta_\nu u(t)|\, dt. \end{aligned}$$

In the first and the third integral we have the (worst) case that ν is approximately the normal direction and thus $\tau(t, \nu, |\rho(t)|) \approx |\rho(t)|$. Nevertheless, we get the estimate

$$\int_{z_0}^{\tilde{z}_0} |\delta_\nu u(t)|\,dt \le \int_0^A |\delta_\nu u(z_0+s\nu)|\,ds \lesssim \int_0^A s^{1/m-1}\,ds \lesssim A^{1/m},$$

and the same is true for the third integral.

To estimate the second integral, first observe that $d(\tilde{z}_0,\tilde{z}_1) \lesssim d(\tilde{z}_0,z_0) + d(z_0,z_1) + d(z_1,\tilde{z}_1) \lesssim A$. Hence $\tilde{z}_1$ and the whole line from $\tilde{z}_0$ to $\tilde{z}_1$ belong to some $P_{CA}(\tilde{z}_0)$ and so $\tau(\zeta,\gamma,A) \approx \tau(\tilde{z}_0,\gamma,A)$ for all those ζ. On the other hand, $|\rho(\tilde{z}_0)| \gtrsim A$ and therefore $|\delta_\gamma u(\zeta)| \lesssim A^{1/m}/\tau(\tilde{z}_0,\gamma,A)$ for all ζ on the line to $\tilde{z}_1$. Therefore,

$$\begin{aligned}\int_{\tilde{z}_0}^{\tilde{z}_1} |\delta_\gamma u(t)|\,dt &\lesssim \int_0^{\tau(\tilde{z}_0,\gamma,A)} |\delta_\gamma u(\tilde{z}_0+s\gamma)|\,ds \\ &\lesssim \int_0^{\tau(\tilde{z}_0,\gamma,A)} \frac{A^{1/m}}{\tau(\tilde{z}_0,\gamma,A)}\,ds \lesssim A^{1/m}.\end{aligned}$$

The same argument applies in the exceptional case, but we first replace $A^{1/m}$ by $A^{(1-\varepsilon)/m}$ and finally get

$$A^{(1-\varepsilon)/m} = d(z_0,z_1)^{(1-\varepsilon)/m} \approx (|z_0-z_1|^m)^{(1-\varepsilon)/m} \approx |z_0-z_1|^{1-\varepsilon}. \qquad \square$$

3. Estimates

In this section we prove Lemma 2.1, which means that we must estimate the integrals $I_a,\dots,I_d$. We will proceed in several steps. In Section 3.1 we estimate Q, $\bar{\partial}Q$, and their derivatives. In Section 3.2 we put these estimates together to obtain estimates for $|(Q\wedge(\bar{\partial}_\zeta Q)^k\wedge\beta)_t|$ and similar terms; we also give an estimate for $|\delta_\gamma S|$. Finally, Section 3.3 contains the estimates of the integrals, which are derived from the estimates of Section 3.2.

In obtaining our estimates, we will frequently use the following setup. Fix a point $\zeta_0 \in \partial D$ and some $\varepsilon > 0$. Denote the ε-extremal coordinates at ζ_0 by w^* and let Φ^* be the unitary transformation such that $w^* = \Phi^*(z-\zeta_0)$. Also define $\eta^* := \Phi^*(\zeta-\zeta_0)$. We say that z and ζ (or w^* and η^*, respectively) satisfy condition $(*)$ if we have $|\eta_j^*| \le C^*\tau(\zeta_0,\varepsilon)$ for all j, $|w_1^*| \le C^*$, and $|w_j^*| \le C^*\tau_j(\zeta_0,\varepsilon)$ for all $j \ge 2$. Note that the condition is satisfied if $\zeta_0 = \pi(z)$ and $\zeta \in P_\varepsilon(\zeta_0)$.

3.1. Q-Estimates

Using the setup just described, we can write Q with respect to the ε-extremal coordinates at ζ_0 as

$$Q^*(w^*,\eta^*) := \bar{\Phi}^* Q(\zeta_0 + (\bar{\Phi}^*)^T w^*, \zeta_0 + (\bar{\Phi}^*)^T \eta^*).$$

The components of this vector will be denoted by $Q_j^*(w^*,\eta^*)$ for $j = 1,\dots,n$. Here is the main result of this section.

LEMMA 3.1. *For all w^* and η^* satisfying condition $(*)$,*

$$|Q_k^*(w^*,\eta^*)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)}, \tag{5}$$

$$\left|\frac{\partial}{\partial w_i^*} Q_k^*(w^*,\eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)}, \tag{6}$$

$$\left|\frac{\partial}{\partial \eta_j^*} Q_k^*(w^*,\eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)}, \tag{7}$$

$$\left|\frac{\partial^2}{\partial w_i^*\partial \eta_j^*} Q_k^*(w^*,\eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)}. \tag{8}$$

Before we start estimating Q and its derivatives, recall that the definition of Q is independent of the special choice of l_ζ because an additional rotation in the complex tangent space does not change it (see [DFFo, Lemma 2.1]). We want to make use of this fact by providing locally a special version of l_ζ that can then be used in the estimates. This is done just as in [DFFo], so we give only a short summary here.

For all η^* satisfying condition $(*)$, there exists a matrix $\Psi(\eta^*)$, depending smoothly on η^*, such that $\Phi(\zeta) := \Psi(\Phi^*(\zeta-\zeta_0))\Phi^*$ has the desired properties and the following proposition holds.

Proposition 3.2. *For all η^* satisfying condition $(*)$,*

$$c \le |\psi_{kk}(\eta^*)| \le 1 \quad \text{and} \quad |\psi_{\nu k}(\eta^*)| \lesssim \frac{\varepsilon^2}{\tau_\nu(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)} \quad \text{for } \nu \neq k$$

and

$$\left|\frac{\partial}{\partial \bar\eta_j^*}\psi_{kk}(\eta^*)\right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial}{\partial \bar\eta_j^*}\psi_{\nu k}(\eta^*)\right| \lesssim \frac{\varepsilon^2}{\tau_j(\zeta_0,\varepsilon)\tau_\nu(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)} \quad \text{for } \nu \neq k.$$

See [DFFo, Lemma 5.2] for a proof of this statement and also observe that we have $\Phi(\zeta_0 + (\bar\Phi^*)^T\eta^*) = \Psi(\eta^*)\Phi^*$. We will also need the following result.

Proposition 3.3. *For all ζ in $P_\varepsilon(\zeta_0)$ and $v_j(\zeta) := \bar\Phi^T(\zeta)e_j$ we have*

$$\tau(\zeta, v_j(\zeta), \varepsilon) \approx \tau_j(\zeta_0,\varepsilon)$$

with constants independent of ζ, ζ_0, and ε.

This proposition has been proved as Lemma 5.3 in [DFFo]. Another basic ingredient in our estimates will be as follows.

Proposition 3.4. *Let w be any orthonormal coordinate system centered at z, and let v_j be the unit vector in the w_j-direction. Then we have*

$$\left|\frac{\partial^{|\alpha+\beta|}\rho(z)}{\partial w^\alpha \partial \bar{w}^\beta}\right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j+\beta_j}}$$

for all multi-indices α and β with $|\alpha+\beta| \geq 1$.

This proposition can be found for instance in [BCDu] or [DFFo].

Now we want to use the definition of Q^*, the definition of Q itself, and our special choice of $\Phi(\zeta)$. Using the abbreviations $\zeta := \zeta_0 + (\bar{\Phi}^*)^T \eta^*$ and $w := \Psi(\eta^*)(w^* - \eta^*)$ yields

$$Q_k^*(w^*, \eta^*) = \sum_{\mu=1}^n \psi_{\mu k}(\eta^*) Q_\zeta^\mu(w),$$

$$\frac{\partial}{\partial w_i^*} Q_k^*(w^*, \eta^*) = \sum_{\mu=1}^n \psi_{\mu k}(\eta^*) \frac{\partial Q_\zeta^\mu(w)}{\partial w} \frac{\partial w}{\partial w_i^*},$$

$$\frac{\partial}{\partial \eta_j^*} Q_k^*(w^*, \eta^*) = \sum_{\mu=1}^n \frac{\partial \psi_{\mu k}(\eta^*)}{\partial \eta_j^*} Q_\zeta^\mu(w) + \psi_{\mu k}(\eta^*) \frac{\partial Q_\zeta^\mu(w)}{\partial \zeta} \frac{\partial \zeta}{\partial \eta_j^*} + \psi_{\mu k}(\eta^*) \frac{\partial Q_\zeta^\mu(w)}{\partial w} \frac{\partial w}{\partial \eta_j^*},$$

$$\frac{\partial^2}{\partial w_i^* \partial \eta_j^*} Q_k^*(w^*, \eta^*) = \sum_{\mu=1}^n \frac{\partial \psi_{\mu k}(\eta^*)}{\partial \eta_j^*} \frac{\partial Q_\zeta^\mu(w)}{\partial w} \frac{\partial w}{\partial w_i^*} + \psi_{\mu k}(\eta^*) \frac{\partial^2 Q_\zeta^\mu(w)}{\partial w \partial \zeta} \frac{\partial \zeta}{\partial \eta_j^*} \frac{\partial w}{\partial w_i^*} + \psi_{\mu k}(\eta^*) \frac{\partial^2 Q_\zeta^\mu(w)}{\partial w^2} \frac{\partial w}{\partial \eta_j^*} \frac{\partial w}{\partial w_i^*} + \psi_{\mu k}(\eta^*) \frac{\partial Q_\zeta^\mu(w)}{\partial w} \frac{\partial^2 w}{\partial w_i^* \partial \eta_j^*}.$$

Moreover, we have

$$\frac{\partial \zeta_i}{\partial \eta_j^*} = \bar{\phi}_{ji}^*, \quad \frac{\partial w_i}{\partial w_j^*} = \psi_{ij}(\eta^*), \quad \frac{\partial^2 w_i}{\partial w_j^* \partial \eta_k^*} = \frac{\partial \psi_{ij}(\eta^*)}{\partial \eta_k^*},$$

and

$$\frac{\partial w_i}{\partial \eta_j^*} = \sum_{\lambda=1}^n \frac{\partial \psi_{i\lambda}}{\partial \eta_j^*}(\eta^*)(w_\lambda^* - \eta_\lambda^*) - \psi_{ij}(\eta^*).$$

Furthermore, if $\mu = 1$ then

$$Q_\zeta^1(w) = 3 + K w_1, \quad \frac{\partial}{\partial w_1} Q_\zeta^1(w) = K, \quad \text{and} \quad \frac{\partial}{\partial w_\nu} Q_\zeta^1(w) = 0 \;\; (\nu > 1) \tag{9}$$

and of course it follows that

$$\frac{\partial}{\partial \zeta} Q^1_\zeta(w) = \frac{\partial^2}{\partial w \partial \zeta} Q^1_\zeta(w) = \frac{\partial^2}{\partial w^2} Q^1_\zeta(w) = 0. \tag{10}$$

For $\mu > 1$ we have

$$Q^\mu_\zeta(w) := \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} c_\mu a_\alpha(\zeta) \frac{w^\alpha}{w_\mu},$$

$$\frac{\partial}{\partial w_i} Q^\mu_\zeta(w) = \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} c_{\mu i} a_\alpha(\zeta) \frac{w^\alpha}{w_i w_\mu},$$

$$\frac{\partial^2}{\partial w_i \partial w_j} Q^\mu_\zeta(w) = \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} c_{\mu ij} a_\alpha(\zeta) \frac{w^\alpha}{w_i w_j w_\mu},$$

$$\frac{\partial}{\partial \zeta_j} Q^\mu_\zeta(w) = \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} c_\mu \frac{\partial a_\alpha(\zeta)}{\partial \zeta_j} \frac{w^\alpha}{w_\mu},$$

$$\frac{\partial^2}{\partial w_i \partial \zeta_j} Q^\mu_\zeta(w) = \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} c_{\mu i} \frac{\partial a_\alpha(\zeta)}{\partial \zeta_j} \frac{w^\alpha}{w_i w_\mu},$$

where the c_* are certain constants that depend on ν, α, and the given indices but are not essential in our estimates. We see that it is important to obtain estimates for $|a_\alpha(\zeta)|$ and $|\partial a_\alpha(\zeta)/\partial \zeta_j|$.

LEMMA 3.5. *For all $\zeta \in P_\varepsilon(\zeta_0)$ we have the estimate*

$$|a_\alpha(\zeta)| \lesssim \frac{\varepsilon}{\tau^\alpha(\zeta_0, \varepsilon)},$$

where $\tau^\alpha(\zeta_0, \varepsilon)$ is shorthand for $\prod_{i=1}^n \tau_i(\zeta_0, \varepsilon)^{\alpha_i}$.

Proof. As in Proposition 3.3, we set $v_j := \bar{\Phi}^T(\zeta) e_j$. Then Proposition 3.4 together with Proposition 3.3 gives

$$|a_\alpha(\zeta)| = \left| \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial w^\alpha} \rho(\zeta + \bar{\Phi}^T(\zeta) w) \big|_{w=0} \right| \lesssim \frac{\varepsilon}{\prod_{i=1}^n \tau(\zeta, v_i, \varepsilon)^{\alpha_i}} \approx \frac{\varepsilon}{\tau^\alpha(\zeta_0, \varepsilon)}. \qquad \square$$

Estimating the derivative of $a_\alpha(\zeta)$ is much more difficult. We get the following result.

LEMMA 3.6. *For all $\zeta \in P\varepsilon(\zeta_0)$ we have the estimate*

$$\left| \sum_{\lambda=1}^{n} \bar{\phi}^*_{j\lambda} \frac{\partial a_\alpha(\zeta)}{\partial \zeta_\lambda} \right| \lesssim \frac{\varepsilon}{\tau^\alpha(\zeta_0, \varepsilon) \tau_j(\zeta_0, \varepsilon)},$$

where again $\tau^\alpha(\zeta_0, \varepsilon)$ is shorthand for $\prod_{i=1}^n \tau_i(\zeta_0, \varepsilon)^{\alpha_i}$.

Proof. Here again we want to make use of Proposition 3.4. In order to do so we first observe that the ζ derivative of ρ can be written in terms of w-derivatives. To be precise, we have

$$\begin{aligned}\frac{\partial a_\alpha(\zeta)}{\partial \zeta_j} &= \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial w^\alpha}\frac{\partial}{\partial \zeta_j}\rho(\zeta+\bar\Phi^T(\zeta)w)\big|_{w=0}\\ &= \sum_{\tau=1}^n \frac{1}{\alpha!}\phi_{\tau j}(\zeta)\frac{\partial^{|\alpha|+1}}{\partial w^{\alpha+(\tau)}}\rho(\zeta+\bar\Phi^T(\zeta)w)\big|_{w=0}\\ &\quad+\sum_{\tau=1}^n\sum_{\substack{i=1\\ \alpha_i>0}}^n\frac{\alpha_i}{\alpha!}\left(\sum_{\lambda=1}^n\phi_{\tau\lambda}(\zeta)\frac{\partial\bar\phi_{i\lambda}(\zeta)}{\partial\zeta_j}\right)\frac{\partial^{|\alpha|}}{\partial w^{\alpha-(i)+(\tau)}}\rho(\zeta+\bar\Phi^T(\zeta)w)\big|_{w=0},\end{aligned}$$

where $\alpha-(i)+(\tau)$ should be the multi-index that is derived from α by decreasing α_i by 1 and increasing α_τ by 1.

Next we remember our special choice of Φ and see that

$$\begin{aligned}&\sum_{\lambda=1}^n\bar\phi^*_{j\lambda}\frac{\partial a_\alpha(\zeta)}{\partial\zeta_\lambda}\\ &\quad=\sum_{\tau=1}^n\psi_{\tau j}(\eta^*)\frac{1}{\alpha!}\frac{\partial^{|\alpha|+1}}{\partial w^{\alpha+(\tau)}}\rho(\zeta+\bar\Phi^T(\zeta)w)\big|_{w=0}\\ &\quad\quad+\sum_{\tau,\mu=1}^n\sum_{\substack{i=1\\ \alpha_i>0}}^n\psi_{\tau\mu}(\eta^*)\frac{\partial\bar\psi_{i\mu}(\eta^*)}{\partial\eta^*_j}\frac{\alpha_i}{\alpha!}\frac{\partial^{|\alpha|}}{\partial w^{\alpha-(i)+(\tau)}}\rho(\zeta+\bar\Phi^T(\zeta)w)\big|_{w=0}.\end{aligned}$$

Finally, the result follows from Proposition 3.4, Proposition 3.3, and Proposition 3.2. □

Now we can start to put together some of the estimates.

LEMMA 3.7. *For all w^* and η^* satisfying condition* $(*)$,

$$|Q^\mu_\zeta(w)|\lesssim\frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial w}{\partial w^*_i}\right|\lesssim\frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial \zeta}\frac{\partial \zeta}{\partial \eta^*_j}\right|\lesssim\frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial w}{\partial \eta^*_j}\right|\lesssim\frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial^2 Q^\mu_\zeta(w)}{\partial w\partial\zeta}\frac{\partial\zeta}{\partial\eta^*_j}\frac{\partial w}{\partial w^*_i}\right|\lesssim\frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial^2 Q^\mu_\zeta(w)}{\partial w^2}\frac{\partial w}{\partial \eta^*_j}\frac{\partial w}{\partial w^*_i}\right| \lesssim \frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial^2 w}{\partial w^*_i\partial \eta^*_j}\right| \lesssim \frac{\varepsilon}{\tau_\mu(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)}.$$

Proof. First we want to establish an estimate for $\partial w_i/\partial\eta^*_j$. Condition $(*)$ implies that $(w^*_\tau - \eta^*_\tau)$ is bounded. From Proposition 3.2 and the fact that $\varepsilon/\tau_\tau(\zeta_0,\varepsilon) \lesssim 1$ it follows that

$$\begin{aligned}\left|\frac{\partial w_\lambda}{\partial \eta^*_j}\right| &\lesssim \left|\sum_{\tau=1}^{n}\frac{\partial\psi_{\lambda\tau}}{\partial\eta^*_j}(\eta^*)(w^*_\tau-\eta^*_\tau) - \psi_{\lambda j}(\eta^*)\right| \\ &\lesssim \left(\delta^\lambda_j + \frac{\varepsilon}{\tau_\lambda(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)}\right). \end{aligned} \tag{11}$$

In all estimates the case $\mu = 1$ must be treated separately. But using (9) and (10) we see that some of the terms vanish, and for the remaining terms we can use the fact that $\tau_1(\zeta_0,\varepsilon) \approx \varepsilon$ and thus every bounded term can be estimated by $\varepsilon/\tau_1(\zeta_0,\varepsilon)$. Together with (11) and Proposition 3.2, this gives the desired result for $\mu = 1$.

In the other cases we have

$$|Q^\mu_\zeta(w)| \lesssim \sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}|a_\alpha(\zeta)|\left|\frac{w^\alpha}{w_\mu}\right|,$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial w}{\partial w^*_i}\right| \lesssim \sum_{\lambda=2}^{n}\sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}|a_\alpha(\zeta)|\left|\frac{w^\alpha}{w_\lambda w_\mu}\right||\psi_{\lambda i}(\eta^*)|,$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial \zeta}\frac{\partial \zeta}{\partial \eta^*_j}\right| \lesssim \sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}\left|\sum_{\lambda=1}^{n}\bar\phi^*_{j\lambda}\frac{\partial a_\alpha(\zeta)}{\partial\zeta_\lambda}\right|\left|\frac{w^\alpha}{w_\mu}\right|,$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial w}{\partial \eta^*_j}\right| \lesssim \sum_{\lambda=2}^{n}\sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}|a_\alpha(\zeta)|\left|\frac{w^\alpha}{w_\lambda w_\mu}\right|\left|\frac{\partial w_\lambda}{\partial\eta^*_j}\right|,$$

$$\left|\frac{\partial^2 Q^\mu_\zeta(w)}{\partial w\partial\zeta}\frac{\partial \zeta}{\partial \eta^*_j}\frac{\partial w}{\partial w^*_i}\right| \lesssim \sum_{\tau=2}^{n}\sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}\left|\sum_{\lambda=1}^{n}\bar\phi^*_{j\lambda}\frac{\partial a_\alpha(\zeta)}{\partial\zeta_\lambda}\right|\left|\frac{w^\alpha}{w_\tau w_\mu}\right||\psi_{\tau i}(\eta^*)|,$$

$$\left|\frac{\partial^2 Q^\mu_\zeta(w)}{\partial w^2}\frac{\partial w}{\partial \eta^*_j}\frac{\partial w}{\partial w^*_i}\right| \lesssim \sum_{\lambda=2}^{n}\sum_{\tau=2}^{n}\sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}|a_\alpha(\zeta)|\left|\frac{w^\alpha}{w_\lambda w_\tau w_\mu}\right|\left|\frac{\partial w_\lambda}{\partial\eta^*_j}\right||\psi_{\tau i}(\eta^*)|,$$

$$\left|\frac{\partial Q^\mu_\zeta(w)}{\partial w}\frac{\partial^2 w}{\partial w^*_i\partial \eta^*_j}\right| \lesssim \sum_{\lambda=2}^{n}\sum_{\nu=2}^{m}\sum_{\substack{|\alpha|=\nu\\ \alpha_1=0}}|a_\alpha(\zeta)|\left|\frac{w^\alpha}{w_\lambda w_\mu}\right|\left|\frac{\partial\psi_{\lambda i}(\eta^*)}{\partial\eta^*_j}\right|.$$

It is easy to see that condition $(*)$ also implies $|w_j| \lesssim \tau_j(\zeta_0, \varepsilon)$ for $j > 1$. Since $\alpha_1 = 0$ a term of the form w^α / w_k can thus be estimated by $\tau^{\alpha-(k)}(\zeta_0, \varepsilon)$ and this cancels nicely with part of the estimate for a_α. Together with (11), Proposition 3.2, and the estimate $\varepsilon/\tau_\tau(\zeta_0, \varepsilon) \lesssim 1$, this yields the desired results—except that, in the forth and sixth estimates, we must also use the fact that $\lambda > 1$ and hence $\varepsilon/\tau_\lambda(\zeta_0, \varepsilon)^2 \lesssim 1$. □

Proof of Lemma 3.1. The result now follows in a straightforward way from Lemma 3.7 and Proposition 3.2. □

3.2. More Estimates

Now we want to put together the estimates of Section 3.1 and so obtain some estimate of part of the kernel. We point out that now it is important to remember that Q was in fact the (1, 0)-form $\sum_{j=1}^n Q_j(z, \zeta) d\zeta_j$. As in the introduction, β should be a form of norm 1 that contains all remaining components such that $Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta$ is always of bidegree $(n, n-1)$ in ζ. Again we denote the tangential component of a differential form α by $(\alpha)_t$.

LEMMA 3.8. *For all z and ζ satisfying condition $(*)$, the term*

$$|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t|$$

can be estimated by a sum of terms of the form

$$E^k_{\mu\nu} = \frac{\varepsilon^k}{\prod_{i=1}^k \tau_{\mu_i}(\zeta_0, \varepsilon)\tau_{\nu_i}(\zeta_0, \varepsilon)},$$

where all μ_i and ν_i are greater than 1 *and each index appears at most once.*

Moreover, if δ_γ is the z-derivative in the γ-direction, then the terms

$$|(\delta_\gamma Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t| \quad \text{and} \quad |(Q \wedge (\delta_\gamma \bar\partial_\zeta Q) \wedge (\bar\partial_\zeta Q)^{k-1} \wedge \beta)_t|$$

can be estimated by a sum of terms of the form $E^k_{\mu\nu}/\tau(\zeta_0, \gamma, \varepsilon)$, where the μ_i and ν_i are as above.

Before we start the proof, we must mention the following fact.

PROPOSITION 3.9. *Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a unit vector written with respect to the ε-extremal coordinates at ζ_0. Then*

$$\frac{1}{\tau(\zeta_0, \gamma, \varepsilon)} \approx \sum_{j=1}^n \frac{|\gamma_j|}{\tau_j(\zeta_0, \varepsilon)}.$$

The proof of this proposition can be found, for instance, in [Mc].

Proof of Lemma 3.8. We can write the term $Q \wedge (\bar\partial_\zeta Q)^k$ with respect to the ε-extremal coordinates at ζ_0 and get

$$Q \wedge (\bar\partial_\zeta Q)^k = \sum_{\nu_0=1}^n Q^*_{\nu_0}(w^*, \eta^*) d\eta^*_{\nu_0} \wedge \bigwedge_{l=1}^k \sum_{\nu_l, \mu_l = 1}^n \frac{\partial Q^*_{\nu_l}(w^*, \eta^*)}{\partial \bar\eta^*_{\mu_l}} d\bar\eta^*_{\mu_l} \wedge d\eta^*_{\nu_l},$$

which shows immediately that the $\nu_0, \dots, \nu_k$ are pairwise distinct and that the $\mu_1, \dots, \mu_k$ are pairwise distinct. Using estimates (5) and (7) from Lemma 3.1 yields the estimate

$$|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t| \lesssim \frac{\varepsilon}{\tau_{\nu_0}(\zeta_0, \varepsilon)} \prod_{l=1}^{k} \frac{\varepsilon}{\tau_{\mu_l}(\zeta_0, \varepsilon)\tau_{\nu_l}(\zeta_0, \varepsilon)}.$$

If one of the $\nu_0, \dots, \nu_l$ is equal to 1 then we simply estimate $\varepsilon/\tau_1(\zeta_0, \varepsilon) \lesssim 1$.

If, however, one of the $\mu_1, \dots, \mu_k$ is equal to 1 then the estimate becomes a little more difficult. In this case the kernel contains the term $d\bar\eta_1^*$, but at ζ_0 this form has no tangential component and for $\zeta \in P_\varepsilon(\zeta_0)$ the tangential component of this term remains small. More precisely, we can write

$$d\bar\eta_1^* = \left(\bar\partial\rho - \sum_{j=2}^{n} \frac{\partial\rho}{\partial\bar\eta_j^*} d\bar\eta_j^*\right)\left(\frac{\partial\rho}{\partial\bar\eta_1^*}\right)^{-1}.$$

The last term is bounded by our assumption on ε (see the definition of Ψ), and the term $\bar\partial\rho$ does not contribute to the tangential component. Assuming without loss of generality that $\mu_1 = 1$, we have

$$\begin{aligned}
\left|\frac{\partial Q_{\nu_1}^*(w^*, \eta^*)}{\partial\bar\eta_1^*} d\bar\eta_1^*\right| &\lesssim \left|\frac{\partial Q_{\nu_1}^*(w^*, \eta^*)}{\partial\bar\eta_1^*} \sum_{j \notin \{\mu_2, \dots, \mu_k\}} \frac{\partial\rho}{\partial\bar\eta_j^*} d\bar\eta_j^*\right| \\
&\lesssim \frac{\varepsilon}{\tau_1(\zeta_0, \varepsilon)\tau_{\nu_1}(\zeta_0, \varepsilon)} \sum_{j \notin \{\mu_2, \dots, \mu_k\}} \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)} \\
&\lesssim \sum_{j \notin \{\mu_2, \dots, \mu_k\}} \frac{\varepsilon}{\tau_{\nu_1}(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)},
\end{aligned}$$

and this saves our estimate also in the case where one of the μ_l is equal to 1.

In order to estimate the remaining two terms, we first consider the case where γ is one of the ε-extremal directions at ζ_0. In this case we can use the same arguments as before, only replacing (5) and (7) from Lemma 3.1 by (6) and (8), respectively. We find that $|(\delta_j Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t|$ and $|(Q \wedge (\delta_j \bar\partial_\zeta Q) \wedge (\bar\partial_\zeta Q)^{k-1} \wedge \beta)_t|$ can be estimated by a sum of terms of the form $E_{\mu\nu}^k/\tau_j(\zeta_0, \varepsilon)$, where μ and ν are as before. Finally, we need only observe that, if $\gamma = (\gamma_1, \dots, \gamma_j)$ with respect to the ε-extremal coordinates at ζ_0, then $\delta_\gamma = \sum \gamma_j \delta_j$ and by Proposition 3.9 we also have

$$\frac{1}{\tau(\zeta_0, \gamma, \varepsilon)} = \sum_{j=1}^{n} \frac{|\gamma_j|}{\tau_j(\zeta_0, \varepsilon)},$$

which completes the proof. □

At this point we also want to give an estimate for $\delta_\gamma S$ that will be needed later. For this we first write S with respect to the ε-extremal coordinates at ζ_0 as

$$S^*(w^*, \eta^*) = S_{\zeta_0 + (\bar\Phi^*)^T \eta^*}(\Psi(\eta^*)(w^* - \eta^*)).$$

LEMMA 3.10. *Let δ_γ be the z-derivative in the γ-direction. Then, for all w^* and η^* satisfying condition $(*)$, we have*

$$|\delta_\gamma S^*(w^*,\eta^*)| \lesssim \frac{\varepsilon}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

Proof. First we consider the case where γ is one of the ε-extremal directions at ζ_0. Using the abbreviations $\zeta = \zeta_0 + (\bar{\Phi}^*)^T\eta^*$ and $w = \Psi(\eta^*)(w^* - \eta^*)$, we have

$$\frac{\partial}{\partial w_j^*} S^*(w^*,\eta^*) = \sum_{\lambda=1}^{n} \frac{\partial S_\zeta(w)}{\partial w_\lambda} \frac{\partial w_\lambda}{\partial w_j^*}$$

with

$$\frac{\partial S_\zeta(w)}{\partial w_1} = 3 + 2Kw_1 \quad \text{and} \quad \frac{\partial S_\zeta(w)}{\partial w_\lambda} = \sum_{\nu=2}^{m} \sum_{\substack{|\alpha|=\nu \\ \alpha_1=0}} \tilde{c}_\lambda a_\alpha(\zeta) \frac{w^\alpha}{w_\lambda}.$$

Using the same arguments as in the proof of Lemma 3.7 now yields

$$\left| \frac{\partial S^*(w^*,\eta^*)}{\partial w_j^*} \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0,\varepsilon)}.$$

To obtain the result of the lemma we need only observe that, if $\gamma = (\gamma_1,\dots,\gamma_j)$ with respect to the ε-extremal coordinates at ζ_0, then $\delta_\gamma = \sum \gamma_j \delta_j$; this, together with Proposition 3.9, completes the proof. □

3.3. Integral Estimates

Finally we come to the estimates of the integrals $I_a,\dots,I_d$. Because the only singularity in these integrals occurs for $\zeta = z$, it is clear that the integrals are uniformly bounded if $\operatorname{dist}(z,\partial D) > c$ or if the integration is only over the boundary outside some small neighborhood of $\pi(z)$. Thus it remains to estimate the integrals over some small neighborhood of $\zeta_0 = \pi(z)$. This neighborhood should be chosen small enough that we can use all the results obtained earlier. For simplicity let us assume that $P_1(\zeta_0)$ is such a neighborhood.

As in [DFFo], we will divide the neighborhood $P_1(\zeta_0)$ into smaller pieces by using certain polyannuli. These are defined by

$$P_\varepsilon^i(\zeta) := C_1 P_{2^{-i}\varepsilon}(\zeta) \setminus \tfrac{1}{2} P_{2^{-i}\varepsilon}(\zeta).$$

The constant C_1 was necessary to make it a covering. In fact, we have

$$\bigcup_{i=0}^{\infty} P_\varepsilon^i(\zeta) \supset P_\varepsilon(\zeta) \setminus \{\zeta\} \quad \text{and} \quad \bigcup_{i=0}^{i_0(\varepsilon)} P_1^i(\zeta) \supset P_1(\zeta) \setminus P_\varepsilon(\zeta), \tag{12}$$

where $i_0(\varepsilon)$ is a finite number depending only on ε and satisfying $i_0(\varepsilon) < -\log_2(c\varepsilon)$ for a certain small constant c.

The following proposition has been proved in [DFFo].

PROPOSITION 3.11. *Let $z \in D$ be close enough to the boundary and assume that ε is small enough. Then we have*

$$|S(z,\zeta)| \gtrsim \begin{cases} \varepsilon & \text{for all } \zeta \in \partial D \cap P_\varepsilon^0(\pi(z)), \\ |\rho(z)| & \text{for all } \zeta \in \partial D \cap P_{|\rho(z)|}(\pi(z)). \end{cases}$$

We must also mention the following proposition, which is implicit in [Mc] and more explicitly stated in [dBF].

PROPOSITION 3.12. *Let γ be an arbitrary direction and write*

$$r(z+\zeta\gamma)-r(z)=\sum_{\mu+\nu=1}^{m} a^{\gamma}_{\mu\nu}(z)\zeta^{\mu}\bar{\zeta}^{\nu}+O(|z|^{m+1}).$$

Now define $A^{\gamma}_k(z):=\max\{|a^{\gamma}_{\mu\nu}(z)|:\mu+\nu=k\}$ *and*

$$s_{\gamma}(z,\varepsilon):=\min_{1\le k\le m}\left\{\left(\frac{\varepsilon}{A^{\gamma}_k(z)}\right)^{1/k}\right\}.$$

Then

$$\tau(z,\gamma,\varepsilon)\approx s_{\gamma}(z,\varepsilon).$$

As a consequence of this proposition we have our next lemma.

LEMMA 3.13. *For every direction γ, every point z, and all values* $0<\varepsilon<\rho$,

$$\frac{\varepsilon}{\rho}\lesssim\frac{\tau(z,\gamma,\varepsilon)}{\tau(z,\gamma,\rho)}.$$

Proof. This result follows immediately from the fact that $s_{\gamma}(z,t)/t$ is a continuous decreasing function in t. □

LEMMA 3.14. *For every σ with* $0<\sigma\le 1$ *we have the estimate*

$$\int_{\partial D\cap P_{\varepsilon}(\zeta_0)}\frac{|(Q\wedge(\bar{\partial}_{\zeta}Q)^k\wedge\beta)_t|}{|\zeta-z|^{2n-2k-3+(1-\sigma)}}\,d\sigma_{2n-1}\lesssim\varepsilon^{\sigma/m+k+1}.$$

Moreover,

$$\int_{\partial D\cap P_{\varepsilon}(\zeta_0)}\frac{|(\delta_{\gamma}Q\wedge(\bar{\partial}_{\zeta}Q)^k\wedge\beta)_t|}{|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1}\lesssim\frac{\varepsilon^{1/m+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)},$$

$$\int_{\partial D\cap P_{\varepsilon}(\zeta_0)}\frac{|(Q\wedge\delta_{\gamma}(\bar{\partial}_{\zeta}Q)\wedge(\bar{\partial}_{\zeta}Q)^{k-1}\wedge\beta)_t|}{|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1}\lesssim\frac{\varepsilon^{1/m+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

Proof. We can write the first integral with respect to the ε-extremal coordinates at ζ_0, which are called $\eta^*_j=\theta^*_j+i\xi^*_j$. We then use the estimate $|\zeta-z|>|\zeta-\zeta_0|$ and Lemma 3.8 to obtain

$$\int_{|\xi^*_1|<\tau_1(\zeta_0,\varepsilon)}\int_{|\eta^*_2|<\tau_2(\zeta_0,\varepsilon)}\cdots\int_{|\eta^*_n|<\tau_n(\zeta_0,\varepsilon)}\frac{\varepsilon^k\,d\xi^*_1\,d\theta^*_2\,d\xi^*_2\cdots d\theta^*_n\,d\xi^*_n}{\prod_{l=1}^{k}\tau_{\mu_l}(\zeta_0,\varepsilon)\tau_{\nu_l}(\zeta_0,\varepsilon)\big(\sum_{j=1}^{n}|\eta^*_j|\big)^{2n-2k-3+(1-\sigma)}},$$

where all μ_l and ν_l are greater than 1 and each index appears at most once.

First we integrate with respect to ξ_1^* and get a constant factor $\tau_1(\zeta_0, \varepsilon) \lesssim \varepsilon$, which together with the other ε already gives us ε^{k+1}. Now we still need to integrate over $n-1$ complex discs; there are exactly $2n-2$ factors of the form τ_j or $|\eta_j^*|$ in the denominator, and only the last factor is of the form $|\eta_j^*|^{1-\sigma}$. Therefore, the following integrals may occur:

$$J_a := \int_{|\eta_l^*|<\tau_l(\zeta_0,\varepsilon)} \frac{d\theta_l^*\, d\xi_l^*}{\tau_l(\zeta_0,\varepsilon)^2} \lesssim 1,$$

$$J_b := \int_{|\eta_l^*|<\tau_l(\zeta_0,\varepsilon)} \frac{d\theta_l^*\, d\xi_l^*}{\tau_l(\zeta_0,\varepsilon)|\eta_l^*|} \lesssim 1,$$

$$J_c := \int_{|\eta_l^*|<\tau_l(\zeta_0,\varepsilon)} \frac{d\theta_l^*\, d\xi_l^*}{\tau_l(\zeta_0,\varepsilon)|\eta_l^*|^{1-\sigma}} \lesssim \tau_l(\zeta_0,\varepsilon)^\sigma \lesssim \varepsilon^{\sigma/m},$$

$$J_d := \int_{|\eta_{l_1}^*|<\tau_{l_1}(\zeta_0,\varepsilon)} \cdots \int_{|\eta_{l_i}^*|<\tau_{l_i}(\zeta_0,\varepsilon)} \frac{d\theta_{l_1}^*\, d\xi_{l_1}^* \ldots d\theta_{l_i}^*\, d\xi_{l_i}^*}{\left(\sum|\eta_{l_j}^*|\right)^{2i-1+(1-\sigma)}}$$

$$\lesssim \int_0^{\varepsilon^{1/m}} \frac{r^{2i-1}dr}{r^{2i-\sigma}} \lesssim \varepsilon^{\sigma/m}.$$

However, J_c and J_d may occur at most once and only one of them will be present. So finally we have our desired result.

To obtain the remaining two estimates, simply repeat the proof with $\sigma = 1$ and then use the second statement from Lemma 3.8 instead of the first part. □

Proof of Lemma 2.1. To achieve the result for $|I_a|$, it remains only to estimate the integral over the set $P_1(\pi(z)) \cap \partial D$. With $\rho = |\rho(z)|$ this set will be split into $P_\rho(\pi(z)) \cap \partial D$ and $P_1(\pi(z)) \setminus P_\rho(\pi(z)) \cap \partial D$, and both subsets will be further subdivided by the coverings given previously. In the first case, Lemma 3.11 gives $|S|^{k+1}|\zeta - z|^\sigma \gtrsim \rho^{k+1}\varepsilon^\sigma \gtrsim \rho^\sigma\varepsilon^{k+1}$ for all $\zeta \in P_\varepsilon^0(\zeta_0)$ with $\varepsilon < \rho$. Together with Lemma 3.14, this means that

$$\int_{\partial D\cap P_\rho(\zeta_0)} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t|}{|S|^{k+1}|\zeta - z|^{2n-2k-2}}\, d\sigma_{2n-1}$$

$$\lesssim \sum_{j=0}^\infty \int_{\partial D\cap P_\rho^j(\zeta_0)} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t|}{|S|^{k+1}|\zeta - z|^\sigma|\zeta - z|^{2n-2k-2-\sigma}}\, d\sigma_{2n-1}$$

$$\lesssim \sum_{j=0}^\infty \frac{1}{\rho^\sigma(2^{-j}\rho)^{k+1}}(2^{-j}\rho)^{\sigma/m+k+1}$$

$$\lesssim \rho^{\sigma(1/m-1)} \sum_{j=0}^\infty (2^{-j})^{\sigma/m} \lesssim \rho^{\sigma(1/m-1)}.$$

In the second case, for $\varepsilon > \rho$ and $\zeta \in P_\varepsilon^0(\zeta_0)$ we have $|S|^{k+1}|\zeta - z|^\sigma \gtrsim \varepsilon^{k+1+\sigma}$. Together with Lemma 3.14, this yields

$$\int_{\partial D\cap P_1(\zeta_0)\setminus P_\rho(\zeta_0)} \frac{|(Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)_t|}{|S|^{k+1}|\zeta-z|^{2n-2k-2}}\,d\sigma_{2n-1}$$
$$\lesssim \sum_{j=0}^{i_0(\rho)} \int_{\partial D\cap P_1^j(\zeta_0)} \frac{|(Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)_t|}{|S|^{k+1}|\zeta-z|^{\sigma}|\zeta-z|^{2n-2k-2-\sigma}}\,d\sigma_{2n-1}$$
$$\lesssim \sum_{j=0}^{i_0(\rho)} \frac{1}{(2^{-j})^{k+1+\sigma}}(2^{-j})^{\sigma/m+k+1} \lesssim \sum_{j=0}^{i_0(\rho)}(2^{-j})^{\sigma(1/m-1)} \lesssim \rho^{\sigma(1/m-1)},$$

where in the last estimate we also used that $i_0(\rho) < -\log_2(c\rho)$.

The integrals I_b and I_c differ only in the place where the δ_γ is applied, but Lemma 3.14 gives the same estimates in both cases. Thus it is enough to consider one of the integrals.

For $\varepsilon < \rho$ and $\zeta \in P_\varepsilon^0(\zeta_0)$ we have $|S|^{k+1} \gtrsim \rho\varepsilon^k$. Therefore,

$$\int_{\partial D\cap P_\rho(\zeta_0)} \frac{|(\delta_\gamma Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)_t|}{|S|^{k+1}|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1}$$
$$\lesssim \sum_{j=0}^{\infty} \frac{1}{\rho(2^{-j}\rho)^k}\frac{(2^{-j}\rho)^{1/m+k+1}}{\tau(\zeta_0,\gamma,(2^{-j}\rho))}$$
$$\lesssim \sum_{j=0}^{\infty} \frac{(2^{-j}\rho)}{\rho}\frac{1}{\tau(\zeta_0,\gamma,(2^{-j}\rho))}(2^{-j}\rho)^{1/m}$$
$$\lesssim \sum_{j=0}^{\infty} \frac{\rho^{1/m}}{\tau(\zeta_0,\gamma,\rho)}(2^{-j})^{1/m} \lesssim \frac{\rho^{1/m}}{\tau(\zeta_0,\gamma,\rho)},$$

where we have also made use of Lemma 3.13.

On the other hand, for $\varepsilon > \rho$ and $\zeta \in P_\varepsilon^0(\zeta_0)$ we have $|S|^{k+1} \gtrsim \varepsilon^{k+1}$. Therefore, with the help of Proposition 3.12 we obtain

$$\int_{\partial D\cap P_1(\zeta_0)\setminus P_\rho(\zeta_0)} \frac{|(\delta_\gamma Q\wedge(\bar\partial_\zeta Q)^k\wedge\beta)_t|}{|S|^{k+1}|\zeta-z|^{2n-2k-3}}\,d\sigma_{2n-1}$$
$$\lesssim \sum_{j=0}^{i_0(\rho)} \frac{1}{(2^{-j})^{k+1}}\frac{(2^{-j})^{1/m+k+1}}{\tau(\zeta_0,\gamma,2^{-j})}$$
$$\lesssim \sum_{j=0}^{i_0(\rho)} \frac{(2^{-j})^{1/m}}{\tau(\zeta_0,\gamma,2^{-j})} \lesssim \sum_{j=0}^{i_0(\rho)}(2^{-j})^{1/m}\max_{1\le k\le m}\{(2^{-j})^{-1/k}A_k^\gamma(\zeta_0)^{1/k}\}$$
$$\lesssim \sum_{1\le k\le m} A_k^\gamma(\zeta_0)^{1/k}\sum_{j=0}^{i_0(\rho)}(2^{-j})^{1/m-1/k} \lesssim |\log\rho| + \sum_{1\le k\le m} A_k^\gamma(\zeta_0)^{1/k}\rho^{1/m-1/k}$$
$$\lesssim |\log\rho| + \rho^{1/m}\max_{1\le k\le m}\{(A_k^\gamma(\zeta_0)/\rho)^{1/k}\} \lesssim \frac{\rho^{1/m}}{\tau(\zeta_0,\gamma,\rho)} + |\log\rho|.$$

Finally, we need only observe that $z \in P_\rho(\zeta_0)$ and hence $\tau(z,\gamma,\rho) \approx \tau(\zeta_0,\gamma,\rho)$.

To estimate I_d, we want to use Lemma 3.14 for $\sigma = 1$. Using Lemma 3.10 and Proposition 3.11 yields $|\delta_\gamma S| \lesssim \varepsilon/\tau(\zeta_0, \gamma, \varepsilon)$ and $|S|^{k+2} \gtrsim \rho\varepsilon^{k+1}$ for all $\zeta \in P^0_\varepsilon(\zeta_0)$ with $\varepsilon < \rho$. Thus we get the same estimate as for I_b, namely,

$$\int_{\partial D \cap P_\rho(\zeta_0)} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t||\delta_\gamma S|}{|S|^{k+2}|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1} \lesssim \sum_{j=0}^{\infty} \frac{(2^{-j}\rho)}{\tau(\zeta_0, \gamma, (2^{-j}\rho))} \frac{1}{\rho(2^{-j}\rho)^{k+1}} (2^{-j}\rho)^{1/m+k+1} \lesssim \frac{\rho^{1/m}}{\tau(\zeta_0, \gamma, \rho)}.$$

On the other hand, for $\varepsilon > \rho$ and $\zeta \in P^0_\varepsilon(\zeta_0)$ we have $|S|^{k+2} \gtrsim \varepsilon^{k+2}$, $|\delta_\gamma S| \lesssim \varepsilon/\tau(\zeta_0, \gamma, \varepsilon)$, and $\tau(\zeta_0, \gamma, \varepsilon) \geq \tau(\zeta_0, \gamma, \rho)$. Therefore, we again get the same result as for I_b:

$$\int_{\partial D \cap P_1(\zeta_0) \setminus P_\rho(\zeta_0)} \frac{|(Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta)_t||\delta_\gamma S|}{|S|^{k+2}|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1} \lesssim \sum_{j=0}^{i_0(\rho)} \frac{(2^{-j})}{\tau(\zeta_0, \gamma, 2^{-j})} \frac{1}{(2^{-j})^{k+2}} (2^{-j})^{1/m+k+1} \lesssim \frac{\rho^{1/m}}{\tau(\zeta_0, \gamma, \rho)} + |\log \rho|.$$

Together with the fact that $z \in P_\rho(\zeta_0)$ and so $\tau(z, \gamma, \rho) \approx \tau(\zeta_0, \gamma, \rho)$, this completes the proof. □

4. Examples

In this section we show that the result obtained in Theorem 1.2 is essentially the best one possible. We do this by providing an example of a domain D and a bounded (0, 1)-form f on D such that the Cauchy–Riemann equation $\bar\partial u = f$ cannot have any solution that is better than expected. In fact, this example is a modification of the well-known example of E. M. Stein.

We consider the domain $D = \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^{2l} + |z_3|^{2m} < 1\}$ with $m > l$. As a complex ellipsoid, this domain is convex and of finite type $2m$. In particular we are interested in a neighborhood of the boundary point $p = (1, 0, 0)$. This point is of type $2m$, and this order of contact is realized by the complex tangential line $\{(1, 0, w) : w \in \mathbb{C}\}$. All other complex tangential lines have order of contact $2l$. For every small ε, the ε-extremal directions at this point are e_1, e_2, and e_3, so we have $\tau_1(p, \varepsilon) = \sqrt{1 + \varepsilon} - 1 \approx \varepsilon$, $\tau_2(p, \varepsilon) = \varepsilon^{1/2m}$, and $\tau_3(p, \varepsilon) = \varepsilon^{1/2l}$. Thus $d(p, (1+\varepsilon, 0, 0)) \approx \varepsilon$, $d(p, (1, \varepsilon, 0)) = \varepsilon^{2l}$, and $d(p, (1, 0, \varepsilon)) = \varepsilon^{2m}$. That means that if we are close to p and measure distances in the Euclidean metric, then we can expect a Hölder exponent of $1/2m$ in the z_1-direction, an exponent of $2l/2m$ in the z_2-direction, and any exponent smaller than 1 in the z_3-direction.

It is well known that the Hölder exponent cannot be larger than 1, and with the given methods it is not possible to show that it must be strictly smaller than 1. Hence we will concentrate on the z_1- and z_2-directions. For this we consider the function

$$v(z) := \frac{\bar z_3}{\log(z_1 + z_2^{2l}/2 - 1)}.$$

Note that the real part of $z_1 + z_2^{2l}/2 - 1$ is negative for all $z \in D$ and that $\log(w)$ should be the branch of the logarithm with $0 < \arg(w) < 2\pi$. In fact, we have the strict negativity of the real part of $z_1 + z_2^{2l}/2 - 1$ for all $z \in \bar{D} \setminus (1, 0, 0)$. Thus, the closed $(0, 1)$-form

$$f(z) = \begin{cases} \dfrac{d\bar{z}_3}{\log(z_1 + z_2^{2l}/2 - 1)} & \text{for } z \in \bar{D} \setminus (1, 0, 0), \\ 0 & \text{for } z = (1, 0, 0) \end{cases}$$

is continuous on $\bar{D}$ and thus also bounded.

Consider the circles

$$C_\varepsilon := \{z \in \mathbb{C}^3 : z_1 = 1 - 2\varepsilon,\ z_2 = 0,\ |z_3| = \varepsilon^{1/2m}\},$$
$$\hat{C}_\varepsilon := \{z \in \mathbb{C}^3 : z_1 = 1 - \varepsilon,\ z_2 = 0,\ |z_3| = \varepsilon^{1/2m}\},$$
$$\tilde{C}_\varepsilon := \{z \in \mathbb{C}^3 : z_1 = 1 - 2\varepsilon,\ z_2 = \varepsilon^{1/2l},\ |z_3| = \varepsilon^{1/2m}\}.$$

It is easy to check that all three circles are contained in D.

For every solution u of the Cauchy–Riemann equation $\bar{\partial} u = f$, the difference $u - v$ is a holomorphic function on D; we therefore have $\int_C u\, dz_3 = \int_C v\, dz_3$ for all of these circles.

Let us first consider the z_1-direction and assume that the Hölder exponent in this direction is α. Then $|(1-\varepsilon, 0, z_3) - (1-2\varepsilon, 0, z_3)| = \varepsilon$ and so $|u(1-\varepsilon, 0, z_3) - u(1-2\varepsilon, 0, z_3)| \leq K\varepsilon^\alpha$. Thus

$$\begin{aligned} \left| \int_{|z_3| = \varepsilon^{1/2m}} u(1-\varepsilon, 0, z_3) - u(1-2\varepsilon, 0, z_3)\, dz_3 \right| &\leq K\varepsilon^\alpha \int_{|z_3| = \varepsilon^{1/2m}} 1\, dz_3 \\ &= K\varepsilon^\alpha 2\pi \varepsilon^{1/2m}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{|z_3| = \varepsilon^{1/2m}} v(1-\varepsilon, 0, z_3) - v(1-2\varepsilon, 0, z_3)\, dz_3 \\ &\quad = \int_{|z_3| = \varepsilon^{1/2m}} \frac{\bar{z}_3}{\log(-\varepsilon)} - \frac{\bar{z}_3}{\log(-2\varepsilon)}\, dz_3 \\ &\quad = \frac{\log(-2\varepsilon) - \log(-\varepsilon)}{\log(-\varepsilon)\log(-2\varepsilon)} \int_{|z_3| = \varepsilon^{1/2m}} \bar{z}_3\, dz_3 \\ &\quad = \frac{\ln(2)}{\log(-\varepsilon)\log(-2\varepsilon)} 2\pi i \varepsilon^{2/2m}. \end{aligned}$$

Hence we conclude that

$$\ln(2)/K \leq |\log(-\varepsilon)\log(-2\varepsilon)| \varepsilon^{\alpha - 1/2m}$$

for all $\varepsilon > 0$, which is impossible for $\alpha > 1/2m$ because $\ln(\varepsilon)\varepsilon^\beta$ tends to zero for $\varepsilon \to 0$ and all positive β. Thus we have proved that, in the z_1-direction, the Hölder exponent α cannot be larger than $1/2m$.

Now consider the z_2-direction and the circles C_ε and $\tilde{C}_\varepsilon$. Assuming that the Hölder exponent is again α, we get $|u(1-2\varepsilon, 0, z_3) - u(1-2\varepsilon, \varepsilon^{1/2l}, z_3)| \leq K(\varepsilon^{1/2l})^\alpha$ and so

$$\left| \int_{|z_3|=\varepsilon^{1/2m}} u(1-2\varepsilon, 0, z_3) - u(1-2\varepsilon, \varepsilon^{1/2l}, z_3)\, dz_3 \right| \leq K\varepsilon^{\alpha/2l} 2\pi\varepsilon^{1/2m}.$$

On the other hand,

$$\begin{aligned}
&\int_{|z_3|=\varepsilon^{1/2m}} v(1-2\varepsilon, 0, z_3) - v(1-2\varepsilon, \varepsilon^{1/2l}, z_3)\, dz_3 \\
&\quad = \int_{|z_3|=\varepsilon^{1/2m}} \frac{\bar{z}_3}{\log(-2\varepsilon)} - \frac{\bar{z}_3}{\log(-2\varepsilon+\varepsilon/2)}\, dz_3 \\
&\quad = \frac{\log(-3\varepsilon/2) - \log(-2\varepsilon)}{\log(-2\varepsilon)\log(-3\varepsilon/2)} \int_{|z_3|=\varepsilon^{1/2m}} \bar{z}_3\, dz_3 \\
&\quad = \frac{-\ln(4/3)}{\log(-2\varepsilon)\log(-3\varepsilon/2)} 2\pi i \varepsilon^{2/2m}.
\end{aligned}$$

We then conclude that

$$\ln(4/3)/K \leq |\log(-2\varepsilon)\log(-3\varepsilon/2)|\varepsilon^{\alpha/2l - 1/2m}$$

for all $\varepsilon > 0$, which is impossible for $\alpha > 2l/2m$. Thus we have proved that, in the z_2-direction, the Hölder exponent α cannot be larger than $2l/2m$, which is exactly what we wanted.

References

[BCDu] J. Bruna, P. Charpentier, and Y. Dupain, *Zero varieties for the Nevanlinna class in convex domains of finite type in* $\mathbb{C}^n$, Ann. of Math. (2) 147 (1998), 391–415.

[ChK] D. E. Chang and S. G. Krantz, *Holomorphic Lipschitz functions and application to the* $\bar\partial$*-problem,* Colloq. Math. 62 (1991), 227–256.

[dBF] F. Di Biase and B. Fischer, *Boundary behaviour of* H^p *functions on convex domains of finite type in* $\mathbb{C}^n$, Pacific J. Math. 183 (1998), 25–38.

[DFFo] K. Diederich, B. Fischer, and J. E. Fornæss, *Hölder estimates on convex domains of finite type,* Math. Z. 232 (1999), 43–61.

[DFo] K. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type,* Math. Z. 230 (1999), 145–164.

[DFoW] K. Diederich, J. E. Fornæss, and J. Wiegerinck, *Sharp Hölder estimates for* $\bar\partial$ *on ellipsoids,* Manuscripta Math. 56 (1986), 399–417.

[DM1] K. Diederich and E. Mazzilli, *Zero varieties for the Nevanlinna class on all convex domains of finite type,* Nagoya Math. J. 163 (2001), 215–227.

[DM2] ———, *Extension of bounded holomorphic functions in convex domains,* Manuscripta Math. 105 (2001), 1–12.

[F] B. Fischer, L^p *estimates on convex domains of finite type,* Math. Z. 236 (2001), 401–418.

[H] T. Hefer, *Hölder and* L^p *estimates for* $\bar\partial$ *on convex domains of finite type depending on Catlin's multitype,* Math. Z. 242 (2002), 367–398.

[Hö] L. Hörmander, *Notions of convexity,* Progr. Math., 127, Birkhäuser, Boston, 1994.

[K] S. G. Krantz, *Intrinsic Lipschitz classes on manifolds with applications to complex function theory and estimates for the $\bar{\partial}$ and $\bar{\partial}_b$ equations,* Manuscripta Math. 24 (1978), 351–378.

[LMi] I. Lieb and J. Michel, *The Cauchy–Riemann complex. Integral formulae and Neumann problem,* Aspects Math., E34, Vieweg, Braunschweig, 2002.

[Mc] J. D. McNeal, *Estimates on the Bergman kernels of convex domains,* Adv. Math. 109 (1994), 108–139.

[R] R. M. Range, *Holomorphic functions and integral representations in several complex variables,* Springer-Verlag, New York, 1986.

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