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Tartar Conjecture and Beltrami Operators

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1. Introduction

The study of the oscillation of sequences of gradients of Sobolev functions is a central topic in the calculus of variations and nonlinear partial differential equations. In this note we examine which subsets of the space of 2×2 matrices $\mathbf{M}^{2\times 2}$ "forbid" the oscillation of sequences of gradients approaching them. In precise words: Let E be a subset of $\mathbf{M}^{2\times 2}$ and let $1 \le p \le \infty$. Is the set stable in the sense that every sequence $\{f_j\}$ such that

$$\operatorname{dist}_E(Df_j) \to 0 \ \text{ in } L^p \tag{1.1}$$

satisfies that $\{Df_j\}$ is compact in L^p? The natural tool for studying this problem is Young measures, which have proved to be an efficient device for capturing properties of oscillating sequences. In particular, we will need the following type of Young measures.

Definition 1.1. Let $1 \le p \le \infty$. A probability measure $\nu \in \mathcal{M}(\mathbf{M}^{2\times 2})$ is a $W^{1,p}$ *homogeneous gradient Young measure* if there exists a sequence $\{f_j\}$, weakly convergent in $W^{1,p}(\Omega, \mathbb{R}^2)$, such that for every $\varphi \in C_0^\infty(\mathbf{M}^{2\times 2})$ we have

$$\varphi(Df_j) \overset{\star}{\rightharpoonup} \int_{\mathbf{M}^{2\times 2}} \varphi(\lambda)\, d\nu(\lambda) \ \text{ in } L^\infty(\Omega).$$

The set of $W^{1,p}$ homogeneous gradient Young measures supported in a closed set E is denoted by $\mathcal{H}^p(E)$.

The sequence $\{Df_j\}$ is called the *generating* sequence. A standard covering argument shows that Definition 1.1 does not depend on the domain Ω. It follows from [P, Prop. 6.12] that, if the sequence $\{|Df_j|^p\}$ is known to be weakly convergent in $L^1(\Omega)$, then we are concerned with the following question.

Question 1.2. Which closed sets $E \in \mathbf{M}^{2\times 2}$ satisfy

$$\mathcal{H}^p(E) = \{\delta_A : A \in E\}? \tag{1.2}$$

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When (1.2) holds we say that $\mathcal{H}^p(E)$ is trivial. Gradient Young measures are closely related to the so-called quasiconvex hulls. If E is compact then E^{qc}, the quasiconvex hull of E, is a crucial notion in the study of martensitic phase transitions [BJ; M1]. For unbounded sets, the quasiconvex hull has turned out to be an important object to be understood in a number of different problems: singular solutions of elliptic systems [Š2], stability of the conformal set [Y; YZh], G-closure problems and control [F1; P2], linear growth minimizing problems [FoM], and others. For unbounded sets, the most-used definition of quasiconvex hull is the following.

DEFINITION 1.3. Let $E \in \mathbf{M}^{2\times 2}$ be a closed set and let $1 \le p < \infty$. Then $E^{p,\mathrm{qc}}$, the p-quasiconvex hull of E, is defined by

$$E^{p,\mathrm{qc}} = \{A \in \mathbf{M}^{2\times 2} : Q\,\mathrm{dist}_E^p(A) = 0\}.$$

Definition 1.3 makes sense after recalling that, for a continuous function f, we have the following.

DEFINITION 1.4. Let f be a continuous function. Then Qf, the quasiconvexification of f, is defined by

$$Qf(A) = \inf_{\varphi \in C_0^\infty(\mathbb{D})} \int_{\mathbb{D}} f(A + D\varphi(z))\,dz.$$

Thorough discussions on quasiconvexification of functions and their relation with the lower semicontinuous envelope of the related functionals can be found in [D; M1; P1].

In our setting, the definition of p-quasiconvex hull immediately yields our next question.

QUESTION 1.5. Which closed sets E have trivial p-quasiconvex hull?

Let us begin with Question 1.2. It goes back at least to Tartar that a necessary condition on a set E to have trivial $\mathcal{H}^p(E)$ is that, for every A and B in E,

$$\det(A - B) \neq 0. \tag{1.3}$$

A set satisfying (1.3) is called a set without rank-1 connections. In 1982, Tartar [Ta] conjectured that this condition might be also sufficient. However, Tartar himself subsequently provided an example of four matrices (independently discovered by other people) without rank-1 connections but supporting a $W^{1,\infty}$ homogeneous gradient Young measure different from a Dirac delta (see [M1, Sec. 2.5] for a detailed description of this example).

On the other hand, Šverák [Š2] proved that if, instead of (1.3), we have

$$\det(A - B) > 0 \quad \text{for every } (A, B) \in E \times E, \tag{1.4}$$

then the weak continuity of the determinant implies that $\mathcal{H}^2(E)$ is trivial. In the same work he showed that, for some sets, (1.4) and (1.3) are equivalent. This is true for connected sets (precisely, we have that the determinant does not change sign) and also for the unions of orthogonal wells, which are fundamental in the study of martensitic phase transitions.

Since a result of Zhang [Z1] states that $\mathcal{H}^p(E) = \mathcal{H}^\infty(E)$ for a compact set E, it follows that Šverák arguments work for each $1 \le p \le \infty$. In addition, the compactness of E implies that

$$E^{p,\mathrm{qc}} = \left\{ A \in \mathbf{M}^{2\times 2} : A = \int_{\mathbf{M}^{2\times 2}} \lambda \nu(\lambda),\ \nu \in \mathcal{H}^p(E) \right\}. \tag{1.5}$$

Thus, compact sets satisfying (1.4) have trivial quasiconvex hulls.

In [Z2; Z3] Zhang extended Šverák's result to unbounded sets. The price to pay was that the set E must now satisfy condition (1.4) in a stronger way. He considered sets E such that, in addition to (1.4), satisfy the condition that the quantity

$$K_E = \sup\left\{ \frac{\|A - B\|^2}{\det(A - B)} : (A, B) \in E \times E \right\} \tag{1.6}$$

is finite (although Zhang used complex notation; see Section 2). Recall that the set of K-quasiconformal matrices is defined by

$$Q(K) = \{A \in \mathbf{M}^{2\times 2} : \|A\|^2 \le K \det(A)\}. \tag{1.7}$$

This motivated us to describe a set E such that $K_E \le K$ as "K-quasiconformal at every point". Therefore, K-quasiconformality at every point is the natural quantitative version of (1.4).

An important feature of such sets is that the answers to Questions 1.2 and 1.5 depend on p. Zhang discovered the existence of a certain threshold $p(K_E)$ such that, for $p > p(K_E)$, the set $\mathcal{H}^p(E)$ is trivial. Unfortunately, in his proof the relation between p and K_E depends on the norm of the Beurling Ahlfors transform, and this norm remains as perhaps the most challenging open problem in planar quasiconformal geometry.

In this paper we circumvent this difficulty by using the invertibility of the Beltrami operators as proved by Astala, Iwaniec, and Saksman [AIS]. We obtain that $p(K_E) = 2K_E/(K_E + 1)$, the Weyl exponent for quasiregular mappings determined by Astala [A]. In fact, we present a more general condition than (1.6) that depends on the complex dilatation of the matrices in E yet still implies the triviality of $\mathcal{H}^p(E)$ (see Remarks 3.7 and 4.4 and condition (3.10)). It is remarkable that our proofs make no appeal to the weak continuity of the determinant. Therefore, we recover the previous results of Šverák and Zhang using a new viewpoint to deal with Questions 1.2 and 1.5. In addition, our method yields a characterization of the behavior of the distance function and its quasiconvexification, extending the work of Iqbal [Iq] and Zhang [Z4].

Zhang proved more recently [Z5] that condition (1.4) and certain control on how the set E behaves near infinity were enough to establish the triviality of $\mathcal{H}^2(E)$. We now introduce the notion of sets that are "asympotically" K-quasiconformal.

DEFINITION 1.6. A closed set $E \subset \mathbf{M}^{2\times 2}$ is said to be *asymptotically K-quasiconformal* if

$$\lim_{M\to\infty} \sup\left\{\frac{\|A\|^2}{\det A},\ A \in E \setminus B(0, M)\right\} \le K \tag{1.8}$$

and

$$\lim_{M\to\infty} \inf\{\det A,\ A \in E \setminus B(0, M)\} > 0. \tag{1.9}$$

We prove $p(E) = \frac{2K}{K+1}$ for sets E that satisfy (1.4) and are also asymptotically K-quasiconformal. The basic ingredient is a higher integrability argument. However, here we need to use the weak continuity of the determinant because we do not have control of the quasiconformality of the set when $\det(A - B)$ is close to zero. This also prevents us from understanding the relaxation of the distance function in this case.

In [Z5] it is asked if (1.4) together with the asymptotic K-quasiconformality for some K implies the triviality of $\mathcal{H}^1(E)$. We shall describe sets E such that $K_E < \infty$ but for which $\mathcal{H}^q(E)$ is not trivial when $q < p(K_E)$. Because they contain zero, our sets are trivially asymptotically K-quasiconformal. Hence the question is solved in negative. In a future project we plan to investigate how some version of the method of convex integration applied to our examples gives new, very weak quasiregular mappings.

Viewing our results as a whole, we see two things. First, regularity results for elliptic PDEs establish restrictions on the kind of sets that can support nontrivial gradient Young measures. Second, if we examine unbounded sets E, the scale of exponents for which the p-quasiconvex hull behaves naturally coincides with the range of the exponents for which the associated PDE can be solved. If we are below these exponents, then very weak solutions give surprising p-quasiconvex hulls and surprising p-quasiconvex hulls give very weak solutions.

Since sharp results for elliptic PDEs in the plane are available owing to Astala's theorem [A], we could obtain sharp results when studying sets that grow linearly at infinity. In order to understand sets that grow nonlinearly at infinity, we should look for and/or prove regularity results for the associated nonlinear elliptic PDE.

The paper is organized as follows. Section 2 reviews some basic notation and facts on Beltrami operators. We will prove a corollary of the Astala–Iwaniec–Saksman theorem on the invertibility of Beltrami operators and so obtain the most general conditions possible (on complex dilatations of matrices in a set E) guaranteeing the triviality of $\mathcal{H}^p(E)$. In Section 3 we deal with sets that are asymptotically K-quasiconformal. We present coercivity results for the distance function as well as higher integrability results for homogeneous gradient Young measures supported in such a set E. We also show that, for asymptotically K-quasiconformal sets, the characterization by equation (1.5) of the p-quasiconvex hull holds—provided p stays in a certain range.

In Section 4 we analyze sets without rank-1 connections: first when they are quasiconformal at every point and then when we have control of their quasiconformality at infinity only. We also present a result that characterizes sequences of gradients approaching to unions of these sets. In Section 5 we construct examples of sets that are K-quasiconformal at every point and such that $E^{q,\mathrm{qc}} = \mathrm{co}(E)$ for every $1 \leq q < \frac{2K}{K+1}$. These examples will show the optimality of our results and answer the questions raised by Zhang.

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2. Notation and Preliminaries

In this paper we will be living in two different worlds: the domain of the functions under consideration (i.e., the complex plane $\mathbb{C}$) and the range of their derivatives (i.e., the space $\mathbf{M}^{2\times 2}$ of 2×2 matrices). In $\mathbb{C}$ we let $\mathbb{D}$ be the unit disc and $\mathbb{D}(a, r)$ the disc centered at a with radius r. The supports of a function f and of a measure ν will be denoted by $\operatorname{spt}(f)$ and $\operatorname{spt}(\nu)$, respectively. For pre-images of Borel sets we will make the standard abuse of notation; for example, if P is a measurable function then the set $\{z \in \mathbb{D} : P(z) \geq M\}$ will be denoted by $\{P(z) \geq M\}$. The hyperbolic distance between two points $\{z_1, z_2\}$ in $\mathbb{D}$ is denoted by $\operatorname{dist}_h(z_1, z_2)$. The length element in the hyperbolic distance is

$$ds = \frac{2}{1 - |z|^2}|dz|.$$

Let $A \in \mathbf{M}^{2\times 2}$. Then $\|A\|$ will stand for the operator norm of A and $|A|$ for the Hilbert–Schmidt norm. The set

$$\mathrm{CO}_+ = \{A \in \mathbf{M}^{2\times 2} : A = \rho R,\ \rho > 0,\ R \in \mathrm{SO}(2)\}$$

is called the set of conformal matrices. Similarly,

$$\mathrm{CO}_- = \left\{A \in \mathbf{M}^{2\times 2} : A = CH,\ H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ C \in \mathrm{CO}_+\right\}$$

is the set of anticonformal matrices. The function $\Pi_{\mathrm{CO}_\pm} : \mathbf{M}^{2\times 2} \to \mathrm{CO}_\pm$ will denote the orthogonal projections on $\mathrm{CO}_\pm$. By means of the natural identification between matrices in SO(2) and complex numbers with modulus 1,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \cos(\theta) + i \sin(\theta), \tag{2.1}$$

the projections $\Pi_{\mathrm{CO}_\pm}$ induce complex coordinates for matrices: $A = (A_+, A_-) : A_+, A_- \in \mathbb{C}$ (see [Ah]). Since the $A_\pm$ are complex numbers we use $|A_\pm|$ to denote their absolute values. Observe that (2.1) implies that $|\Pi_{\mathrm{CO}_\pm}(A)| = \sqrt{2}|A_\pm|$. In complex coordinates we have the identities

$$\begin{aligned} \det(A) &= |A_+|^2 - |A_-|^2, \\ \|A\| &= |A_+| + |A_-|, \\ |A|^2 &= 2(|A_+|^2 + |A_-|^2). \end{aligned} \tag{2.2}$$

Given two matrices A, B, we say that $[A, B] = \{tA + (1-t)B : t \in [0, 1]\}$ is a rank-1 segment if $\det(A - B) = 0$. In conformal coordinates the latter condition is equivalent to $|A_+ - B_+| = |A_- - B_-|$. Complex coordinates give rise to the concept of complex dilatation of a matrix A: $\mu_A = A_-/A_+ \in \mathbb{C}$. Let $k = \frac{K-1}{K+1}$; it then follows from (2.2) that $A \in Q(K)$ if and only if $|\mu_A| \le k$. Let $E \subset \mathbf{M}^{2\times 2}$. Then $\mathrm{co}(E)$ denotes the closed convex hull of E.

Let us now turn our attention to mappings. The space $C_0^\infty(\mathbb{D})$ is the space of infinitely differentiable functions with compact support. The space of vector-valued Sobolev mappings may be denoted either by $W^{1,p}(\mathbb{D},\mathbb{C})$ or by $W^{1,p}(\mathbb{D},\mathbb{R}^2)$. If $f \in W^{1,2}(\mathbb{D},\mathbb{C})$ and $Df \in Q(K)$ almost everywhere in $\mathbb{D}$, then f is said to be K-quasiregular. If, in addition, f is a homeomorphism then it is said to be K-quasiconformal. We will also use complex coordinates to study the Jacobian derivative of a Sobolev mapping f:

$$Df(z) = (\partial_z f(z), \partial_{\bar z} f(z)), \quad \mu_f = \frac{\partial_{\bar z} f(z)}{\partial_z f(z)}, \quad J_f(z) = \det(Df(z)).$$

For functions that are supported in the whole space, the complex derivatives of a (nonhomogeneous) Sobolev function are related by the Beurling–Ahlfors transform

$$S(\partial_{\bar z} f) = \partial_z f,$$

where S is a Calderon–Zygmund operator with Fourier multiplier equal to $\bar\xi/\xi$. Hence, the general theory of these operators states that S is bounded from $L^p(\mathbb{C})$ to $L^p(\mathbb{C})$ if $1 < p < \infty$. One can interpret the boundedness of S as a coercivity result for the PDE

$$\partial_{\bar z} f(z) = h,$$

where $h \in L^p(\mathbb{C})$. This is the simplest example of the so-called Beltrami equations. In general, each function $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$ determines an elliptic equation. The function μ is called a Beltrami coefficient and the equation is known as the Beltrami equation. For a given μ, the associated Beltrami equation is

$$\partial_{\bar z} f - \mu \partial_z f = h \quad \text{for } h \in L^p(\mathbb{C}).$$

These equations are well understood thanks to [AIS], whose key result is the coercivity of the Beltrami operator $I - \mu S$ (see [AIS, Lemma 14]).

THEOREM 2.1. *Let $g \in L^p(\mathbb{C},\mathbb{C})$ and $\mu \in L^\infty(\mathbb{C}, B(0,k))$. Let $1 + k < p < 1 + 1/k$. Then*

$$\int_{\mathbb{C}} |g(z)|^p \, dz \le C(k,p) \int_{\mathbb{C}} |g(z) - \mu(z) S(g(z))|^p \, dz.$$

Theorem 2.1 will be the main tool used in this paper. However, in some places we will need more general versions. The main ingredient in the proof of Proposition 4.5 is the analogous result to Theorem 2.1 where the associated equation is nonlinear. Astala, Iwaniec, and Saksman [AIS, Thm. 1] also established the following invertibility result for nonlinear Beltrami operators.

THEOREM 2.2. *Let H be a measurable function $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that H is k-Lipschitz ($k < 1$) in the second variable and $H(w, 0) = 0$ for almost every w.*

Let $B(g) = g - H(z, S(g))$. Then $B\colon L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is an invertible operator provided that $1 + k < p < 1 + 1/k$.

In order to derive optimal answers to the questions studied in this work, it is better to express Theorem 2.1 in the following more general version.

COROLLARY 2.3. *Let $K > 1$ and let $\mu \in L^\infty(\mathbb{C})$ be such that there exists a complex number $\mu_0 \in \mathbb{D}$ with $\mathrm{dist}_h(\mu(z), \mu_0) \leq \log(K)$. Let $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Then the operator $(I - \mu S)^{-1}$ is bounded from $L^p(\mathbb{C})$ into itself.*

Notice that if $K = \frac{k+1}{1-k}$ and $\mu_0 = 0$ then the corollary simply rephrases Theorem 2.1.

Proof of Corollary 2.3. The proof of Theorem 2.1 in [AIS] uses $|\mu(z)| \leq k$ only to obtain that, for every K-quasiconformal homeomorphic solution to the homogeneous Beltrami equation

$$\partial_{\bar{z}} f = \mu \partial_z f,$$

$w = |J_f|^{1-p/2}$ is an A_p-weight for $2 \leq p < 1 + 1/k$ (see [AIS, Lemma 6]). It is therefore enough to prove that, if μ is as in Corollary 2.3, then every quasiconformal solution to

$$\partial_{\bar{z}} F = \mu \partial_z F$$

gives rise to the corresponding A_p-weights. This is a consequence of the appropriate linear change of variables. Consider the linear mapping $A_0(z) = z + \mu_0 \bar{z}$. Then the chain rule gives that

$$\mu_{F \circ A_0^{-1}}(A_0(z)) = \frac{\mu_F(z) - \mu_0}{1 - \mu_F(z)\overline{\mu_0}}1.$$

The point is that $T(z) = (z - \mu_0)/(1 - z\overline{\mu_0})$ is a Möbius transformation from the disc to the disc mapping μ_0 to 0. Hence, it is an isometry with respect to the hyperbolic metric of the disc. Thus, for every $w \in \mathbb{C}$, it follows that $\mathrm{dist}_h(\mu_{F \circ A_0^1}(w), 0) \leq \log K$ and hence

$$\|\mu_{F \circ A_0^{-1}}\|_\infty \leq k;$$

that is, $F \circ A_0^{-1}$ is K-quasiconformal. So by setting $f = F \circ A_0^{-1}$ we have shown that $F(z) = f(A_0(z))$, where f is K-quasiconformal. Since A_p properties are not affected by a linear change of variables, $|J_F|^{1-p/2}$ is also an A_p-weight. Similarly, $|J_{F^{-1}}|^{1-p/2}$ is an A_p-weight because $|J_{f^{-1}}|^{1-p/2}$ is an A_p- weight. $\square$

Theorem 2.2 is almost sharp if we look at the regularity of the solutions to the Beltrami equations in terms of the range of the Beltrami coefficient μ. Take two complex numbers $w_1, w_2 \in \mathbb{D}$ such that $\mathrm{dist}_h(w_1, w_2) = 2\log(K)$. Pick the point w_0 in the hyperbolic geodesic γ joining w_1 and w_2 such that $\mathrm{dist}_h(w_0, w_1) = \mathrm{dist}_h(w_0, w_2) = \log(K)$. Consider the Möbius transformation T_{w_0} from the disc into itself such that $T_{w_0}(w_0) = 0$. Because $T_{w_0}(\gamma)$ is a hyperbolic geodesic passing through zero, it must be a straight line. Moreover, $\mathrm{dist}_h(T_{w_0}(w_1), 0) = \mathrm{dist}_h(T_{w_0}(w_2), 0) = \log(K)$. Hence, if composing with a rotation we can assume there is a Möbious transformation T mapping w_1 and w_2 to k and $-k$, respectively. It follows from the ideas in [F2] that one can saturate the bounds for the integrability of the gradient of quasiregular mappings with mappings f such that $\mu_f \in \{k, -k\}$ almost everywhere. Let f be one of these mappings.

Now, if A is a linear mapping then the chain rule gives

$$\mu_{f\circ A^{-1}}(A(z)) = \frac{\mu_f(z) - \mu_A}{1 - \mu_f(z)\overline{\mu_A}} \frac{A_+}{\bar{A}_+}.$$

This implies that we can find a matrix A_0 such that $\mu_{f\circ A_0^{-1}} = T^{-1}(\mu_f(z))$. Therefore, the mapping $F(z) = f(A_0^{-1}z)$ satisfies $\mu_F(z) = \{w_1, w_2\}$ almost everywhere and F behaves as badly as f.

3. Quasiconvex Hulls and Quasiconformality of Sets

Quasiconvex hulls have arisen as an appropriate analogy to convex hulls in the vectorial calculus of variations. After proper interpretation, the quasiconvex hull of a closed set E expresses the possible macroscopic effects of a system constrained by the set E at the microscopic level. There are several definitions of quasiconvex hulls in use [Y; Z5]. Definition 1.3 is the least restrictive, whereas the characterization (1.5) is the most. In other words, for a general set E we always have

$$\left\{A \in \mathbf{M}^{2\times 2} : A = \int_{\mathbf{M}^{2\times 2}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\right\} \subset E^{p,\mathrm{qc}},$$

and the other existing definitions of p-quasiconvex hull (see [Z5]) give sets in between.

In [Z5] Zhang presented examples where the inclusion just displayed is strict. His sets are asymptotically close to rank-1 lines. The following less subtle example illustrates the idea behind them.

EXAMPLE 3.1. *The set E will be contained in the plane of all diagonal matrices in $\mathbf{M}^{2\times 2}$. Hence we use the identification*

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = (d_1, d_2).$$

In this notation let $E = \{(d_1, 0) \cup (0, d_2)$ with $0 \le d_1, d_2 < \infty\}$. Then $E^{p,\mathrm{qc}} = \mathrm{co}(E)$ for every $1 \le p < \infty$. Nevertheless, if $\nu \in \mathcal{H}^2(E)$ then $\int_{\mathbf{M}^{2\times 2}} \lambda\, d\nu(\lambda) \in E$.

Proof. Let $A \in \mathrm{co}(E)$. Then $A = (d_1, d_2)$ with $0 < d_1, d_2 < \infty$. We use that finite quasiconvex functions are convex along rank-1 segments and also that, in the diagonal plane, vertical segments are rank-1 segments. Hence, for every natural number j,

$$Q\operatorname{dist}_E^p(A) \le \theta_j \operatorname{dist}_E^p((d_1, j)) + (1-\theta_j)\operatorname{dist}_E^p((d_1, 0)) = \theta_j \operatorname{dist}_E^p(d_1, j),$$

where $\theta_j j = d_2$. Since $\operatorname{dist}_E(d_1, j) = \min\{d_1, j\}$, we have

$$Q\,\mathrm{dist}_E^p(A) \le \frac{d_1^p d_2}{j}$$

for every j, whence $\mathrm{co}(E) \in E^{p,\mathrm{qc}}$. The claim is proved because $E^{p,\mathrm{qc}}$ is always included in $\mathrm{co}(E)$.

However, if $A \in \mathrm{co}(E)\setminus E$ were the center of mass of a measure $\nu \in \mathcal{H}^2(E)$ then, by [M1, Thm. 1], ν would satisfy Jensen inequality for separately convex functions with quadratic growth. The function $f(d_1, d_2) = \max(d_1, 0)\max(d_2, 0)$ yields a contradiction. □

Observe that here we have found a sequence $\{g_j\} \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$ that yields the infimum in the definition of $Q\,\mathrm{dist}^p$ but such that $\mathrm{dist}_E(A + Dg_j) \in \{0, d_1\}$ and $\lim_{j\to\infty}\int_{\mathbb{D}}|Dg_j(z)|^2\,dz = \infty$. When the sequence giving the infimum in the definition of the quasiconvexification is bounded in $W^{1,p}$, the characterization (1.5) for the p-quasiconvex hull holds.

LEMMA 3.2. *Let $E \in \mathbf{M}^{n\times m}$ be a closed set such that for every matrix $A \in \mathbf{M}^{n\times m}$ there exists a constant C_A such that, for every function $\phi \in C_0^\infty(\Omega, \mathbb{R}^m)$,*

$$\int_\Omega |D\phi(z)|^p\,dx \le C_A\left(1 + \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\phi(z))\,dz\right).$$

Then

$$E^{p,\mathrm{qc}} = \left\{A \in \mathbf{M}^{2\times 2} : A = \int_{\mathbf{M}^{2\times 2}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\right\}.$$

Proof. This is a well-known result. Under our assumptions one can argue as in [P1, Chap. 4, Sec. 3]. More precisely, the claim is a corollary of Pedregal's Theorem 4.4, Corollary 4.5 and Corollary 4.7. □

Beltrami operators yield natural sets that satisfy the preceding coercivity condition. Using them as the main tool, we can show that the examples containing rank-1 lines or asymptotic to rank-1 lines [Z5] are quite extremal. Roughly speaking, if a set $E \subset \mathbf{M}^{2\times 2}$ is bounded away from the rank-1 lines then both previous notions of p-quasiconvex hulls agree for a range of exponents depending on how much we are off the rank-1 lines. This can be made quantitatively precise as follows.

PROPOSITION 3.3. *Let $M > 0$, $K > 0$, and $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Set $E = B(0, M) \cup Q(K)$. Then for every $A \in \mathbf{M}^{2\times 2}$ there exists a constant C_A depending on A, p, M, and K such that, for every function $\phi \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$,*

$$\int_{\mathbb{D}} |D\phi(z)|^p\,dz \le C_A\left(1 + \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\phi(z))\,dz\right).$$

Proof. Let $v(z) = Az + \phi(z)$. By smoothness there exists $R(v) > 0$ such that $|Dv(z)| \le R$. Thus, the so-called measurable selection lemma [ET, p. 236] provides us with a measurable function $P : \mathbb{D} \to E \cap B(0, 2R)$ such that, for every z,

$$|Dv(z) - P(z)| = \mathrm{dist}_E(Dv(z)). \tag{3.1}$$

We declare

$$\mu(z) = \begin{cases} \frac{P(z)_-}{P(z)_+} & \text{if } z \in \mathbb{D} \text{ and } |P(z)| \geq M, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We need a bound only for the L^p norm, so it is enough to control $D\phi$ in some ball independent of ϕ that is centered at ∞. Since $\phi(z)$ has zero boundary values, its zero extension belong to $W^{1,p}(\mathbb{C}, \mathbb{C})$. In addition, it follows from the definition of μ that $\|\mu\|_\infty \leq k$. Hence, we can use the key Theorem 2.1 and the L^p boundedness of the Beurling transform to obtain

$$\int_{\mathbb{D}} |D\phi(z)|^p \, dz \leq C(k, p) \int_{\mathbb{D}} |\partial_{\bar{z}}\phi(z) - \mu\partial_z\phi(z)|^p \, dz. \tag{3.3}$$

Therefore, if we find a constant C such that

$$\int_{\mathbb{D}} |\partial_{\bar{z}}\phi(z) - \mu\partial_z\phi(z)|^p \, dz \leq C\left(1 + \int_{\mathbb{D}} \operatorname{dist}_E^p(Dv(z)) \, dz\right)$$

then the proposition will be proved. Let us plug in the definition of μ. Then

$$\begin{aligned} \int_{\mathbb{D}} |\partial_{\bar{z}}\phi(z) - \mu\partial_z\phi(z)|^p \, dz \leq & \int_{\{P(z) \leq M\}} |\partial_{\bar{z}}\phi(z)|^p \, dz \\ & + \int_{\{P(z) \geq M\}} |\partial_{\bar{z}}\phi(z) - \mu\partial_z\phi(z)|^p \, dz. \end{aligned} \tag{3.4}$$

Now, we estimate each term separately. For the first term we have

$$\begin{aligned} \int_{\{|P(z)| \leq M\}} |D\phi(z)|^p \, dz &\leq C(p)\left(\int_{\{|P(z)| \leq M\}} \operatorname{dist}_E^p(Dv(z)) + |P(z)|^p \, dz + |A|^p\right) \\ &\leq C(p, A, M)\left(1 + \int_{\{|P(z)| \leq M\}} \operatorname{dist}_E^p(Dv(z)) \, dz\right). \end{aligned} \tag{3.5}$$

For the second term, use (3.1) to get

$$\begin{aligned} \int_{\{|P(z)| \geq M\}} |\partial_{\bar{z}}\phi(z) - \mu\partial_z\phi(z)|^p \, dz &\leq \int_{\{|P(z)| \geq M\}} |\partial_{\bar{z}}v(z) - \mu\partial_z v(z)|^p \, dz \\ &\quad + \int_{\{|P(z)| \geq M\}} |A_- - \mu(z)A_+|^p \, dz \\ &\leq C\left(1 + \int_{\mathbb{D}} \operatorname{dist}_E^p(Dv(z)) \, dz\right). \end{aligned} \tag{3.6}$$

We have merely added and subtracted the appropriate quantities and used the respective domains of integration. Combining (3.5) and (3.6) with (3.4) and (3.3) yields the desired result. □

COROLLARY 3.4. *Let $K > 1$ and $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Let $E = B(0, M) \cup Q(K)$. Then*

$$E^{p,\mathrm{qc}} = \left\{A \in \mathbf{M}^{2\times 2} : A = \int_{\mathbf{M}^{2\times 2}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\right\}.$$

Proof. The proof readily follows from Lemma 3.2 and Proposition 3.3. □

One of the remarkable features of quasiregular mappings is the automatic improvement of their regularity. Precisely, a mapping $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{D}, \mathbb{R}^2)$ with $q > \frac{2K}{K+1}$ and such that $Df(z) \in Q(K)$ belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{D}, \mathbb{R}^2)$ for every $p < \frac{2K}{K-1}$. This was generalized to gradient Young measures in [AF], where it was proved that $\mathcal{H}^q(Q(K)) = \mathcal{H}^p(Q(K))$ for $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. In this work we shall need the following sharper result. The idea of the proof is push the sequence to the quasiconformal cone when it is close to the cone. We thereby find another generating sequence $\{D\varphi_j\}$ that is either quasiconformal or close to the ball.

This implies that the elements of the new generating sequence satisfy Beltrami equations whose right-hand sides are uniformly bounded in $L^\infty(\mathbb{D})$. Then, by means of cutoff functions, the new sequence is shown to have local higher integrability. Since the Young measure is homogeneous, this is enough to imply the correct integrability for the measure.

Proposition 3.5. *Let $M > 0$, $K > 1$, and $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. Set $E = B(0, M) \cup Q(K)$. Then*

$$\mathcal{H}^q(E) = \mathcal{H}^p(E).$$

Proof. Let $\nu \in \mathcal{H}^q(E)$. A well-known property of gradient Young measures (see [P1, Prop. 8.15]) is that there exists a generating sequence $\varphi_j \in W^{1,\infty}(\mathbb{D}, \mathbb{R}^2)$ such that $|D\varphi_j|^q$ is equi-integrable. Therefore,

$$\int_{\mathbb{D}} \operatorname{dist}^q_E(D\varphi_j(z))\, dz \to 0.$$

This clearly implies that

$$\int_{\{|D\varphi_j(z)| \geq 2M\}} \operatorname{dist}^q_{Q(K)}(D\varphi_j(z))\, dz \to 0. \tag{3.7}$$

Now let $P_j(z)$ be the measurable function obtained from the measurable selection lemma such that

$$\operatorname{dist}_{Q(K)}(D\varphi_j(z)) = |D\varphi_j(z) - P_j(z)|, \quad P_j(z) \in Q(K)$$

(we need here that $\varphi_j \in W^{1,\infty}(\mathbb{D}, \mathbb{R}^2)$). Declare $\mu_j(z) = \mu_{P_j(z)} \chi_{\{|D\varphi_j(z)| \geq 2M\}}(z)$ and $h_j = (\partial_{\bar{z}}\varphi_j - \mu_j \partial_z \varphi_j)\chi_{\{|D\varphi_j(z)| \geq 2M\}}$. Arguing as in the proof of Proposition 3.3, we find a constant C such that $|\partial_{\bar{z}}\varphi_j - \mu_j \varphi_j|^q \leq C \operatorname{dist}^q_{Q(K)}(D\varphi_j)$. Hence (3.7) implies that

$$\|h_j\|_{L^q(\mathbb{D})} \to 0.$$

By the invertibility of the Beltrami operators and using the Cauchy transform, we find a sequence $\{F_j\} \in L^q(\mathbb{C})$ such that

$$\partial_{\bar{z}} F_j - \mu_j \partial_z F_j = h_j \tag{3.8}$$

in the whole complex plane $\mathbb{C}$ and such that $\|DF_j\|_{L^q(\mathbb{C})} \leq C\|h_j\|_{L^q(\mathbb{D})}$ (for a related argument see [AF, Thm. 1.2]). Thus the sequence $\{D(\varphi_j - F_j)\}$ also generates the measure ν. Set $g_j = \varphi_j - F_j$. Combining (3.8) and the definition of h_j yields that, in $\mathbb{D}$,

$$\partial_{\bar z} g_j - \mu_j \partial_z g_j = \partial_{\bar z} \varphi_j \chi_{\{|D\varphi_j(z)|<2M\}}. \tag{3.9}$$

That is, g_j satisfies locally a nonhomogeneous Beltrami equation whose right-hand side is bounded by $2M$ and compactly supported.

In order to apply Theorem 2.1 we must extend g_j to the whole plane. Formally, let $\eta_1 \in C_0^\infty(\mathbb{D})$ be a cutoff function such that $\eta(z) = 1$ if $|z| \le \frac{1}{2}$. Then, by the chain rule,

$$\partial_{\bar z} \eta_1 g_j - \mu_j \partial_z \eta_1 g_j = \eta_1(\partial_{\bar z} g_j - \mu_j \partial_z g_j) + g_j(\partial_{\bar z}\eta_1 - \mu_j \partial_z \eta_1)$$

in the distributional sense.

Let us denote the new sequence of right-hand sides by $\tilde h_j$. Since Dg_j is uniformly bounded in L^q, the Sobolev embedding theorem implies that g_j is uniformly bounded in $L^{q^*}(\mathbb{D}, \mathbb{C})$, where q^* is the Sobolev conjugate exponent of q (notice that $\mathbb{D}$ is an extension domain).

Because η_1 is compactly supported, $\|\eta_1(\partial_{\bar z} g_j - \mu_j \partial_z g_j)\|_{L^p(\mathbb{C})} \le 2M|\mathrm{spt}(\eta_1)|^{1/p}$ for every p and thus $\tilde h_j$ is uniformly bounded in $L^{q^*}(\mathbb{C})$. Now we need to consider two different cases.

1. If $q^* \ge \frac{2K}{K-1}$ then, given that $\partial_{\bar z}(\eta_1 g_j) = (I - \mu_j S)^{-1} \tilde h_j$, we can use Theorem 2.1 once more to obtain that $\{D(\eta_1 g_j)\}$ is uniformly bounded in $L^p(\mathbb{C})$ for every $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Hence $\{Dg_j\}$ is uniformly bounded in $L^p\big(\mathbb{D}\big(0, \frac{1}{2}\big)\big)$.

2. If $q^* < \frac{2K}{K-1}$ then the preceding argument gives only that the sequence $\{Dg_j\}$ is uniformly bounded in $L^{q^*}\big(\mathbb{D}\big(0, \frac{1}{2}\big)\big)$. Next, we take a second cutoff function $\eta_2 \in C_0^\infty\big(\mathbb{D}\big(0, \frac{1}{2}\big)\big)$ where $\eta_2(z) = 1$ if $|z| \le \frac{1}{4}$. Proceeding as before, we find that $\eta_2 g_j$ satisfies a nonhomogeneous Beltrami equation with the same Beltrami coefficient μ_j but with a new right-hand side H_j,

$$H_j(z) = \eta_2(\partial_{\bar z} g_j - \mu_j \partial_z g_j) + g_j(\partial_{\bar z}\eta_2 - \mu_j \partial_z \eta_2).$$

The key is that now $g_j \in W^{1,q^*}\big(\mathbb{D}\big(0, \frac{1}{2}\big)\big)$. Since $q^* > 2$, the Sobolev embedding implies the existence of representatives of g_j that are uniformly Hölder continuous (again we use that discs are extension domains). There is no loss of generality in assuming that $g_j(0) = 0$ and so, in particular, the g_j are uniformly bounded in $L^\infty\big(\mathbb{D}\big(0, \frac{1}{4}\big)\big)$. Therefore, since we already knew that $\partial_{\bar z} g_j - \mu_j \partial_z g_j$ were uniformly bounded in $L^\infty(\mathbb{D})$, we deduce that the sequence $\{H_j\}$ is uniformly bounded in $L^p(\mathbb{C})$ for each $p > 1$. Now, as before, the invertibility of the Beltrami operators gives that $\{D(\eta_2 g_j)\}$ is uniformly bounded in $L^p(\mathbb{C})$ for $p < \frac{2K}{K-1}$, which implies that $\{Dg_j\}$ is uniformly bounded in $L^p\big(\mathbb{D}\big(0, \frac{1}{4}\big)\big)$. Finally, the homogeneity of ν implies that both $\{Dg_j|_{\mathbb{D}(0,\frac{1}{2})}\}$ and $\{Dg_j|_{\mathbb{D}(0,\frac{1}{4})}\}$ generate ν. This gives the desired fact: $\nu \in \mathcal{H}^p(E)$ as well. □

Let us combine the foregoing theorems and extend the result to sets that are asymptotically K-quasiconformal.

THEOREM 3.6. *Let $E \in \mathbf{M}^{2\times 2}$ be a set that is asymptotically K-quasiconformal in the sense of Definition 1.6, and let $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. Then E enjoys the following properties.*

(i) *For every* $A \in \mathbf{M}^{2\times 2}$ *there exists a constant* $C(K, A)$ *such that, for every* $\varphi \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$,

$$\int_{\mathbb{D}} |D\phi(z)|^p \, dz \leq C(K, A)\left(1 + \int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\phi(z))\, dz\right).$$

(ii) $E^{p,\mathrm{qc}} = \left\{A \in \mathbf{M}^{2\times 2} : A = \int_{\mathbf{M}^{2\times 2}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\right\}$.

(iii) $\mathcal{H}^q(E) = \mathcal{H}^p(E)$.

Proof. Let $\frac{2K}{K+1} < p$, and choose $\varepsilon > 0$ such that $\frac{2K}{K+1} < \frac{2(K+\varepsilon)}{K+\varepsilon+1} < p$. If E is asymptotically K-quasiconformal then by definition we have $M(\varepsilon)$ such that $E \subset B(0, M(\varepsilon)) \cup Q(K + \varepsilon)$. We can therefore apply Proposition 3.3, Corollary 3.4, and Corollary 3.5 to such a set and such a p. The result follows. □

REMARK 3.7. Theorem 3.6 concerns the quasiconformal set of matrices $Q(K)$. However it should be clear that Corollary 2.3 immediately yields the following more general result. Let p and K be as before, and let E be a closed set such that $K = \frac{k+1}{1-k}$. If now

$$\lim_{M\to\infty}\left\{\inf_{\mu_0\in\mathbb{D}}\left\{\sup_{\|A\|>M,\, A\in E} \operatorname{dist}_h(\mu_A, \mu_0)\right\}\right\} \leq \log(K), \tag{3.10}$$

then Theorem 3.6 holds.

4. Sets with Trivial Quasiconvex Hulls

Let E be a set such that $\det(A - B) > 0$ for $A, B \in E$. In [Z3], Zhang discovered that an enlightening way to express this fact is to describe E as the graph of a Lipschitz function $H\colon \Pi_{\mathrm{CO}_+}(E) \mapsto \mathrm{CO}_-$. Specifically, H is defined by $H(A) = \Pi_{\mathrm{CO}_-} \circ \Pi^{-1}_{\mathrm{CO}_+}(A)$. Furthermore, by a straightforward calculation we can see that E is K-quasiconformal at every point (see Definition 1.6) if and only if the Lipschitz norm of H is at most $k = \frac{K-1}{K+1}$. We combine Beltrami operators with the ideas of Zhang [Z4] and Iqbal [Iq] for the case $K = 1$ to obtain the following theorem.

THEOREM 4.1. *Let* $K > 1$ *and let* $E \subset \mathbf{M}^{2\times 2}$ *be K-quasiconformal at every point in the sense of Definition 1.6. Let* $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. *Then E has the following properties*:

(i) $\mathcal{H}^p(E)$ *is trivial*;

(ii) $E^{p,\mathrm{qc}} = E$; *and*

(iii) *there exists a constant* $C(p, k)$ *such that*

$$\operatorname{dist}_E^p(A) \leq C(p, k) Q \operatorname{dist}_E^p(A).$$

Proof. Let $H\colon \Pi_{\mathrm{CO}_+}(E) \mapsto \mathrm{CO}_-$ be the k-Lipschitz function such that E is the graph of H. We extend H (using Kirzsbraun's extension theorem) to a Lipschitz map, still denoted by H, that is defined on the whole conformal set $\mathrm{CO}_+(2)$.

Let $A \in \mathbf{M}^{2\times 2}$ and let $P(A)$ be such $|A - P(A)| = \operatorname{dist}_E(A)$. Then

$$|A_- - H(A_+)| = |A - P(A) + P(A)_+ - A_+ + H(P(A)_+) - H(A_+)|.$$

The Lipschitz condition implies that

$$\begin{aligned}|A_- - H(A_+)| &\le |A - P(A)| + (1+k)|P(A)_+ - A_+| \\ &\le (2+k)\operatorname{dist}_E(A). \end{aligned} \tag{4.1}$$

Let $A \in \mathbf{M}^{2\times 2}$ and let $\{\varphi_j\} \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$ be a sequence such that

$$\lim_{j\to\infty} \int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz = Q\operatorname{dist}_E^p(A).$$

A rescaling argument as in [Z4] shows that we can assume

$$\varphi_j \rightharpoonup 0 \ \text{ in } W_0^{1,p}(\mathbb{D}, \mathbb{R}^2). \tag{4.2}$$

Let φ_j, φ_i be two elements in the sequence. Using the estimate (4.1) at almost every point z yields

$$\begin{aligned}&\int_{\mathbb{D}} |A_- + \partial_{\bar z}\varphi_j(z) - H(A_+ + \partial_z\varphi_j(z))|^p\,dz \\ &\qquad + \int_{\mathbb{D}} |A_- + \partial_{\bar z}\varphi_i(z) - H(A_+ + \partial_z\varphi_i(z))|^p\,dz \\ &\qquad\qquad \le (2+k)^p\bigg(\int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz + \int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_i(z))\,dz\bigg).\end{aligned}$$

By the convexity of the pth power, the left-hand side of the inequality is larger than

$$\frac{1}{2^{p-1}}\int_{\mathbb{D}} |\partial_{\bar z}(\varphi_j - \varphi_i)(z) - H(A_+ + \partial_z\varphi_j(z)) + H(A_+ + \partial_z\varphi_i(z))|^p.$$

The important point here is choosing an appropriate sequence of Beltrami coefficients. Let

$$\mu_j(z) = \begin{cases} \frac{H(A_+ + \partial_z\varphi_j(z)) - H(A_+ + \partial_z\varphi_i(z))}{\partial_z(\varphi_j(z) - \varphi_i(z))} & \text{if } \varphi_j(z) - \varphi_i(z) \neq 0, \\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

Putting these equations together, we arrive at the following inequality:

$$\begin{aligned}&\int_{\mathbb{D}} |\partial_{\bar z}(\varphi_j - \varphi_i)(z) - \mu_j(z)\partial_z(\varphi_j - \varphi_i)(z)|^p\,dz \\ &\qquad \le 2^{p-1}(2+k)^p\bigg(\int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz + \int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz\bigg).\end{aligned}$$

The Lipschitz condition on H implies that $\mu_j(z) \le k$ and that $(\varphi_j - \varphi_i)(z)$ belongs to $C_0^\infty(\mathbb{D}, \mathbb{R}^2)$. Hence we can apply Theorem 2.1. Together with the boundedness of the Beurling–Alhfors transform, this yields

$$\begin{aligned}&\int_{\mathbb{D}} |D(\varphi_j - \varphi_i)(z)|^p\,dz \\ &\qquad \le C(k,p)\bigg(\int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz + \int_{\mathbb{D}} \operatorname{dist}_E^p(A + D\varphi_j(z))\,dz\bigg).\end{aligned} \tag{4.4}$$

Now we compute the limit when i tends to infinity. On the right-hand side we obtain $Q\,\mathrm{dist}_E^p(A)$. On the left-hand side we use that the functional $I(\varphi) = \int_{\mathbb{D}} |D\varphi_j - D\varphi|^p$ is sequentially weakly lower semicontinuous in $W^{1,p}$ owing to the convexity of $F_z(A) = |D\varphi_j(z) - A|^p$ for every z. We obtain

$$C(k,p)\bigg(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))\,dz + Q\,\mathrm{dist}_E^p(A)\bigg) \\ \geq \liminf_{i\to\infty} \int_{\mathbb{D}} |D(\varphi_j - \varphi_i)(z))|^p\,dz \geq \int_{\mathbb{D}} |D\varphi_j(z)|^p\,dz. \quad (4.5)$$

Finally, we use the following norm estimate. Let M_1, M_2 be two matrices in $\mathbf{M}^{2\times 2}$ and let E be a closed set. Then

$$|M_1|^p \geq \frac{1}{2^{p-1}}\,\mathrm{dist}_E^p(M_2) - \mathrm{dist}_E^p(M_1 + M_2). \quad (4.6)$$

Take $M_1 = D\varphi_j$ and $M_2 = A$ at every z in $\mathbb{D}$. Rearranging inequality (4.5), we have

$$\frac{1}{2^{p-1}}\,\mathrm{dist}_E(A) \leq C(k,p)\bigg(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))\,dz + Q\,\mathrm{dist}_E^p(A)\bigg).$$

Letting $j \to \infty$, claim (iii) is proved.

The definition of p-quasiconvex hull (see Definition 1.3) immediately yields claim (ii).

Let $\nu \in \mathcal{H}^p(E)$. Then by general principles there exists a sequence $\{\phi_j\} \in W^{1,\infty}(\mathbb{D}, \mathbb{R}^2)$ generating ν and such that

$$\lim_{j\to\infty} \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\phi_j(z))\,dz = 0.$$

Applying the argument of the proof (in particular, inequality (4.4)) to $\{\phi_j\}$ gives that the sequence $\{D\phi_j\}$ is a Cauchy sequence in L^p. This implies that ν is a Dirac Delta and hence proves claim (i). The proof of Theorem 4.1 is complete. □

REMARK 4.2. Sets that are quasiconformal at every point enjoy further interesting properties. It was proved by Zhang that Theorem 4.1(iii) implies that, if we denote by E_ε the epsilon neighborhood of E, there exists a constant C such that $E_\varepsilon^{p,\mathrm{qc}} \subset E_{C\varepsilon}$.

In a different direction, Theorem 4.1 implies that the only mappings $\varphi \in W^{1,p}(\mathbb{D}, \mathbb{C})$ affine in the boundary and such that $D\varphi \in E$ almost everywhere are affine. Šverák [Š1] proved that, if E is compact and K-quasiconformal at every point, then there exists a $p > 2$ such that every Lipschitz mapping φ with $D\varphi(z) \in E$ a.e. $z \in \mathbb{D}$ actually belongs to $W^{2,p}_{\mathrm{loc}}(\mathbb{D}, \mathbb{C})$. He studied the behavior of the quotients $\varphi_h = \frac{\varphi(x+ah)-\varphi(x)}{h}$ (where a is any unit vector) and proved that they satisfy Cacciopoli inequalities. Our notation simplifies his proof and extends the result, removing the compactness assumption and yielding the precise value of p. The idea is that E being K-quasiconformal at every point implies that, for every h, the φ_h are K-quasiregular mappings. This observation rather easily yields the following result.

Let $\frac{2K}{K+1} < q$, and let $\varphi \in W^{1,q}_{\mathrm{loc}}(\mathbb{D}, \mathbb{C})$ be such that $D\varphi \in E$, where E is K-quasiconformal at every point. Then $\varphi \in W^{2,p}_{\mathrm{loc}}$ for every $p < \frac{2K}{K-1}$ and $D\varphi$ is locally $\frac{1}{K}$ Hölder continuous. The argument combines a simple integration by parts with Astala's theorem [A].

If our control on the quasiconformality of E is only at infinity, then we can not estimate the norm of the Lipschitz function H whose graph is E and hence the foregoing proof does not work. However, results from previous sections—together with the weak continuity of the determinant when the integrability is large enough—immediately yield the following weaker theorem.

THEOREM 4.3. *Let $E \subset \mathbf{M}^{2\times 2}$ be an asymptotically K-quasiconformal set* (*cf. Definition 1.6*) *such that*

$$\det(A - B) > 0 \quad \textit{for all } A, B \in E. \tag{4.7}$$

Then, for every $q > \frac{2K}{K+1}$, we have that
(1) *$\mathcal{H}^q(E)$ is trivial and*
(2) *$E^{q,\mathrm{qc}} = E$.*

Proof. It is enough to prove the theorem for $\frac{2K}{K+1} < q < 2$, since Hölder's inequality implies that $E^{q,\mathrm{qc}} \subset E^{p,\mathrm{qc}}$ and that $\mathcal{H}^q(E) \subset \mathcal{H}^p(E)$ when $q \geq p$. By Theorem 3.6(i) it is enough to prove claim (1). But by Theorem 3.6(ii), $\mathcal{H}^q(E) = \mathcal{H}^p(E)$ for some $p > 2$. The determinant is weakly continuous in $W^{1,p}$ and hence every $\nu \in \mathcal{H}^p(E)$ satisfies the so-called minor relations:

$$\det\left(\int_{\mathbf{M}^{2\times 2}} \lambda \, d\nu(\lambda)\right) = \int \det(\lambda)\, d\nu(\lambda).$$

Then we can apply the arguments from [Š1] to conclude that ν must be a Dirac mass. □

REMARK 4.4. Corollary 2.3 gives generalizations of the results just described. In Theorem 4.1 we could replace requiring the K-quasiconformality of E at every point by instead requiring that E belong to the graph of a Lipschitz function H such that, for every A, B, the complex number $\frac{H(A)-H(B)}{A-B}$ belongs to some hyperbolic ball in the unit disc with radius $\log(K)$. Similarly, the requirement of asymptotic quasiconformality can be replaced by condition (3.10) in Remark 3.7. This should be compared with the laminates of infinite rank constructed in Section 5.

The content of the final result in the section is that, if we are studying gradient Young measures supported in a union of graphs of Lipschitz functions, then we can use the nonlinear Beltrami operators to obtain a generating sequence with gradients almost everywhere in the graphs.

PROPOSITION 4.5. *Let $\{H_i\}_{i=1}^N$ with $H_i \colon \mathrm{CO}_+(2) \to \mathrm{CO}_-(2)$ be a collection of k-Lipschitz functions with $k < 1$. Denote by $E_i \subset \mathbf{M}^{2\times 2}$ the graph of H_i. Let*

$q > 1+k$ and $E = \bigcup_{i=1}^{n} E_i$. Then every $\nu \in \mathcal{H}^q(E)$ can be generated by gradients Dg_j such that $Dg_j(z) \in E$ for a.e. z.

Proof. Let $\nu \in \mathcal{H}^q$ and let $D\varphi_j$ be the usual equi-integrable generating sequence. Then as always we have that $\lim_{j\to\infty} \int_{\mathbf{M}^{2\times 2}} \operatorname{dist}_E^q(D\varphi_j(z))\, dz = 0$. By the pointwise estimate (4.1) we obtain that

$$\lim_{j\to\infty} \int_{\mathbf{M}^{2\times 2}} |\partial_{\bar z}\varphi_j(z) - H_j(\partial_z \varphi_j(z), z)|^p \, dz = 0, \tag{4.8}$$

where $H_j(A, z) = H_{i(z)}(A)$ for some i. The existence and measurability of H_j is guaranteed by the measurable selection lemma.

Denote $h_j = \partial_{\bar z}\varphi_j(z) - H_j(\partial_z\varphi_j(z), z)$. We want to correct φ_j in order to obtain another sequence that satisfies the homogeneous elliptic PDE determined by H_j and with the same oscillating behavior. It clearly suffices to find, for every j, a solution F_j to the equation

$$\partial_{\bar z} F_j - H_j(\partial_z\varphi_j + \partial_z F_j, z) + H_j(\partial_z\varphi_j, z) = -h_j. \tag{4.9}$$

Then $g_j = F_j + \varphi_j$ would satisfy

$$\partial_{\bar z} g_j - H_j(\partial_z g_j, z) = 0$$

in $\mathbb{D}$. To solve (4.9) we observe that, for almost every z,

$$\hat{H}_j(A, z) = H_j(\partial_z\varphi_j(z) + A, z) - H_j(\partial_z\varphi_j(z), z)$$

is a k-Lipschitz map and $\hat{H}_j(0, z) = 0$. Hence, we can apply Theorem 2.2 and obtain that the operator $I - \hat{H}_j(S, z)$ is invertible for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. We can then follow the proof of [AF, Thm. 1.2] with $I - \hat{H}_j(S, z)$ in place of $I - \mu_j S$ and so obtain a solution to (4.9) in the whole complex plane. Furthermore, the solution comes with the estimate $\|DF_j\|_p \le C\|h_j\chi_{\mathbb{D}}\|_p$. From here on we can repeat the arguments in [AF] line by line to obtain that the sequence $\{Dg_j|_{\mathbb{D}}\}$ is a generating sequence for ν. □

5. Examples

In this section we present examples of connected sets E without rank-1 connections, and even K-quasiconformal at every point, but with nontrivial p-quasiconvex hulls. More precisely, for every $1 < q < 1+k$ we construct nontrivial measures $\nu \in \mathcal{H}^q(E)$. We use the two known methods for constructing homogeneous gradient Young measures. The first, known in the literature as the Riemann–Lebesgue lemma, consists of homogenizing a function with affine boundary values. It is of interest to observe that the appropriate function is a so-called very weak quasiregular mapping.

EXAMPLE 5.1. *Let $E \subset \mathbf{M}^{2\times 2}$ be defined by*

$$E \equiv \{A : A_- = k|A_+|,\ A_- \in \mathbb{R}\}$$

and $1 < q < 1+k$. Then

$$E^{q,\mathrm{qc}} = \mathrm{co}(E).$$

Proof. The example lives in the space of reflected symmetric matrices. We shall use conformal coordinates throughout the example. In these coordinates a matrix A is the composition of a reflection and a symmetric matrix if and only if its anticonformal coordinate is real. Consider the quasiconformal inversion

$$f(z) = \bar{z}|z|^{2K/(K+1)}.$$

A direct computation shows that the mapping belongs to the class $W^{1,q}(\mathbb{D}, \mathbb{C})$ and that $Df(z) \in E$ a.e. z. Moreover, on the boundary of the unit disc we have $f(z) = \bar{z}$. Since f is affine on the boundary of the unit disc, we can apply the classical result from gradient Young measure theory [P1, Lemma 8.2], which states that the probability measure ν_f defined by

$$\int_{\mathbf{M}^{2\times 2}} \varphi(\lambda)\, d\nu_f(\lambda) = \int_{\mathbb{D}} \varphi(Df(z))\, dz$$

belongs to $\mathcal{H}^q(E)$, where q is the integrability of Df. Moreover, ν_f's center of mass is equal to the affine boundary values. Therefore, in conformal coordinates we have that $\int_{\mathbf{M}^{2\times 2}} \lambda\, d\nu_f(\lambda) = (0, 1)$. Thus, $(0, 1)$ is in $E^{q,\mathrm{qc}}$. Similarly, $(0, \rho) \in E^{q,\mathrm{qc}}$ for every $\rho > 0$.

To deal with the rest of $\mathrm{co}(E)$, we observe that $A \in \mathrm{co}(E)$ if and only if $A_- \geq k|A_+|$. Drawing a picture on the diagonal plane illustrates that if $A = (A_+, A_-)$ with $A \in \mathrm{co}(E)$ and $A_+ > 0$ then A belongs to a rank-1 segment between the set E and the half-line $\{(0, \rho),\ \rho < 0\}$. Hence if $A \in \mathrm{co}(E)$ then we can find positive numbers $t, \rho, \tilde{B}_+$ such that

$$(|A_+|, A_-) = t(0, \rho) + (1 - t)(\tilde{B}_+, k\tilde{B}_+), \tag{5.1}$$

with $\tilde{B}_+ = \rho - k\tilde{B}_+$. The last follows because we must have that $\det(\tilde{B}_+, \rho - k\tilde{B}_+) = 0$. It is easy to see that this implies that A belongs to a rank-1 segment between $(0, \rho)$ and $\left(\frac{A_+}{1-t}, k\frac{A_+}{1-t}\right)$ with t as in (5.1). Since $Q\,\mathrm{dist}^q$ is convex along rank-1 segments, we obtain that $A \in E^{q,\mathrm{qc}}$. $\square$

This example answers in the negative a question raised in [Z5]. There it was asked whether E being asymptotically quasiconformal for some K is enough to guarantee $E^{1,\mathrm{qc}} = E$. As observed in [Z4] the result is true if $K = 1$, but Example 5.1 shows that it is false in general.

The next example is devoted to showing that our Theorem 4.1 is almost sharp. There are sets, consisting only of two lines, that saturate the bounds between the integrability of the approaching sequence and the quasiconformality of the set. We use the second method to construct homogeneous gradient Young measures, the so-called p-laminates. The literature on these measures is already extensive (see [K; P1] for the case $p = \infty$). Laminates and integrability issues were first considered in [F1]. In fact, the next example might be considered as dual to the one in [F1].

Let us recall that, in the plane of diagonal matrices, we use the notation

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = (d_1, d_2).$$

We also recall the definition of laminates and prelaminates.

DEFINITION 5.2. The family of prelaminates $\mathcal{PL}$ is the smallest family of probability measures on $\mathbf{M}^{n\times m}$ such that the following statements hold.

(1) $\mathcal{PL}$ contains all Dirac masses in $\mathbf{M}^{n\times m}$.
(2) Let $\nu = \sum_{i=1}^{k} \lambda_i \delta_{A_i} \in \mathcal{PL}$ and let $A_1 = \lambda B + (1-\lambda)C$, where $\lambda \in [0,1]$ and $[B, C]$ is a rank-1 segment. Then the probability measure $\sum_{i=2}^{k} \lambda_i \delta_{A_i} + \lambda_1(\lambda\delta_B + (1-\lambda)\delta_C) \in \mathcal{PL}$.

DEFINITION 5.3. Let ν be a probability measure on $\mathbf{M}^{n\times m}$ and let $1 \le p < \infty$. Then ν is said to be a p-laminate if there exists a sequence of prelaminates ν_j such that

(a) $\sup_j \int_{\mathbf{M}^{n\times m}} |\lambda|^p \, d\nu_j(\lambda) < \infty$ and
(b) $\nu_j \overset{\star}{\rightharpoonup} \nu$ in $\mathcal{M}(\mathbf{M}^{n\times m})$.

EXAMPLE 5.4. *Let $E \subset D$, where D is the diagonal plane, be defined by*

$$E = \begin{cases} (a, Ka) & \text{if } a \ge 0, \\ (Ka, a) & \text{if } a < 0. \end{cases}$$

Let $q < 1 + k = \frac{2K}{K+1}$. Then

$$E^{q,\mathrm{qc}} = \mathrm{co}(E).$$

Proof. For the proof we construct a laminate ν with center of mass $(-1, 1)$. Since a dual construction was explained in detail in [F1], we shall briefly sketch the main steps in the construction of ν. Consider the sequence $A_n = (-n, n)$. We shall build prelaminates ν_n supported in $E \cup A_n$ and then take the weak-star limit when $n \to \infty$. A computation of the integrability of limit measure ν will yield the result. The key is use of the following equalities:

$$(-n, n) = \frac{K}{n(K+1)+K}\left(\frac{n}{K}, n\right) + \left(1 - \frac{K}{n(K+1)+K}\right)(-(n+1), n);$$

$$\begin{aligned}(-(n+1), n) = {} & \frac{K}{(n+1)(1+K)}\left(-(n+1), \frac{-(n+1)}{K}\right) \\ & + \left(1 - \frac{K}{(n+1)(1+K)}\right)(-(n+1), n+1).\end{aligned} \tag{5.2}$$

In Figure 1 we use the following notation for the auxiliary matrices $B_n = \left(\frac{n}{K}, n\right)$, $C_n = (-(n+1), n)$, and $D_n = \left(-(n+1), \frac{-(n+1)}{K}\right)$. Because the convex combinations in (5.2) occur along rank-1 segments (see the figure), we can use the closedness of prelaminates under splitting; see Definition 5.2(2). The basic idea is as follows.

The matrix A_1 is the center of mass of a laminate ν_1 supported on B_1 and C_1. Then δ_{C_1} can be split into a measure supported in A_2 and D_1. This gives as the measure ν_2 supported on $\{B_1, D_1, A_2\}$. However, δ_{A_2} can be split into a measure supported on B_2 and C_2. Furthermore, δ_{C_2} can be split into a measure supported on D_2 and A_3, so we have a prelaminate ν_3 supported on $\{B_1, D_1, B_2, D_2, A_3\}$.

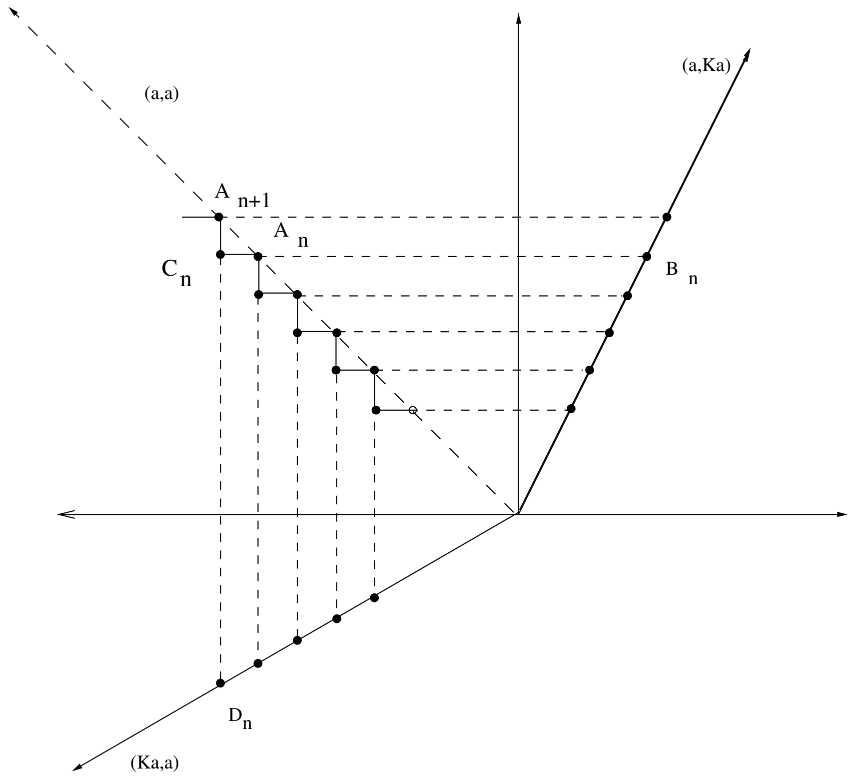

Figure 1 Black dots denote the support of the measure ν_n; the white dot is the center of mass

Figure 1 and equalities (5.2) indicate how the construction can be iterated to obtain the required sequence of measures ν_n.

It is easy to check that these prelaminates will enjoy the following properties. First, they can be written as

$$\nu_n = \mu_n + \lambda_n \delta_{A_n},$$

where

$$\lambda_n = \prod_{i=1}^{n-1}\left(1 - \frac{K}{i(K+1)+K}\right)\left(1 - \frac{K}{(i+1)(1+K)}\right) \tag{5.3}$$

and ν_n is supported in $E \cap B\big(0, n\big|\big(1, \frac{1}{K}\big)\big|\big)$. Second, the center of mass is fixed: $\int_{\mathbf{M}^{2\times 2}} \lambda \, d\nu(\lambda) = (-1, 1)$ for every n. The measure ν is then defined as the weak-star limit of the measures ν_n. The integrability of ν follows from standard arguments in measure theory and the basic properties of logarithms. The details are explained in the dual construction in [F1]. Let us state the result: For every $q < \frac{2K}{K+1}$,

$$\int_{\mathbf{M}^{2\times 2}} |\lambda|^q \, d\nu(\lambda) < \infty$$

but

$$\int_{\mathbf{M}^{2\times 2}} |\lambda|^{2K/(K+1)} \, d\nu(\lambda) = \infty.$$

Thus we have proved that $(-1, 1) \in E^{q,\mathrm{qc}}$. It is clear that our construction remains the same if we multiply everything by a scalar $\rho > 0$, whence it follows that $(-\rho, \rho) \in E^{q,\mathrm{qc}}$. As in the previous example, this suffices to conclude that $E^{q,\mathrm{qc}} = \mathrm{co}(E)$ as desired. □

REMARK 5.5. Let μ_1, μ_2 be such that $\mathrm{dist}_h(\mu_1, \mu_2) = 2\log(K)$. The affine change of variables used in Section 2 gives us the natural plane of matrices where a q-laminate supported in two lines with dilatations μ_1, μ_2 lives. The change of variables is given by the unique matrix A such such that if $\mu_f \in \{k, -k\}$ then $\mu_{f\circ A} \in \{\mu_1, \mu_2\}$. Here, of course, $k = \frac{K+1}{K-1}$.

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