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Elementary Transversality in the Schubert Calculus in Any Characteristic

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Introduction

Schubert [15] declared enumerative geometry to be concerned with all questions of the following form: How many geometric figures of some type satisfy certain given conditions? For conditions imposed by general fixed figures, the approach—then, as now—is to interpret the conditions as subvarieties of a parameter space of figures, which give cycle classes in the Chow ring of that parameter space. Then the degree of the product of these cycle classes provides an algebraic count of the solutions, counting the solutions weighted with certain intersection multiplicities. Thus this degree solves the original problem of enumeration—if each solution occurs with multiplicity 1, that is, if the subvarieties meet transversally at each point of intersection.

This demonstrates how the validity of the standard approach to enumerative geometry via multiplying cycle classes in a Chow ring rests upon the following basic premise in enumerative geometry: General subvarieties of the parameter space meet transversally at each point of intersection. An intersection number is enumerative if this basic premise holds for the corresponding intersection. Kleiman [8] established this basic premise in characteristic 0, when an algebraic groups acts transitively on the parameter space. The restriction to characteristic 0 is necessary. General translates of arbitrary subvarieties simply do not meet transversally in positive characteristic. Kleiman [8] exhibits a subvariety of a grassmannian that does not meet any translate of a particular codimension-2 Schubert variety transversally.

However, Kleiman's example does not arise in an enumerative geometric problem. In fact, in every known case, general Schubert subvarieties of flag varieties meet generically transversally (transverse along a dense subset of each component) in any characteristic. Thus the question remains: To what extent does this basic principle of enumerative geometry hold in positive characteristic? Laksov and Speiser develop a general theory for transversality: they give a condition, using tangent spaces to families of subvarieties, that implies a general member of a family meets any fixed subvariety transversally [9; 10; 11; 25]. By Kleiman's example, families of codimension-2 Schubert subvarieties do not satisfy this condition.

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By Theorem E of [17], general Schubert subvarieties of a grassmannian of 2-planes in a vector space meet transversally in any characteristic. Here, we give an elementary and characteristic-free proof that general simple (codimension-1 or divisorial) Schubert varieties meet transversally when the ambient space is one of the following: (i) the grassmannian, (ii) the flag manifold for the special linear group, (iii) the orthogonal grassmannian, or (iv) the space of parameterized rational curves of a fixed degree in a grassmannian. These results are rather special in that they involve only simple Schubert varieties, but such enumerative problems are in fact quite natural. More importantly, these results suggest that transversality is ubiquitous in enumerative geometry.

Thus, the corresponding intersection numbers are enumerative for fields of any characteristic (except characteristic 2 for the orthogonal grassmannian). This includes some genus-0 Gromov–Witten invariants of the grassmannian, by (iv). For example, given 12 general 4-planes in 7-dimensional space, there are exactly 462 3-planes that meet all 12. (This number was computed by Schubert [16].) Similarly, given $N = 5q + 6$ general points $s_1, s_2, \ldots, s_N$ in $\mathbb{P}^1$ and N general 3-planes $K_1, K_2, \ldots, K_N$ in 5-dimensional space, there are exactly F_{5+5q} (the $(5 + 5q)$th Fibonacci number) degree-q maps M from $\mathbb{P}^1$ to the grassmannian of 2-planes in 5-space satisfying $M(s_i) \cap K_i \neq \{0\}$ for each $i = 1, \ldots, N$ [7; 13].

We use the formalism of [21] to show transversality. Although it was developed to construct real-number solutions to these enumerative problems, it can also be used to show transversality. We state our results in Section 1, review the formalism of [21] in Section 2, and prove our results in Section 3. In Section 4, we strengthen the real enumerative geometric results of [19; 20; 21]. Finally, in Section 5 we show that families of simple Schubert varieties do not satisfy the condition of Laksov and Speiser.

Although the simple Schubert varieties we study are members of a linear series of sections of an ample line bundle, the general section is not a Schubert variety and so standard Bertini-type theorems in positive characteristic do not apply. Our results, together with the inapplicability of the general theory of Laksov and Speiser, emphasize the need for new, perhaps less general or more specific, criteria that imply transversality. For example, is there a Bertini-type theorem for the grassmannian concerning intersections with a general codimension-1 Schubert variety?

Similarly, our results and extensive computer calculations (see e.g. [22, Sec. 4.3]) suggest that the classical Schubert calculus of enumerative geometry (and the quantum Schubert calculus) is enumerative in all characteristics (except characteristic 2 for the orthogonal groups). Specifically, we make the following conjecture, for which we feel there is ample support (from both theoretical and practical examples).

CONJECTURE 1. *Let $\mathbb{K}$ be an algebraically closed field, and let X be a flag variety for a reductive algebraic group defined over $\mathbb{K}$. If $X_1, X_2, \ldots, X_s$ are Schubert varieties of X in general position and if the sum of their codimensions equals the dimension of X, then the intersection*

$$X_1 \cap X_2 \cap \cdots \cap X_s$$

is transverse.

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1. Statement of Results

Let $\mathbb{K}$ be any infinite field. We describe the spaces and simple Schubert varieties for which we prove transversality.

(i) Let $0 < r < n$ be integers. Let $\mathbf{G}(r, n)$ denote the grassmannian of r-planes in $\mathbb{K}^n$. An $(n-r)$-dimensional subspace ($(n-r)$-plane) K defines a simple (codimension-1) Schubert subvariety of $\mathbf{G}(r, n)$,

$$\Omega(K) := \{H \in \mathbf{G}(r, n) \mid H \cap K \neq \{0\}\}.$$

(ii) For a sequence $\mathbf{r} : 0 < r_1 < r_2 < \cdots < r_m < n$ of integers, let $\mathbb{F}\ell_{\mathbf{r}}$ be the variety of m-step flags

$$E_\bullet : E_1 \subset E_2 \subset \cdots \subset E_m \subset \mathbb{K}^n$$

with $\dim E_i = r_i$ for $i = 1, \ldots, m$. An $(n - r_i)$-plane K defines a simple Schubert subvariety of $\mathbb{F}\ell_{\mathbf{r}}$

$$\Phi_i(K) := \{E_\bullet \in \mathbb{F}\ell_{\mathbf{r}} \mid E_{r_i} \cap K \neq \{0\}\}.$$

(iii) Suppose that the characteristic of $\mathbb{K}$ is not 2 and that $n = 2r + 1$ is odd. Let $\langle\cdot, \cdot\rangle$ be a nondegenerate split symmetric bilinear form on $\mathbb{K}^n$—that is, one for which there is a basis $e_1, \ldots, e_n$ of $\mathbb{K}^n$ such that

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i + j = n + 1 = 2r + 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The resulting quadratic form is

$$q\Big(\sum x_i e_i\Big) = 2x_1x_{2n-1} + 2x_2x_{2n-2} + \cdots + 2x_r x_{r+2} + x_{r+1}^2. \tag{1.2}$$

A subspace H of $\mathbb{K}^n$ is *isotropic* if $\langle\cdot, \cdot\rangle|_H \equiv 0$, and isotropic subspaces have dimension at most r. The orthogonal grassmannian $\mathrm{OG}(r)$ is the space of maximal (dimension-r) isotropic subspaces of $(\mathbb{K}^n, \langle\cdot, \cdot\rangle)$. This has dimension $\binom{r+1}{2}$ and, by our choice of the form (1.1), the $\mathbb{K}$-rational points are dense. (This may be deduced, for example, from the coordinates for Schubert cells of the orthogonal flag manifold given in [5, p. 67].) An isotropic r-plane K of $\mathbb{K}^n$ defines a simple Schubert subvariety of $\mathrm{OG}(r)$ as follows:

$$\Psi(K) := \{H \in \mathrm{OG}(r) \mid H \cap K \neq \{0\}\}.$$

(iv) For an integer $q \geq 0$, let $\mathcal{M}^q_{r,n}$ be the space of degree-q maps $M\colon \mathbb{P}^1 \to \mathbf{G}(r,n)$. A point $s \in \mathbb{P}^1$ and an $(n-r)$-plane K define a simple (quantum) Schubert subvariety of $\mathcal{M}^q_{r,n}$,

$$Z(s,K) := \{M \in \mathcal{M}^q_{r,n} \mid M(s) \cap K \neq \{0\}\}.$$

Each of these four spaces have more general *Schubert varieties.* We will prove the following theorem.

THEOREM 1.1. *Suppose $\mathbb{K}$ is infinite. Let X be either $\mathbf{G}(r,n)$, $\mathbb{F}\ell_{\mathbf{r}}$, $\mathrm{OG}(r)$, or $\mathcal{M}^q_{r,n}$, and suppose that $Z \subset X$ is any Schubert variety of X. Then general simple Schubert varieties $Y_1, Y_2, \ldots, Y_{\dim Z}$ meet Z transversally on X.*

We prove Theorem 1.1 by exhibiting 1-parameter families of simple Schubert varieties having properties which imply that general members of these families intersect transversally. We remark that in each case (except for the classical flag manifold) there is only one simple Schubert variety, up to the action of the relevant group ($\mathrm{GL}_n(\mathbb{K})$, $O_{2n+1}(\mathbb{K})$, or $\mathrm{GL}_n(\mathbb{K}) \times P\mathrm{GL}_2(\mathbb{K})$). Since we use an action of the 1-dimensional torus $\mathbb{G}_m$ to construct our 1-parameter families, there are in fact large moduli of different families. The careful reader will notice that in all arguments (except those for the classical flag manifold) we could dispense with multiple families and use only one family but with different fibres.

2. The Method of Schubert Induction

We review the method of Schubert induction introduced in [21]. Let $\mathbb{G}_m$ be the group scheme whose $\mathbb{K}$-rational points are the invertible elements $\mathbb{K}^\times$ of $\mathbb{K}$, considered as a group under multiplication. While this theory holds for families of subvarieties over any curve, we use $\mathbb{G}_m$ as the base for our families. Suppose that X is a projective variety and that we have a subvariety $E \subset X \times \mathbb{G}_m$ where the fibres $E \to \mathbb{G}_m$ are equidimensional, that is, a family $E \to \mathbb{G}_m$ of subvarieties of X. Then the scheme-theoretic limit $\lim_{s\to 0} E(s)$ is defined to be the fibre over 0 of the closure $\bar{E}$ of E in $X \times \mathbb{K}$. Since $\mathbb{G}_m$ is a curve, this fibre has the expected dimension (see [6, Rem. 9.8.1]).

A *Bruhat decomposition* of an irreducible variety X defined over a field $\mathbb{K}$ is a finite decomposition

$$X = \coprod_{w \in I} X^\circ_w$$

satisfying the following conditions.

(1) Each stratum X°_w is a locally closed irreducible subvariety defined over $\mathbb{K}$.
(2) The closure $\overline{X^\circ_w}$ of a stratum is a union of some strata X°_v.
(3) There is a unique 0-dimensional stratum $X^\circ_{\hat{0}}$.

For $w \in I$, define the *Schubert variety* X_w to be $\overline{X^\circ_w}$. The space X is the top-dimensional Schubert variety. By condition (2), the intersection $X_w \cap X_v$ of two Schubert varieties is a union of some Schubert varieties. The *Bruhat order* on I

is the order induced by inclusion of Schubert varieties: $u \le v$ if $X_u \subseteq X_v$. Set $|w| := \dim X_w$.

Let $\mathcal{Y} \to \mathbb{G}_m$ be a family of divisors on X. For $s \in \mathbb{G}_m$, let $Y(s)$ be the fibre of $\mathcal{Y}$ over the point s. We say that $\mathcal{Y}$ *respects* the Bruhat decomposition if, for every $w \in I$, the (scheme-theoretic) limit $Z := \lim_{s\to 0}(Y(s) \cap X_w)$ is supported on a union of Schubert subvarieties X_v of codimension 1 in X_w. Write $v \prec_{\mathcal{Y}} w$ when X_v is the support of a component of the limit scheme Z. This relation generates a suborder of the Bruhat order and, as we shall explain, the number of solutions to an enumerative problem given by simple Schubert varieties equals the number of saturated chains in such a suborder. Our notation for this suborder, $\prec_{\mathcal{Y}}$, reflects its dependence on the family $\mathcal{Y}$. A family $\mathcal{Y} \to \mathbb{G}_m$ of divisors of X is *multiplicity-free* if it respects the Bruhat decomposition and if each component of the scheme-theoretic limit Z is reduced at its generic point.

The general fibre $Y(s)$ of such a multiplicity-free family of divisors on X meets each Schubert variety X_w generically transversally. Indeed, if the general fibre of $\mathcal{Y}$ did not meet X_w generically transversally, then every intersection $Y(s)\cap X_w$ would have a nonreduced component. This would imply that $\lim_{s\to 0} Y(s) \cap X_w$ has a nonreduced component, violating our assumption that the family $\mathcal{Y}$ is multiplicity-free.

Thus when X is smooth and $Y(s)$ is a general member of the family $\mathcal{Y}$, we have the formula

$$[X_w] \cdot [Y(s)] = [X_w \cap Y(s)] = \sum_{v \prec_{\mathcal{Y}} w} [X_v] \tag{2.1}$$

in the Chow ring of X. The second equality in (2.1) expresses the rational equivalence of the fibres of the family $\mathcal{Y} \cap (X_w \times \mathbb{K})$, and the first equality is a basic property of any intersection theory—namely, that a generically transverse intersection of subvarieties of X represents the product of their cycle classes because the intersection multiplicities are equal to 1 [4, Rem. 8.2, p. 138].

A collection $\mathcal{Y}_1, \dots, \mathcal{Y}_l$ of families of divisors of X "meets the Bruhat decomposition of X properly" (or is "in general position with respect to the Bruhat decomposition" [21]) if for all $w \in I$, for general $s_1, s_2, \dots, s_l \in \mathbb{G}_m$, for each $1 \le k \le l$, and for every k-subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, l\}$, the intersection

$$Y_{i_1}(s_{i_1}) \cap Y_{i_2}(s_{i_2}) \cap \cdots \cap Y_{i_k}(s_{i_k}) \cap X_w \tag{2.2}$$

is *proper* in that either it is empty or else it has (the expected) dimension $|w| - k$. This definition makes sense even if some (or all) of the families $\mathcal{Y}_i$ happen to be repeated.

Suppose that we have a collection $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_l$ ($l = \dim X$) of multiplicity-free families of divisors of X with each meeting the Bruhat decomposition of X properly. Recall that $\hat{0}$ is the index of the minimal Schubert variety, which is a single point. For $w \in I$, let $\deg(w)$ count the number of (saturated) chains in the Bruhat order

$$\hat{0} \prec_1 w_1 \prec_2 w_2 \prec_3 \cdots \prec_{k-1} w_{k-1} \prec_k w_k = w, \tag{2.3}$$

where $\prec_i = \prec_{\mathcal{Y}_i}$ and $|w| = k$. This is the degree of the intersection,

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_k(s_k) \cap X_w. \tag{2.4}$$

A result of [21] asserts that this intersection is free of multiplicities.

PROPOSITION 2.1 [21]. *Suppose X has a Bruhat decomposition. Let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_l$ be multiplicity-free families of divisors in X over $\mathbb{G}_m$ meeting the Bruhat decomposition of X properly, and suppose that each respects the Bruhat decomposition. Then, for every $1 \le k \le l$ and every $w \in I$ with $|w| = k$, the intersection* (2.4) *is transverse for general $s_1, s_2, \ldots, s_k \in \mathbb{G}_m$ and has degree* $\deg(w)$. *In particular, when $\mathbb{K}$ is algebraically closed, such an intersection consists of* $\deg(w)$ *reduced points.*

The point of Proposition 2.1 is that while we assume only that the intersections (2.2) have the expected dimension, we are able to conclude that (2.4) is transverse and to compute the number of solutions without reference to the Chow ring. There is a simple criterion (Lemma 2.4 of [21]) which implies that a collection of families meets the Bruhat decomposition properly.

LEMMA 2.2. *Suppose a variety X has a Bruhat decomposition. Let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_l$ be a collection of families of divisors of X. If each family $\mathcal{Y}_k \to \mathbb{G}_m$ satisfies*

$$\bigcap_{s \in \mathbb{G}_m} \mathcal{Y}_k(s) = \emptyset,$$

then the collection of families $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_l$ meets the Bruhat decomposition properly.

Proof. Here we illustrate the elementary methods of [21].

If the families $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_l$ do not meet the Bruhat decomposition properly, then (after possibly reindexing) there exist a $w \in I$, an integer k, and points $s_1, s_2, \ldots, s_k \in \mathbb{G}_m$ such that

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_k(s_k) \cap X_w \tag{2.5}$$

has (the expected) dimension of $|w| - k \ge 0$; yet for every $s \in \mathbb{G}_m$, the intersection

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_k(s_k) \cap Y_{k+1}(s) \cap X_w$$

also has dimension $|w| - k$. But then some component of (2.5) lies in every subvariety $Y_{k+1}(s)$, contradicting the assumption of the lemma. $\square$

3. Proof of Theorem 1.1

By Proposition 2.1, for each space $\mathbf{G}(r, n)$, $\mathbb{F}\ell_{\mathbf{r}}$, $\mathrm{OG}(r)$, and $\mathcal{M}^q_{r,n}$, we need only construct multiplicity-free families of simple Schubert subvarieties over $\mathbb{G}_m$ such that the entire collection meets the Bruhat decomposition properly. For the space $\mathcal{M}^q_{r,n}$ of rational curves in a grassmannian, we work in Drinfel′d's compactification $\mathcal{K}^q_{r,n}$, also called the quantum grassmannian.

3.1. The Grassmannian

Let $e_1, e_2, \ldots, e_n$ be an ordered basis for $\mathbb{K}^n$. For a sequence $\alpha\colon 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r \le n$, the Schubert variety Ω_α is

$$\Omega_\alpha := \{H \in \mathbf{G}(r, n) \mid \dim H \cap F_{\alpha_j} \ge j \ \text{ for } \ j = 1, 2, \ldots, r\},$$

where F_i is the linear span of $e_1, e_2, \ldots, e_i$. Set $\Omega^\circ_\alpha := \Omega_\alpha - \{\Omega_\beta \mid \beta < \alpha\}$. Here, $\le$ is given by componentwise comparison: $\alpha \le \beta$ if and only if $\alpha_i \le \beta_i$ for all i. For example, $(1, 3, 5) \le (1, 4, 5) \le (2, 5, 6)$, but $(1, 4, 5) \not\le (2, 3, 6)$. Write $\binom{[n]}{r}$ for the resulting partial order on this set of sequences. With these definitions, the grassmannian $\mathbf{G}(r, n)$ has a Bruhat decomposition

$$\mathbf{G}(r, n) = \coprod_{\alpha \in \binom{[n]}{r}} \Omega^\circ_\alpha,$$

indexed by sequences $\alpha \in \binom{[n]}{r}$. For $\alpha \in \binom{[n]}{r}$, set $|\alpha| = \sum(\alpha_i - i)$, which is the dimension of Ω_α. Write $\beta \lessdot \alpha$ when $\beta < \alpha$ with $|\beta| = |\alpha| - 1$.

If H is the row space of an $r \times n$ matrix, then the $\binom{n}{r}$ maximal minors of that matrix are the Plücker coordinates of H. Write these as p_α for $\alpha \in \binom{[n]}{r}$. These Plücker coordinates give an embedding of $\mathbf{G}(r, n)$ into $\mathbb{P}^{\binom{n}{r}-1}$.

We use the following elementary set-theoretic fact, originally due to Schubert [16]:

$$\Omega_\alpha \bigcap \{p_\alpha = 0\} = \bigcup_{\beta \lessdot \alpha} \Omega_\beta, \tag{3.1}$$

and the intersection is generically transverse. This is a consequence of the related ideal- (or scheme-) theoretic fact, which may be deduced from the combinatorics of the Plücker ideal of $\mathbf{G}(r, n)$. There are, however, very elementary reasons that intersection (3.1) is as claimed (set-theoretically) and is generically transverse. Indeed, if we set

$$K := \operatorname{Span}\langle e_i \mid i \in \{1, 2, \ldots, n\} - \{\alpha_1, \ldots, \alpha_r\}\rangle,$$

then $\Omega(K)$ has equation $p_\alpha = 0$ in the Plücker coordinates, and a simple computation in local coordinates proves the equality in (3.1) and shows as well that it is generically transverse. (For a synthetic argument, see [18, Thm. 2.4(2)].)

Proof of Theorem 1.1 for $\mathbf{G}(r, n)$. Let $K \subset \mathbb{K}^n$ be an $(n - r)$-plane, none of whose Plücker coordinates vanish. Since no Plücker coordinate vanishes identically on the grassmannian, such a plane exists because $\mathbb{K}$ is infinite. Let $\mathbb{G}_m$ act on $\mathbb{K}^n$ by $s \cdot e_j = s^j e_j$ for $j = 1, \ldots, n$, and set $K(s) := s \cdot K$. Let $\mathcal{Y}(K) \to \mathbb{G}_m$ be the family of simple Schubert varieties whose fibre over $s \in \mathbb{G}_m$ is the Schubert variety $\Omega(K(s))$. Recall that $\Omega(K(s))$ is the set of those r-planes H that have a nontrivial intersection with the $(n - r)$-plane $K(s)$.

Theorem 1.1 is a consequence of Proposition 2.1 and the following claim.

CLAIM. *Let $K_1, K_2, \ldots, K_l$ be $(n-r)$-planes, each with no vanishing Plücker coordinates.*

(i) *Each family $\mathcal{Y}(K_i)$ preserves the Bruhat decomposition of $\mathbf{G}(r,n)$ and is multiplicity-free.*

(ii) *The collection of families $\mathcal{Y}(K_1), \mathcal{Y}(K_2), \ldots, \mathcal{Y}(K_l)$ meets the Bruhat decomposition properly.*

Proof. Represent K_i as the row space of an $(n-r)\times n$ matrix K_i. Then $K_i(s)$ is represented by the same matrix but with column j multiplied by s^j. If an r-plane H is the row space of an $r\times n$ matrix H, then

$$H \cap K_i(s) \neq \{0\} \iff \det\begin{bmatrix} K_i(s) \\ H \end{bmatrix} = 0.$$

Laplace expansion of the determinant along the rows of H gives the equation in Plücker coordinates for H to lie in $\Omega(K_i(s))$:

$$0 = \sum_{\beta\in\binom{[n]}{r}} p_\beta k_\beta s^{r(n-r)+\binom{n-r+1}{2}-|\beta|}, \tag{3.2}$$

where k_β is the appropriately signed maximal minor of K_i for the columns that are complementary to β. Up to a sign, this is a Plücker coordinate of K_i.

If we restrict (3.2) to the Schubert variety Ω_α and divide by the common factor $s^{r(n-r)+\binom{n-r+1}{2}-|\alpha|}$, the result is a polynomial with constant term $p_\alpha k_\alpha$. Thus, the limit scheme $\lim_{s\to 0}(\Omega_\alpha \cap \Omega(K_i(s)))$ is defined in Ω_α by $p_\alpha = 0$ and so, by (3.1),

$$\lim_{s\to 0}(\Omega_\alpha \cap \Omega(K_i(s))) = \sum_{\beta \lessdot \alpha} \Omega_\beta,$$

which proves claim (i).

For a fixed $H \in \mathbf{G}(r,n)$, (3.2) is a nonzero polynomial in s and so there are finitely many $s\in\mathbb{G}_m$ with $H\in\Omega(K_i(s))$. In particular, $\bigcap_{s\in\mathbb{G}_m}\Omega(K_i(s)) = \emptyset$. By Lemma 2.2, this implies claim (ii) and thus completes the proof of Theorem 1.1 for $\mathbf{G}(r,n)$. □

3.2. *The Flag Manifold*

Fix an ordered basis $e_1, e_2, \ldots, e_n$ for $\mathbb{K}^n$. Let $\mathbf{r}: 0 < r_1 < r_2 < \cdots < r_m < n$ be integers. The flag manifold $\mathbb{F}\ell_{\mathbf{r}}$ has a Bruhat decomposition

$$\mathbb{F}\ell_{\mathbf{r}} = \coprod X_w^\circ,$$

indexed by those permutations $w = w_1 w_2 \cdots w_n$ in the symmetric group $\mathcal{S}_n$ on n letters whose descent set $\{i \mid w_i > w_{i+1}\}$ is a subset of $\{r_1, r_2, \ldots, r_m\}$. Here, X_w° is a Schubert cell of $\mathbb{F}\ell_{\mathbf{r}}$ (see [3, Sec. 9]).

Let $\mathbb{G}_m$ act on $\mathbb{K}^n$ by $s\cdot e_j = s^j e_j$ for $j = 1, \ldots, n$. Suppose K is an $(n-r_i)$-plane, none of whose Plücker coordinates vanish. For $s\in\mathbb{G}_m$, set $K(s) := s\cdot K$. Let $\mathcal{Y}(K)\to\mathbb{G}_m$ be the family whose fibre over $s\in\mathbb{G}_m$ is the simple Schubert variety $\Phi_i(K(s))$. (Recall that this is the collection of flags $E_\bullet$ whose r_i-dimensional component E_i has a nontrivial intersection with the $(n-r_i)$-plane $K(s)$.) The projection $\mathbb{F}\ell_{\mathbf{r}} \to \mathbf{G}(r_i, n)$ sends a flag $E_\bullet \in \mathbb{F}\ell_{\mathbf{r}}$ to its ith component $E_i \in \mathbf{G}(r_i, n)$, and $\Phi_i(K(s))$ is the inverse image of $\Omega(K(s))$. It is an easy consequence (see [21, Sec. 2] for details) of the results proven in Section 3.1 that the family $\mathcal{Y}(K)$ preserves the Bruhat decomposition of the flag variety $\mathbb{F}\ell_{\mathbf{r}}$ and is multiplicity-free. (This last fact follows from Monk's formula [12].) Since the intersection $\bigcap_{s\in\mathbb{G}_m} \Omega(K(s))$ of Schuberet varieties in the grassmannian is empty, we have $\bigcap_{s\in\mathbb{G}_m} \Phi_i(K(s)) = \emptyset$. Hence we conclude that, if we have any $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, m\}$ and $(n - r_{i_j})$-planes K_j for $j = 1, \ldots, l$, none of whose Plücker coordinates vanish, then the families $\mathcal{Y}(K_1), \mathcal{Y}(K_2), \ldots, \mathcal{Y}(K_l)$ meet the Bruhat decomposition properly.

These facts together establish Theorem 1.1 for the flag manifold $\mathbb{F}\ell_{\mathbf{r}}$.

3.3. The Orthogonal Grassmannian

Suppose that the characteristic of $\mathbb{K}$ is not 2 and that $n = 2r + 1$ is odd. Let $e_1, e_2, \ldots, e_n$ be a basis for $\mathbb{K}^n$ for which the symmetric bilinear form is as given by (1.1). The orthogonal grassmannian has a Bruhat decomposition

$$\mathrm{OG}(r) = \coprod X_\lambda^\circ,$$

indexed by decreasing sequences λ of integers $n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. This decomposition is induced from that of $\mathbf{G}(r, n)$ by the inclusion $\iota\colon \mathrm{OG}(r) \hookrightarrow \mathbf{G}(r, n)$. As before, X_λ° is a Schubert cell of $\mathrm{OG}(r)$.

Let $\mathbb{G}_m$ act on $\mathbb{K}^n$ by $s \cdot e_i = s^i e_i$. This induces an action of $\mathbb{G}_m$ on the orthogonal grassmannian, since the quadratic form (1.2) is homogeneous of degree $2n+2$ under this action. Given $K \in \mathrm{OG}(r)$ with no vanishing Plücker coordinates, set $K(s) := s \cdot K$ and let $\mathcal{Y}(K) \to \mathbb{G}_m$ be the family of simple Schubert varieties with fibres $\Psi(K(s))$. Recall that the simple Schubert variety $\Psi(K(s))$ consists of those isotropic planes $H \in \mathrm{OG}(r)$ having a nontrivial intersection with $K(s)$. Since $K(s)$ is isotropic, it is a subset of its annihilator $K(s)^\perp$, which has dimension $r + 1$. Furthermore, $K(s) = K(s)^\perp \cap Q$, where Q is the quadric of isotropic points in $\mathbb{K}^n$. It follows that $\Psi(K(s))$ is the intersection of $\mathrm{OG}(r)$ with the simple Schubert variety $\Omega(K(s)^\perp)$ of $\mathbf{G}(r, n)$; that is, $\Psi(K(s)) = \iota^{-1}(\Omega(K(s)))$. Note that $K(s)^\perp = s \cdot K^\perp$, since $\mathbb{G}_m$ preserves the quadratic form.

As in Section 3.2 (see [21, Sec. 3]), the results proven for the family $\Omega(K(s)^\perp)$ in Section 3.1 imply the corresponding results for the family $\mathcal{Y}(K)$ and for any collection of such families. (Multiplicity-freeness follows from a cohomological formula due to Chevalley [2].)

We have thus established Theorem 1.1 for the orthogonal grassmannian $\mathrm{OG}(r)$.

3.4. The Space of Rational Curves in the Grassmannian

The space $\mathcal{M}_{r,n}^q$ of degree-q maps $M\colon \mathbb{P}^1 \to \mathbf{G}(r, n)$ is a smooth quasi-projective variety of dimension $qn + r(n - r)$. The Plücker coordinates of such a map are homogeneous forms of degree q. Choosing $\mathbb{K} \subset \mathbb{P}^1$, these forms are polynomials of degree q in the parameter $s \in \mathbb{K}$. Let $z_{\alpha^{(a)}}$ be the coefficient of s^{q-a} in the αth

Plücker coordinate of a map M. These coefficients give quantum Plücker coordinates for $\mathcal{M}^q_{r,n}$, determining the Plücker embedding of $\mathcal{M}^q_{r,n}$ into the projective space $\mathbb{P}\big(\bigwedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1}\big)$. The closure of $\mathcal{M}^q_{r,n}$ in this embedding is the singular Drinfel′d compactification or quantum grassmannian $\mathcal{K}^q_{r,n}$.

Let $\mathcal{C}^q_{r,n} := \big\{\alpha^{(a)} \mid \alpha \in \binom{[n]}{r} \text{ and } 0 \le a \le q\big\}$ be the set of indices of quantum Plücker coordinates. This set is partially ordered as follows:

$$\alpha^{(a)} \le \beta^{(b)} \iff a \le b \text{ and } \alpha_i \le \beta_{b-a+i} \text{ for } i = 1, 2, \dots, p-b+a.$$

The quantum Schubert varieties

$$\overline{Z_{\alpha^{(a)}}} := \{z \in \mathcal{K}^q_{r,n} \mid z_{\beta^{(b)}} = 0 \text{ if } \beta^{(b)} \not\le \alpha^{(a)}\}$$

are the Schubert varieties of a Bruhat decomposition of $\mathcal{K}^q_{r,n}$,

$$\mathcal{K}^q_{r,n} = \coprod_{\alpha^{(a)} \in \mathcal{C}^q_{r,n}} Z^\circ_{\alpha^{(a)}}$$

(see [20]). Here $Z^\circ_{\alpha^{(a)}}$ is the set of points in $\overline{Z_{\alpha^{(a)}}}$ with nonvanishing coordinate $z_{\alpha^{(a)}}$.

Let $I^q_{r,n}$ be the ideal of the quantum grassmannian. The ideals of quantum Schubert varieties have a very simple description, which generalizes (3.1).

PROPOSITION 3.1 [24].

(i) *The ideal* $I_{\alpha^{(a)}}$ *of* $\overline{Z_{\alpha^{(a)}}}$ *is* $I^q_{r,n} + \langle z_{\beta^{(b)}} \mid \beta^{(b)} \not\le \alpha^{(a)}\rangle$.

(ii) $\langle I_{\alpha^{(a)}}, z_{\alpha^{(a)}}\rangle = \bigcap_{\beta^{(b)} \lessdot \alpha^{(a)}} I_{\beta^{(b)}}$.

This was established modulo embedded primes of lower dimension (which is sufficient for our purposes) in [13; 14].

Proof of Theorem 1.1 for $\mathcal{M}^q_{r,n}$. Let $K \subset \mathbb{K}^n$ be an $(n-r)$-plane, none of whose Plücker coordinates vanish. Let $\mathbb{G}_m$ act on $\mathbb{K}^n$ by $s \cdot e_i = s^i e_i$ and set $K(s) := s \cdot K$. Consider the family of simple Schubert varieties of $\mathcal{M}^q_{r,n}$,

$$Z(s, K) := \{M \in \mathcal{M}^q_{r,n} \mid M(s^n) \cap K(s) \ne \{0\}\}. \tag{3.3}$$

Let $\mathcal{Y}(K) \to \mathbb{G}_m$ be the family of subvarieties of $\mathcal{K}^q_{r,n}$ whose fibre $\overline{Z(s, K)}$ over $s \in \mathbb{G}_m$ is the closure of $Z(s, K)$. As in [20, Sec. 3], expanding the determinantal equation for M to lie in $Z(s, K)$ gives the linear equation for this fibre:

$$0 = \sum_{\alpha^{(a)} \in \mathcal{C}^q_{r,n}} z_{\alpha^{(a)}} k_\alpha s^{qn + r(n-r) + \binom{n-r+1}{2} - |\alpha^{(a)}|}. \tag{3.4}$$

As in Section 3.1, the form of this equation and Proposition 3.1 show that the family $\mathcal{Y}(K)$ respects the Bruhat decomposition of $\mathcal{K}^q_{r,n}$ and is multiplicity-free. Furthermore, given any $(n-r)$-planes $K_1, K_2, \dots, K_l$ in $\mathbb{K}^n$, none of whose Plücker coordinates vanish, the resulting families $\mathcal{Y}(K_1), \mathcal{Y}(K_2), \dots, \mathcal{Y}(K_l)$ meet the Bruhat decomposition properly. Thus general members of these families meet transversally on the quantum grassmannian $\mathcal{K}^q_{r,n}$ and hence on its dense subset $\mathcal{M}^q_{r,n}$. This completes the proof of Theorem 1.1. □

In general, all points of intersection of general members of the families $\mathcal{Y}(K_1)$, $\mathcal{Y}(K_2), \ldots, \mathcal{Y}(K_{\dim \mathcal{M}^q_{r,n}})$ lie in the space $\mathcal{M}^q_{r,n}$ of curves. An argument given in [20] uses work of Bertram [1] concerning the quot scheme compactification of $\mathcal{M}^q_{r,n}$. This argument is valid here, as the pertinent results of Bertram hold over arbitrary fields. (See [23] for a survey of this quantum intersection problem.)

4. Some More Reality

In this section we strengthen the results of [19; 20; 21].

PROPOSITION 4.1. *Suppose* $\mathbb{K} = \mathbb{R}$. *Let* X *be one of the spaces* $\mathbf{G}(r, n)$, $\mathbb{F}\ell_{\mathbf{r}}$, $\mathrm{OG}(r)$, *or* $\mathcal{M}^q_{r,n}$. *There exist simple Schubert varieties* $Y_1, Y_2, \ldots, Y_{\dim X}$ *that meet transversally in the complexification* $X_{\mathbb{C}}$ *of* X, *and every point of intersection is real.*

In [19; 20; 21] we constructed families of simple Schubert varieties defined by subspaces $K(s)$ that osculate a given rational normal curve. Unlike the families constructed in Section 3, those families respect the Bruhat order only in characteristic 0. The calculations of Section 3 enable a more flexible choice of subspaces.

An action of the torus $\mathbb{R}^\times$ on a real vector space V of dimension n is *general* if V is a direct sum of 1-dimensional eigenspaces, each with a different character. Given such an action, we say that a linear subspace $L \subset V$ is *general* if none of its Plücker coordinates vanishes, where we define Plücker coordinates with respect to a basis of eigenvectors. When $n = 2r+1$ and V is equipped with a nondegenerate (split) symmetric bilinear form of signature ± 1, we require the form to be homogeneous with respect to this action. (We require the form to be split so that $\mathrm{OG}(r)$ has sufficiently many real points.) See (1.1) for an example of such a split form for the general action $s \cdot e_i = s^i \cdot e_i$. A general action induces actions of the torus on the spaces $\mathbf{G}(r, n)$, $\mathbb{F}\ell_{\mathbf{r}}$, and $\mathrm{OG}(r)$. We obtain a torus action on $\mathcal{M}^q_{r,n}$ by having $\mathbb{R}^\times$ act on the source $\mathbb{P}^1$ of the maps via $s \cdot [a, b] := [a, s^N b]$ for some integer N. This action on $\mathbb{P}^1$ will be general (and induce a general action on $\mathcal{M}^q_{r,n}$) when $N > N_0$, for some N_0 (described below) depending upon the given general torus action.

THEOREM 4.2. *Let* V *be an* n*-dimensional real vector space equipped with a general action of the torus, and equip* $\mathbb{P}^1$ *with a general action of the torus. Let* X *be one of the spaces* $\mathbf{G}(r, n)$, $\mathbb{F}\ell_{\mathbf{r}}$, $\mathrm{OG}(r)$, *or* $\mathcal{M}^q_{r,n}$. *For any simple Schubert varieties* $Y_1, Y_2, \ldots, Y_{\dim X}$ *defined by general linear subspaces of* V, *there exist real numbers* $s_1, s_2, \ldots, s_{\dim X}$ *such that the translates* $s_1 \cdot Y_1, s_2 \cdot Y_2, \ldots, s_{\dim X} \cdot Y_{\dim X}$ *meet transversally in the complexification* $X_{\mathbb{C}}$ *of* X, *and every point of intersection is real.*

There is a second part to Proposition 2.1 of [21], which we shall use for the proof of this theorem.

PROPOSITION 4.3 [21]. *Under the same hypotheses as Proposition 2.1 but with* $\mathbb{K} = \mathbb{R}$, *there exist points* $s_1, s_2, \ldots, s_l \in \mathbb{R}$ *such that, for every* $1 \leq k \leq l$ *and every* $w \in I$ *with* $|w| = k$, *the intersection*

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_k(s_k) \cap X_w$$

is transverse, and each of its $\deg(w)$ *points are real.*

Proof of Theorem 4.2. The characters of $\mathbb{R}^\times$ are the monomials s^i for i an integer. We use the integer i to represent the character s^i. Suppose V is an n-dimensional real vector space equipped with a general action of $\mathbb{R}^\times$. Assume that $e_1, e_2, \ldots, e_n$ is a basis of eigenvectors of V with respective characters $i_1, i_2, \ldots, i_n$, where $i_1 < i_2 < \cdots < i_n$. Then the arguments of Sections 3.1, 3.2, and 3.3 remain valid with this action in place of the action $s \cdot e_i = s^i e_i$. This action induces the action on Plücker coordinates $s \cdot p_\alpha = s^{J(\alpha)} p_\alpha$, where $J(\alpha) := \sum_j i_{\alpha_j}$. The key facts are that the map $\alpha \mapsto J(\alpha)$ is an order-preserving map from the poset $\binom{[n]}{r}$ to the integers and that the exponent $r(n-r) + \binom{n-r+1}{2} - |\alpha|$ of (3.2) is replaced by $i_{r+1} + \cdots + i_n - J(\alpha)$. The theorem follows from Proposition 4.3 for the spaces $\mathbf{G}(r,n)$, $\mathbb{F}\ell_{\mathbf{r}}$, and $\mathrm{OG}(r)$.

For the space of rational curves $\mathcal{M}^q_{r,n}$, set $N_0 := i_n - i_1 + 1$ and suppose $N > N_0$ for the torus action on $\mathbb{P}^1$. Thus the action of $s \in \mathbb{R}^\times$ on the map M is given by the map $t \mapsto s \cdot [M(s^N t)]$. On the quantum Plücker coordinates, this is $s \cdot z_{\alpha^{(a)}} = s^{I(\alpha^{(a)})} z_{\alpha^{(a)}}$, where $I(\alpha^{(a)}) = Na + J(\alpha)$. Again the map $\alpha^{(a)} \mapsto I(\alpha^{(a)})$ is an order-preserving map from the poset $\mathcal{C}^q_{r,n}$ to the integers. The arguments of Section 3.4 extend to this more general action, a key point being that the exponent $qn + r(n-r) + \binom{n-r+1}{2} - |\alpha^{(a)}|$ of (3.4) is now replaced by $qN + i_{r+1} + \cdots + i_n - I(\alpha^{(a)})$. Thus the theorem follows for the space $\mathcal{M}^q_{r,n}$ by Proposition 4.3. $\square$

The proof of Proposition 4.3 in [21] (see [20, Sec. 4]) gives further information about the choice of the points $s_1, s_2, \ldots, s_l$. By "$\forall s_1 \gg s_2 \gg \cdots \gg s_l$" we mean

$$\forall s_1 > 0 \ \exists \varepsilon_2 > 0 \text{ such that } \forall s_2 < \varepsilon_2 \cdots \exists \varepsilon_l > 0 \text{ such that } \forall s_l < \varepsilon_l.$$

Thus the existential statement "there exist points $s_1, s_2, \ldots, s_{\dim X} \in \mathbb{R}$" of Theorem 4.2 may be replaced by "$\forall s_1 \gg s_2 \gg \cdots \gg s_{\dim X}$".

5. Not a Determinantal Pair

Laksov and Speiser [9; 10; 11; 25] have developed the notion of a determinantal family of subvarieties, thereby giving a general criterion for proving that a general member of a family meets arbitrary subvarieties transversally. This approach studies the tangent spaces of members of the family with respect to possible tangent spaces for arbitrary subvarieties. To facilitate applications of their theory, they give a local condition which implies that a family is determinantal. We show that this local condition does not hold for families of simple Schubert varieties in $\mathbf{G}(r,n)$ and thus does not hold for the other families of simple Schubert varieties of Theorem 1.1. This is not implied by Kleiman's example [8] of a subvariety in a Grassmannian not meeting any translate of a particular Schubert variety transversally, for his Schubert variety has codimension 2. This emphasizes the need for new, perhaps less general or more specific, criteria that imply transversality.

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ S & \end{tikzcd}$$

Figure 1 Intersection with a family

Let X, Y, Z, and S be smooth equidimensional varieties with π, f, and g the maps of Figure 1, where π is smooth, g is unramified, and f is flat. The fibres of $X \times_Z Y \to S$ are the intersections of the fibres of $X \to S$ with Y (along the maps f and g).

Consider the bundles $E := p_1^* T(X/S)$, the pullback of the relative tangent bundle of $X \to S$, and $F := p_2^*(g^* TZ)/TY$, the pullback of the conormal bundle of g. Let

$$E \xrightarrow{\alpha} F$$

be the map induced by $df\colon T(X/S) \to TZ$ and define $\Delta \subset X \times_Z Y$ to be the degeneracy locus of the map α, the set of points where the map α does not have full rank. This is, in fact, the locus where the intersection is not transverse. Set

$$\begin{aligned} \rho &:= |\operatorname{rank} E - \operatorname{rank} F| + 1 \\ &= \dim X \times_Z Y - \dim S + 1 \quad (\text{when } X \times_Z Y \text{ dominates } S). \end{aligned}$$

Then Δ is either empty or has codimension at most ρ in $X \times_Z Y$.

PROPOSITION 5.1 [11]. *Suppose either that $\Delta = \emptyset$ or else the codimension of Δ in $X \times_Z Y$ is equal to ρ. Then there is a nonempty open subset U of S such that, for $s \in U$, either $X_s \times_Z Y$ is empty or it is smooth of the expected dimension $\rho - 1$.*

Laksov and Speiser define the family X to be *determinantal* if,

> for every unramified map $f\colon Y \to Z$ from a smooth variety Y, either Δ is empty or else it has codimension ρ in $X \times_Z Y$ (with ρ as defined previously). (5.1)

This condition is quite strong, as it implies that the general member of the family $X \to S$ meets every unramified map $Y \to Z$ from a smooth variety Y transversally. Since enumerative geometry is concerned only with those maps $Y \to Z$ that are "geometrically meaningful" (an admittedly ill-defined class), this condition is perhaps stronger than needed by enumerative geometry.

Laksov and Speiser introduce a local condition that implies (5.1). Fix $z \in Z$ and a linear subspace $L \subset T_z Z$. For any $x \in f^{-1}(z)$, we have the map

$$T_x(X/S) \xrightarrow{\alpha_{L,x}} T_z Z/L. \tag{5.2}$$

Set $\rho_L := \dim X - \dim S + \dim L - \dim Z + 1$ and let $\Delta_L \subset f^{-1}(z)$ be the locus of points x, where $\alpha_{L,x}$ has less than maximal rank. Then either $\Delta_L = \emptyset$ or else it has codimension at most ρ_L. The family X is *determinantal at* z if, for every linear subspace $L \subset T_z Z$, either $\Delta_L = \emptyset$ or else it has codimension ρ_L in $f^{-1}(z)$.

PROPOSITION 5.2 [11]. *If the family X is determinantal at every $z \in Z$, then is it determinantal.*

Consider the enumerative problem (i) of Section 1 involving simple Schubert varieties of $\mathbf{G}(r, n)$. Let $l \geq 1$ be an integer, $Y := \mathbf{G}(r, n)$, and $Z := [\mathbf{G}(r, n)]^l$, with the map $g \colon Y \to Z$ the diagonal embedding. Let $S := [\mathbf{G}(n-r, n)]^l$, and set

$$X := \{(K_1, K_2, \dots, K_l, H_1, H_2, \dots, H_l) \in S \times Z \mid \dim K_i \cap H_i = 1\},$$

which consists of the smooth points in the l-fold product of the universal simple Schubert variety in $\mathbf{G}(n-r, n) \times \mathbf{G}(r, n)$,

$$\{(K, H) \in \mathbf{G}(n-r, n) \times \mathbf{G}(r, n) \mid \dim K \cap H \neq \{0\}\}.$$

In this case, the maps π and f are just the projections, and the fibre of $X \times_Y Z$ over a point $(K_1, K_2, \dots, K_l)$ of S is the intersection

$$\Omega^{\mathrm{sm}}(K_1) \cap \Omega^{\mathrm{sm}}(K_2) \cap \cdots \cap \Omega^{\mathrm{sm}}(K_l), \tag{5.3}$$

where $\Omega^{\mathrm{sm}}(K)$ consists of the smooth points of $\Omega(K)$.

Theorem 1.1 asserts that, when $l = r(n-r)$ (and hence for all $l \leq r(n-r)$), there exists an open subset U of S consisting of points $(K_1, K_2, \dots, K_l)$ such that the intersection (5.3) is nonempty and is transverse at the generic point of each component. This fact is not implied by the theory of Laksov and Speiser.

THEOREM 5.3. *If $r > 1$ and $n - r > 1$, then the family X is not determinantal at any $z \in Z$.*

Similar arguments show that transversality in the other enumerative problems of Theorem 1.1 cannot be obtained from the theory of Laksov and Speiser. For those, we replace Y by one of $\mathbb{F}\ell_{\mathbf{r}}$, $\mathrm{OG}(r)$, or $\mathcal{M}^q_{r,n}$ and then possibly modify X, S, and Z.

Let $V = \mathbb{K}^n$. If $H \in \mathbf{G}(r, n)$ then it follows that the tangent space $T_H\mathbf{G}(r, n)$ equals $\mathrm{Hom}(H, V/H)$. Similarly, if $K \in \mathbf{G}(n-r, n)$ and $\dim H \cap K = 1$, then

$$T_H\Omega(K) = \{\varphi \in \mathrm{Hom}(H, V/H) \mid \varphi(K \cap H) \subset (H + K)/H\}$$

(see e.g. [18, Sec. 2.9]). If we let $v = K \cap H \in \mathbb{P}(H)$ and $\Lambda = K + H \in \mathbb{P}^\vee(V/H)$, a hyperplane containing H, then $T_H\Omega(K) = \tau_{v,\Lambda}$, where

$$\tau_{v,\Lambda} := \{\varphi \in \mathrm{Hom}(H, V/H) \mid \varphi(v) \subset \Lambda/H\}.$$

The key point is that the set of hyperplanes $T_H\Omega(K)$ for $K \in \Omega^{\mathrm{sm}}(H)$ forms the proper subvariety of the space $\mathbb{P}^\vee(\mathrm{Hom}(H, V/H))$ consisting of hyperplanes $\tau_{v,\Lambda}$ for $(v, \Lambda) \in \mathbb{P}(H) \times \mathbb{P}^\vee(V/H)$. This also implies that the grassmannian hypothesis [10, Thm. 3.3] fails to hold. This set of hyperplanes is a single orbit of the stabilizer of H in $\mathrm{GL}(V)$ of dimension $n-2$. (Compare with the hypotheses of [8, Thm. 10].)

Proof of Theorem 5.3. Let $z = (K_1, K_2, \ldots, K_l) \in Z$. Then

$$f^{-1}(z) = \{(K_1, K_2, \ldots, K_l) \in S \mid \dim K_i \cap H_i = 1\} = \prod_{i=1}^{l} \Omega^{\mathrm{sm}}(H_i),$$

which has dimension $l[r(n-r)-1]$. Fix any $(v, \Lambda) \in \mathbb{P}(H) \times \mathbb{P}^{\vee}(V/H)$ and set

$$L := \tau_{v,\Lambda} \times \prod_{i=2}^{l} T_{H_i}\mathbf{G}(r, n).$$

For $x = (K_1, K_2, \ldots, K_l) \in f^{-1}(z)$ we have

$$T_x(X/S) = \prod_{i=1}^{l} T_{H_i}\Omega(K_i) \subset \prod_{i=1}^{l} T_{H_i}\mathbf{G}(r, n) = T_z Z,$$

and the map $\alpha_{L,x}$ (5.2) is the composition

$$T_x(X/S) \hookrightarrow T_z Z \twoheadrightarrow T_z Z/L \simeq \mathbb{K}.$$

This map drops rank precisely when $T_{H_1}\Omega(K_1) = \tau_{v,\Lambda}$, that is, when $v \in K_1 \subset \Lambda$ (and $v = K_1 \cap H_1$). This defines an open subset $\Omega'(v, \Lambda)$ of a Schubert subvariety of $\mathbf{G}(n-r, n)$ isomorphic to $\mathbf{G}(n-r-1, r-1)$, which has dimension $(n-r-1)(r-1) = r(n-r) - 1 - (n-2)$. Thus

$$\Delta_L = \Omega'(v, \Lambda) \times \prod_{i=2}^{l} \Omega^{\mathrm{sm}}(K_i)$$

has dimension $l[r(n-r)-1] - (n-2)$ and hence codimension $n-2$ in $f^{-1}(z)$. However,

$$\begin{aligned}\rho_L &= \dim X - \dim S + \dim L - \dim Z + 1\\ &= l[r(n-r)-1],\end{aligned}$$

which exceeds $n-2$ for all $l > 0$ and for all $r, n-r > 1$. □

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