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Measures of Transcendency for Entire Functions

Dan Coman & Evgeny A. Poletsky

1. Introduction

If X is a non-pluripolar compact set in $\mathbb{C}^k$ and P is a polynomial of degree n on $\mathbb{C}^k$, then the Bernstein–Walsh inequality (see [K]) is

$$|P(z)| \le \|P\|_X e^{nV_X(z)}, \tag{1}$$

where $\|P\|_X$ is the uniform norm of P on X and $V_X(z)$ is the extremal function of X. For example, if $z = (z_1, \dots, z_k)$ and $X = \Delta^k = \{z \in \mathbb{C}^k : |z_j| \le 1,\ 1 \le j \le k\}$ is the unit polydisk, then

$$V_X(z) = L(z) = \max\{\log^+|z_1|, \dots, \log^+|z_k|\}.$$

If X is pluripolar then, in general, such estimates are impossible.

Let $\mathcal{P}_n$ be the space of polynomials of degree at most n on $\mathbb{C}^2$ and let $\Delta(z,r)$ denote the closed disk centered at z and of radius r in $\mathbb{C}$. For an entire function f on $\mathbb{C}$ we define

$$M_n(f,r) = M_n(r) = \sup\{\|P_f\|_{\Delta_r} : P \in \mathcal{P}_n,\ \|P_f\|_{\Delta} \le 1\},$$

where $P_f(z) = P(z, f(z))$, $\Delta_r = \Delta(0,r)$, and $\Delta = \Delta(0,1)$.

If f is a polynomial of degree d, then it follows from the Bernstein–Walsh inequality that $M_n(f,r) \le r^{dn}$ for $r \ge 1$. To prove certain deep theorems in transcendental number theory, Tijdeman [T] showed that $M_n(f,r) \le \exp(Cn^2 \log^+ r)$ when $f(z) = e^z$ (see also [B]). Later we proved in [CP] that, in this case,

$$M_n(f,r) = \exp\big(\tfrac{1}{2}n^2 \log^+ r + o(n^2)\big).$$

The latter estimate is asymptotically sharp. In both papers the authors used highly specific features of the exponential function.

In this paper (see the inequality (6) and Theorem 4.2) we give an effective method to obtain upper estimates for $M_n(f,r)$ when f is an arbitrary entire transcendental function.

The results in [T] and [CP] can be viewed as a generalization of the classical Bernstein–Walsh inequality to the case of polynomials of variables z and $f(z)$. When f is a polynomial,

$$m_n(f,r) = m_n(r) = \log M_n(f,r)$$

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grows like a linear function of n, but in the transcendental case we have by Corollary 2.6 that

$$\liminf_{n\to\infty} \frac{m_n(f,r)}{n^2} \geq \frac{1}{2}\log r.$$

Thus, $m_n(f,r)$ grows at least like n^2.

For many applications it is useful to know the order of the asymptotics of $m_n(f,r)$ in n, that is, to find the minimal $\alpha(f) > 0$ such that

$$\limsup_{n\to\infty} \frac{m_n(f,r)}{n^{\alpha(f)}} < \infty.$$

Until now even the existence of functions f with $2 < \alpha(f) < \infty$ was not known. An example in [BLMT] shows that there is an entire function f such that

$$\limsup_{n\to\infty} \frac{m_n(f,r)}{n^{\alpha}} = \infty$$

for all $\alpha > 0$, so in this case $\alpha(f) = \infty$. The method developed in Section 4 allows us in Section 6 to prove, for every $\alpha \geq 3$, the existence of entire functions f having order of growth 1 and type $1/e$ such that $\alpha - 1 \leq \alpha(f) \leq \alpha$. More precisely,

$$0 < \limsup_{n\to\infty} \frac{m_n(f,r)}{n^{\alpha-1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{m_n(f,r)}{n^{\alpha}} < \infty.$$

This shows that the order of the asymptotics of m_n does not depend only on the order of f.

Another interesting characteristic of the function f is the maximal number of zeros $Z_n(f,r) = Z_n(r)$ of $P_f(z)$ in the disk Δ_r when $P \in \mathcal{P}_n$. This is closely related to $m_n(f,r)$. The function $Z_n(f,r)$ counts the maximal number of intersection points of the graph of f in $\mathbb{C}^2$ with algebraic curves of degree at most n over the disk Δ_r. In Section 2 we show that

$$\frac{m_n(f,r/3)}{\log r + 16} \leq Z_n(f,r) \leq 2m_n(f,3r).$$

The characteristic $M_n(f,r)$ gives a bound on the growth of the restriction of $|P(z,w)|$, $P \in \mathcal{P}_n$, to the graph of f in $\mathbb{C}^2$. To obtain estimates on $|P|$ at every point, we introduce

$$E_n(f) = E_n = \sup\{\|P\|_{\Delta^2} : P \in \mathcal{P}_n,\ \|P_f\|_{\Delta} \leq 1\}$$

and we let $e_n = \log E_n$. A simple normal family argument shows that $E_n(f)$ is finite if f is transcendental. Then, by (1) we have

$$|P(z,w)| \leq \|P\|_K E_n(f) e^{nL(z,w)}, \tag{2}$$

where $P \in \mathcal{P}_n$ and $K = \{(z, f(z)) : |z| \leq 1\}$. In [CP] we proved the asymptotically sharp estimate

$$e_n = \tfrac{1}{2}n^2 \log n + O(n^2)$$

when $f(z) = e^z$. In this paper we prove upper estimates on E_n for general transcendental functions. Thus one can obtain a Bernstein–Walsh-type inequality (2) for the special pluripolar set K, provided there are effective estimates on the constants $E_n(f)$.

Toward this end, we give in Section 3 an appropriate version of Cartan's lemma for polynomials. This lemma allows us to obtain in Section 4 an upper estimate for $E_n(f)$ in terms of $M_n(f,r)$ and the nth diameter (see Section 3) of the set $D(\theta,r) = \{z \in \Delta_r : f(z) = e^{i\theta}\}$. Roughly speaking, $E_n(f) \le M_n(f,r)r^n\delta^{-n}(r)$ if, for every θ, the set $D(\theta,r)$ has at least $n+1$ points and $\delta(r)$ is the minimal distance between these points. Combining this with the upper estimate on $M_n(f,r)$ in terms of $E_n(f)$, which follows from (1), we obtain effective estimates on $E_n(f)$ in terms of $M(f,r) = \|f\|_{\Delta_r}$ and the nth diameter of the set $D(\theta,r)$. In Section 5 we apply these estimates to the case of $f(z) = e^{p(z)}$, where p is a polynomial.

If f is a polynomial of degree d, then E_n and Z_n are equal to ∞ when $n \ge d$ and $M_n(f,r) \le r^{dn}$ for all n. The exponential function realizes the least bounds for $Z_n(r)$ and $M_n(r)$ in the class of transcendental functions. In our point of view, the characteristics of f studied in this paper measure how far f is from polynomials—that is, the *transcendency* of f. This explains the title of the paper.

2. Relationship between $Z_n(r)$ and $m_n(r)$

For an entire function g and $r > 0$, we let $Z(g,r)$ be the number of zeros of g in the disk Δ_r and let $M(g,r) = \max_{|z|=r}|g(z)|$.

Let $\mathcal{F}$ be a finite-dimensional linear subspace of entire functions containing constants. We introduce

$$M_{\mathcal{F}}(r) = \sup\{M(f,r) : f \in \mathcal{F},\ \|f\|_\Delta \le 1\}$$

and

$$Z_{\mathcal{F}}(r) = \sup\{Z(f,r) : f \in \mathcal{F}\}.$$

Since $\mathcal{F}$ contains constants, $Z_{\mathcal{F}}(r)$ coincides with the maximal valency in Δ_r of functions from $\mathcal{F}$.

We will need the following theorem of Jenkins and Oikawa (see [JO, p. 670]).

THEOREM 2.1. *Let*

$$f = \sum_{k=0}^{\infty} a_k z^k$$

be a p-valent holomorphic function in the unit disk, $\mu_p = \max_{k \le p}|a_k|$, *and* $0 < r < 1$. *Then*

$$M(f,r) < A(p)\mu_p(1-r)^{-2p},$$

where $A(p) = (p+2)2^{3p-1}\exp(p\pi^2 + 1/2)$.

The following theorem establishes the relationship between $M_{\mathcal{F}}(r)$ and $Z_{\mathcal{F}}(s)$.

THEOREM 2.2. *If* $1 < s < r$ *and* $f \in \mathcal{F}$, *then*

$$M_{\mathcal{F}}(r) \ge \left(\frac{r^2+s^2}{2rs}\right)^{Z_{\mathcal{F}}(s)}$$

and

$$M_{\mathcal{F}}(r) \le e^{(\log r+16)Z_{\mathcal{F}}(3r)}.$$

Proof. Let us take $f \in \mathcal{F}$ with $\|f\|_\Delta = 1$ and let $a_1, \dots, a_m$, where $m = Z_{\mathcal{F}}(s)$, be the zeros of f in Δ_s. We form a Blaschke product

$$B(z) = r^m \prod_{j=1}^{m} \frac{z - a_j}{r^2 - \bar{a}_j z}$$

on Δ_r. If $g = f/B$ then $M(g, r) = M(f, r)$. By the maximum modulus principle, $M(g, s) \le M(f, r)$. Hence

$$M(f, s) \le M(B, s)M(f, r).$$

Since $|z| \le s$ and $|a_j| \le s$ for $s < r$, we have

$$\left|\frac{z - a_j}{r^2 - \bar{a}_j z}\right| \le \frac{s + |a_j|}{r^2 + |a_j|s} \le \frac{2s}{r^2 + s^2},$$

so

$$1 \le M(f, s) \le \left(\frac{2rs}{r^2 + s^2}\right)^m M(f, r) \le \left(\frac{2rs}{r^2 + s^2}\right)^m M_{\mathcal{F}}(r).$$

For the second inequality we take any $f \in \mathcal{F}$ such that $M(f, 1) = 1$. Let $t = 3r$. As noted after the definition of $Z_{\mathcal{F}}(r)$, the function $g(z) = f(tz) = \sum a_k z^k$ is at most p-valent in Δ, where $p = Z_{\mathcal{F}}(t)$. Direct computation shows that the constant $A(p)$ in Theorem 2.1 does not exceed $\exp(14p)$. Hence, by this theorem and the Cauchy inequality we have

$$M(f, r) = M\left(g, \frac{r}{t}\right) < e^{14p} M\left(g, \frac{1}{t}\right) t^p \left(1 - \frac{r}{t}\right)^{-2p}.$$

Since $M(g, 1/t) = M(f, 1) = 1$ and $t = 3r$, it follows that

$$M(f, r) < e^{14p} r^p \left(\frac{27}{4}\right)^p < e^{(\log r+16)p}. \qquad \square$$

COROLLARY 2.3. *For all $r > 1$ we have*

$$\frac{m_{\mathcal{F}}(r/3)}{\log r + 16} \le Z_{\mathcal{F}}(r) \le 2m_{\mathcal{F}}(3r),$$

where

$$m_{\mathcal{F}}(r) = \log M_{\mathcal{F}}(r).$$

Proof. By the first inequality from Theorem 2.2,

$$M_{\mathcal{F}}(3r) \ge \left(\frac{10}{6}\right)^{Z_{\mathcal{F}}(r)}.$$

Therefore,

$$2m_{\mathcal{F}}(3r) \ge Z_{\mathcal{F}}(r).$$

If $r > 3$ then, by the second inequality from Theorem 2.2,

$$M_{\mathcal{F}}(r/3) \leq e^{(\log r + 16) Z_{\mathcal{F}}(r)}.$$

If $r \leq 3$ then $m_{\mathcal{F}}(r/3) = 0$ and the inequality is evident. $\square$

Let $\mathcal{F}_k$ be an increasing sequence of linear subspaces of entire functions of finite dimensions N_k and containing constants. Since the dimension of $\mathcal{F}_k$ is finite, the function

$$W_k(z) = \sup\{\log|f(z)| : f \in \mathcal{F}_k,\ \|f\|_\Delta = 1\}$$

is continuous. As a consequence, we conclude that W_k is subharmonic. Moreover, letting $f \equiv 1$ we see that $W_k \equiv 0$ on Δ and $W_k \geq 0$ on $\mathbb{C}$.

We say that the sequence of $\mathcal{F}_k$ satisfies the *Bernstein–Walsh inequality* if there are numbers a_k $(k = 1, 2, \dots)$ and a function $\omega(r)$ such that $W_k(z) \leq a_k \omega(|z|)$. If the function

$$W(z) = \limsup_{k \to \infty} \frac{W_k(z)}{a_k}$$

is finite when $1 < |z| < \infty$ and is not identically equal to 0, then we say that W is the *Bernstein–Walsh extremal function* of the sequence $\mathcal{F}_k$.

The following corollary follows immediately from Corollary 2.3.

COROLLARY 2.4. *A sequence $\mathcal{F}_k$ satisfies the Bernstein–Walsh inequality for some choice of numbers a_k if and only if*

$$Z_{\mathcal{F}_k}(r) \leq a_k G(r), \quad r > 1,\ k \geq 0,$$

for some function $G(r)$. It has a Bernstein–Walsh extremal function for a choice of numbers a_k if and only if the function

$$\limsup_{k \to \infty} \frac{Z_k(r)}{a_k}$$

is finite when $1 < r < \infty$ and is not identically equal to zero.

The following theorem provides *a priori* lower estimates for $Z_{\mathcal{F}}$ and $M_{\mathcal{F}}$.

THEOREM 2.5. *If the dimension of $\mathcal{F}$ is N, then*

$$M_{\mathcal{F}}(r) \geq e^{(N-1)\log r}.$$

Proof. Let us take a basis $f_1, \dots, f_N$ in $\mathcal{F}$ and consider the matrix A consisting of the first $N-1$ rows of the Wronskian for this system of functions at $z = 0$. There is a nonzero vector $c = \{c_j\}$ such that $Ac = 0$. If

$$f(z) = \sum_{j=1}^{N} c_j f_j(z)$$

then $f \not\equiv 0$, because the functions f_j are linearly independent over $\mathbb{C}$ and all the first $N-2$ derivatives of f at the origin are equal to 0. Hence $Z(f, 1) \geq N - 1$.

Dividing f by $M(f,1)$ yields a function g with $M(g,1)=1$. Applying the maximum principle to $g(z)/z^{N-1}$, we obtain

$$M(g,r) \geq r^{N-1} = e^{(N-1)\log r}. \qquad \square$$

Our main interest in this paper is in the case when $\mathcal{F} = \{P_f : P \in \mathcal{P}_n\}$. Then the dimension of $\mathcal{F}$ is $N = (n+1)(n+2)/2$ and we have both $Z_n(r) = Z_{\mathcal{F}}(r)$ and $m_n(r) = m_{\mathcal{F}}(r)$. The results so far yield the following corollary.

COROLLARY 2.6. *For every integer $n \geq 0$ and for $r \geq 1$,*

$$\frac{m_n(r/3)}{\log r + 16} \leq Z_n(r) \leq 2m_n(3r), \qquad m_n(r) \geq \frac{n^2+3n}{2}\log r.$$

3. Cartan's Lemma and the *n*th Diameter

We refer to [L] for the following estimate, due to H. Cartan.

LEMMA 3.1. *Let P be a monic polynomial of degree n on $\mathbb{C}$. For every $h > 0$ there exist at most n closed disks C_j of radii r_j ($j = 1, \dots, k$, $k \leq n$) such that $r_1 + \cdots + r_k \leq 2eh$ and $|P(z)| > h^n$ for all $z \in \mathbb{C} \setminus \bigcup C_j$.*

That the number of these exceptional disks C_j is at most $n = \deg P$ follows from the proof of this lemma (see [L]).

For a set $G \subset \mathbb{C}$ and an integer $n \geq 1$, we define the nth diameter of G by

$$\operatorname{diam}_n(G) = \inf\left\{r_1 + \cdots + r_k : k \leq n,\ G \subset \bigcup_{j=1}^{k} C_j(r_j)\right\},$$

where $C_j(r_j)$ are closed disks of radii $r_j > 0$. It is clear that $\operatorname{diam}_n(G) = 0$ if $|G| \leq n$, where $|G|$ is the number of points in G, and that $\operatorname{diam}_n(F) \leq \operatorname{diam}_n(G)$ if $F \subset G$. We have the following simple fact.

LEMMA 3.2. *If $n < |G| < +\infty$ and $\delta = \min\{|z-w| : z, w \in G, z \neq w\}$, then* $\operatorname{diam}_n(G) \geq \delta/2$.

Using the nth diameter, Cartan's lemma can be stated as follows: If P is a monic polynomial of degree n on $\mathbb{C}$ and if $h > 0$, then

$$\operatorname{diam}_n(\{z \in \mathbb{C} : |P(z)| \leq h^n\}) \leq 2eh.$$

It is essential for this estimate that P be monic. Otherwise, we prove the following Cartan-type result.

LEMMA 3.3. *Let P be a polynomial of degree n on $\mathbb{C}$. Then, for every* $0 < h \leq 1/(8e)$ *and every $t \geq 2$, the* nth *diameter of the set*

$$G_t = \left\{z \in \mathbb{C} : |P(z)| \leq \|P\|_\Delta \left(\frac{h|z|}{t^2}\right)^n,\ 2 \leq |z| \leq t\right\}$$

does not exceed $36eh$.

Proof. Let $z_0 \in \partial\Delta$ be such that $|P(z_0)| = \|P\|_\Delta$, and let $\alpha = P(z_0)$. The polynomial $Q(w) = w^n P(t^2/w + z_0)/\alpha$ is monic and, by Cartan's lemma, the set

$$F = \{w \in \mathbb{C} : |Q(w)| \leq (2h)^n\}$$

is contained in the union of $l \leq n$ closed disks $C_j(w_j, r_j)$ of centers w_j and radii r_j $(j = 1, \ldots, l)$, with $r_1 + \cdots + r_l \leq 4eh$. We consider the set

$$F_t = F \cap \{w \in \mathbb{C} : |w| \geq t^2/(t+1)\}$$

and those circles C_j that intersect $\{|w| \geq t^2/(t+1)\}$. After renumbering these circles we have for some $k \leq l \leq n$ that $F_t \subset C_1(w_1, r_1) \cup \cdots \cup C_k(w_k, r_k)$, where $r_1 + \cdots + r_k \leq 4eh$ and

$$|w_j| \geq t^2/(t+1) - r_j > r_j, \quad j = 1, \ldots, k,$$

since $r_j \leq 4eh \leq 1/2$ and $t \geq 2$. It follows that $0 \notin C_j(w_j, r_j)$.

Let ϕ be the map $\phi(w) = t^2/w + z_0$. If $z \notin \phi(F)$ and $|z| \geq 2$, then

$$|P(z)| > \|P\|_\Delta (2h)^n \frac{|z - z_0|^n}{t^{2n}} \geq \|P\|_\Delta \left(\frac{h|z|}{t^2}\right)^n.$$

Hence $G_t \subset \phi(F_t) \subset \phi(C_1) \cup \cdots \cup \phi(C_k)$. Since $0 \notin C_j$, it follows that $\phi(C_j)$ is a closed disk of radius

$$\begin{aligned} R_j &= \frac{t^2 r_j}{|w_j|^2 - r_j^2} \leq \frac{t^2 r_j}{(t^2/(t+1) - r_j)^2 - r_j^2} = \frac{(t+1)^2 r_j}{t^2 - 2r_j(t+1)} \\ &\leq \frac{(t+1)^2 r_j}{t^2 - t - 1} \leq 9 r_j, \end{aligned}$$

since $2r_j \leq 1$ and $t \geq 2$. Hence $\operatorname{diam}_n(G_t) \leq R_1 + \cdots + R_k \leq 36eh$. $\square$

4. Relationship between E_n and $m_n(r)$

Let f be a transcendental entire function, and let

$$m(r) = m_f(r) = \max\{\log^+ M(f, r), \log^+ r\}.$$

If $P \in \mathcal{P}_n$ and $\|P_f\|_\Delta \leq 1$, then by (2) we have

$$\log|P(z, f(z))| \leq e_n + n \max\{\log^+|z|, \log^+|f(z)|\},$$

where $e_n = \log E_n$. Thus

$$m_n(r) \leq e_n + n m(r). \tag{3}$$

Combining (3) with Corollary 2.6, we obtain an *a priori* estimate

$$e_n \geq \frac{n^2}{4} \log r_n, \tag{4}$$

where r_n is the solution of the equation $m(r_n) = (n/4) \log r_n$. Choosing a transcendental entire function f such that $m(f, r) \leq \phi(r) \log r$, where $\phi(r)$ grows to

∞ as slow as we want, we see that e_n can increase arbitrarily fast. This happens because slow-growing functions are close to polynomials.

Since $m_n(r)$ is a convex function of $\log r$ and since $m_n(1) = 0$, we have

$$m_n(r) \le \frac{e_n + nm(t)}{\log t} \log r, \quad 1 \le r \le t. \tag{5}$$

Substituting for t the solution t_n of the equation $nm(t) = e_n$ yields

$$m_n(r) \le \frac{2e_n}{\log t_n} \log r, \quad 1 \le r \le t_n. \tag{6}$$

Now, to get effective estimates for $m_n(r)$ we will obtain bounds for e_n. For $\theta \in [0, 2\pi]$ and $r \ge 2$ we let

$$\begin{aligned} D(\theta, r) &= \{z \in \mathbb{C} : 2 \le |z| \le r,\ f(z) = e^{i\theta}\}, \\ d_n(\theta, r) &= \min\{1, \operatorname{diam}_n(D(\theta, r))\}. \end{aligned}$$

If $P(z, w)$ is a polynomial we let $P_\theta(z) = P(z, e^{i\theta})$. We denote by $S = \partial\Delta$ the unit circle and by $S_r = \partial\Delta_r$ the circle of radius r centered at the origin.

LEMMA 4.1. *Let $P \in \mathcal{P}_n$ with $\|P_f\|_\Delta \le 1$ and $r \ge 2$. Then, for any θ,*

$$\log\|P_\theta\|_\Delta \le m_n(r) + n \log \frac{36er}{d_n(\theta, r)}.$$

Proof. Without loss of generality we may assume that $P_\theta \not\equiv 0$ and $d_n(\theta, r) > 0$. Since $|P(z, f(z))| \le e^{m_n(|z|)}$, we know that $|P_\theta(z)| \le e^{m_n(|z|)}$ when $z \in D(\theta, r)$. We take $h < d_n(\theta, r)/(36e)$. Then $36eh < \operatorname{diam}_n(D(\theta, r))$ and so, by Lemma 3.3, there is a point $z \in D(\theta, r)$ where

$$\|P_\theta\|_\Delta \left(\frac{h|z|}{r^2}\right)^n < |P_\theta(z)| \le e^{m_n(|z|)}.$$

Thus

$$\log\|P_\theta\|_\Delta \le m_n(|z|) - n \log|z| + 2n \log r - n \log h.$$

By Corollary 2.6, the function $g(t) = m_n(t) - n \log t$ is a convex function of $\log t$, $g(1) = 0$, and $g(t) \ge 0$ when $t \ge 1$. It follows that g is increasing and therefore

$$\log\|P_\theta\|_\Delta \le m_n(r) + n \log r - n \log h.$$

Since $h < d_n(\theta, r)/(36e)$ was arbitrary, this implies the lemma. $\square$

THEOREM 4.2. *Suppose that $r \ge 2$ and $d_n(\theta, r) \ge a$ on a set $E \subset S$ of length l. Then*

$$e_n \le nm(er) \log r + n \log(er)\left(\log \frac{36er}{a} + \frac{4\pi}{l}\right).$$

The proof requires some auxiliary lemmas.

LEMMA 4.3. *Let $0 < \rho < 1 < R$ and let $E \subset S$ be a set of length l. Then, for $z \in \Delta$ the regularized relative extremal function,*

$$\omega(z, E, \Delta_R) \le -\frac{(1-\rho)\log R}{(1+\rho)\log(R/\rho)}\frac{l}{2\pi}.$$

Proof. The (regularized) relative extremal function $\omega(z) = \omega(z, E, \Delta_R)$ is the maximal negative subharmonic function in Δ_R that does not exceed -1 on a subset $E' \subset E$ of length l (see e.g. [K]). So, for every $z \in S_\rho$ we have

$$\omega(z) \le \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\omega(e^{i\theta})\,d\theta \le -C = -\frac{l(1-\rho)}{2\pi(1+\rho)},$$

since the Poisson kernel $P(z,\theta) \ge (1-\rho)/(1+\rho)$. Hence, for $z \in \Delta_R$,

$$\frac{\omega(z)}{C} \le \omega(z, S_\rho, \Delta_R) = \max\left\{\frac{\log(|z|/R)}{\log(R/\rho)}, -1\right\}.$$

Therefore on S we have $\omega(z) \le -C\log R/\log(R/\rho)$. $\square$

LEMMA 4.4. *Let $P(z,w)$ be a polynomial of degree n, and let $E \subset S$ be a set of length l. Then*

$$\sup\{\|P_\theta\|_\Delta : e^{i\theta} \in E\} \ge \|P\|_{\Delta^2}\exp\left(-\frac{4\pi n}{l}\right).$$

Proof. Suppose that $\|P\|_{\Delta^2} = 1$, and let h be the supremum of $\|P_\theta\|_\Delta$ when $e^{i\theta} \in E$. Let $E' = \Delta \times E$ and $0 < \rho < 1 < R$. Since

$$\omega((z,w), E', \Delta_R \times \Delta_R) = \max\{\omega(z, \Delta, \Delta_R), \omega(w, E, \Delta_R)\},$$

by Lemma 4.3 for $(z,w) \in \Delta^2$ we obtain

$$\omega((z,w), E', \Delta_R \times \Delta_R) \le -C = -\frac{(1-\rho)\log R}{(1+\rho)\log(R/\rho)}\frac{l}{2\pi}.$$

By (1) in the case of the bidisk,

$$\log|P(z,w)| \le n\max\{\log|z|, \log|w|\};$$

hence $\log|P(z,w)| \le n\log R$ on $\Delta_R \times \Delta_R$. Therefore,

$$\frac{\log|P(z,w)| - n\log R}{n\log R - \log h} \le \omega((z,w), E', \Delta_R \times \Delta_R) \le -C$$

when $(z,w) \in \Delta^2$. But $\|P\|_{\Delta^2} = 1$ and so

$$-n\log R \le \frac{(1-\rho)\log R}{(1+\rho)\log(R/\rho)}\frac{l}{2\pi}(\log h - n\log R)$$

for every $0 < \rho < 1 < R$. We divide this by $\log R$ and then let $R \searrow 1$. It follows that

$$\log h \ge \frac{2\pi n}{l}\frac{(1+\rho)\log\rho}{1-\rho}.$$

The lemma follows by letting $\rho \nearrow 1$. $\square$

Proof of Theorem 4.2. Let us take $P \in \mathcal{P}_n$ such that $\|P_f\|_\Delta = 1$ and $\|P\|_{\Delta^2} = E_n$. By Lemma 4.4 there exists, for each $\varepsilon > 0$, a number $e^{i\theta} \in E$ such that

$$\|P_\theta\|_\Delta \geq E_n \exp\left(-\frac{4\pi n}{l} - \varepsilon\right).$$

By Lemma 4.1,

$$e_n - \frac{4\pi n}{l} - \varepsilon \leq m_n(r) + n \log \frac{36er}{d_n(\theta, r)}.$$

By (5) we have

$$m_n(r) \leq (e_n + nm(er)) \frac{\log r}{\log(er)}.$$

Using this and $d_n(\theta, r) \geq a$ yields

$$e_n \leq (e_n + nm(er)) \frac{\log r}{\log(er)} + n \log \frac{36er}{a} + \frac{4\pi n}{l} + \varepsilon.$$

Letting $\varepsilon \searrow 0$, it follows that

$$\frac{e_n}{\log(er)} \leq nm(er) \frac{\log r}{\log(er)} + n\left(\log \frac{36er}{a} + \frac{4\pi}{l}\right),$$

which implies the conclusion. □

Theorem 4.2 has the following corollary, which will be employed in studying concrete classes of functions in Sections 5 and 6. For $r \geq 2$ and $0 < \delta \leq 2$, we define $A_{r\delta}$ as the subset of those $e^{i\theta} \in S$ such that (i) $D(\theta, r)$ has at least $n + 1$ elements and (ii) the minimal distance between them is greater than or equal to δ.

COROLLARY 4.5. *Let $r \geq 2$ and $0 < \delta \leq 2$. If $\lambda(A_{r\delta})$ is the length of $A_{r\delta}$ in S, then*

$$e_n \leq nm(er) \log r + n \log(er)\left(\log \frac{72er}{\delta} + \frac{4\pi}{\lambda(A_{r\delta})}\right).$$

Proof. We can assume that $\lambda(A_{r\delta}) > 0$. The corollary follows directly from Theorem 4.2 and Lemma 3.2, since for $e^{i\theta} \in A_{r\delta}$ we have $d_n(\theta, r) \geq \delta/2$. □

5. The Case $f(z) = e^{p(z)}$

We consider here entire functions of the form $f(z) = e^{p(z)}$, where p is a polynomial of degree $\rho \geq 1$. The special case $p(z) = z$ was studied by different techniques in [T] and [CP].

THEOREM 5.1. *There exist constants $C, C', C'' > 0$ depending only on the polynomial p such that, for every $n \geq 1$ and $r \geq 1$, we have*

$$\frac{n^2 \log n}{2\rho} - C'n^2 \leq e_n(f) \leq Cn^2 \log n,$$

$$m_n(r) \leq C''(n^2 \log r + nr^\rho).$$

In particular, if P is a polynomial of degree n on $\mathbb{C}^2$ then

$$|P(z,w)| \le \|P\|_K e^{Cn^2 \log n + n \max\{\log^+|z|, \log^+|w|\}} \quad \forall (z,w) \in \mathbb{C}^2,$$

$$|P(z, e^{p(z)})| \le \|P\|_K e^{C''(n^2 \log^+|z| + n|z|^\rho)} \quad \forall z \in \mathbb{C},$$

where $K = \{(z, e^{p(z)}) : |z| \le 1\}$.

In the proof we need the following lemma.

LEMMA 5.2. *There exist constants $0 < c_1 < 1 < C_1$ depending only on p such that if $r = C_1 n^{1/\rho}$ and $\delta = c_1 n^{-1+1/\rho}$ then $A_{r\delta} = S$, where $A_{r\delta}$ is the set from Corollary 4.5.*

Proof. Let $e^{i\theta} \in S$. For $k = 0, \dots, m$ we choose z_k such that

$$p(z_k) = i(\theta + 2k\pi).$$

Since $\deg p = \rho$ and since $k \le m$, it follows that $|z_k| \le Cm^{1/\rho}$ for some constant $C > 1$. If $|z| \le Cm^{1/\rho}$ we have $|p'(z)| \le C'm^{1-1/\rho}$, where C' is a constant. Therefore, if $k, l \le m$ $(k \ne l)$ then

$$2\pi \le |p(z_k) - p(z_l)| \le C'm^{1-1/\rho}|z_k - z_l|.$$

Hence there exists a constant $C_1 > 1$ such that, if $r = C_1 n^{1/\rho}$, then the annulus $\{2 \le |z| \le r\}$ contains at least $n+1$ points z_k. Moreover, if $k \ne l$ then $|z_k - z_l| > c_1 n^{-1+1/\rho}$, where $c_1 > 0$ is a constant. □

Proof of Theorem 5.1. There exists a constant C' such that, for $r \ge 1$,

$$m(r) = \max\{\log r, \log M(f,r)\} \le C' r^\rho.$$

By (3) and Corollary 2.6 we obtain

$$e_n \ge \frac{n^2}{2} \log r - C' n r^\rho.$$

Taking $r = n^{1/\rho}$ yields the lower bound on e_n.

In order to prove the upper estimate, we let $r = C_1 n^{1/\rho}$ and $\delta = c_1 n^{-1+1/\rho}$ be as in Lemma 5.2. Since $A_{r\delta} = S$, by Corollary 4.5 we have

$$\begin{aligned} e_n &\le nm(er)\log r + n\log(er)\left(\log\frac{72er}{\delta} + 2\right) \\ &= nm(C_1 e n^{1/\rho})\log(C_1 n^{1/\rho}) + n\log(C_1 e n^{1/\rho})(\log(C_1' n) + 2) \\ &\le Cn^2 \log n, \end{aligned}$$

where C is a constant.

For the estimate on $m_n(r)$, we can use (6) with $t_n = n^{1/\rho}$ and the upper bound on e_n to get

$$m_n(r) \le C_2 n^2 \log r, \quad 1 \le r \le n^{1/\rho},$$

where C_2 is a constant. Combining this with (3) yields

$$m_n(r) \le \begin{cases} C_2 n^2 \log r & \text{if } 1 \le r \le n^{1/\rho}, \\ Cn^2 \log n + C'nr^\rho & \text{if } r \ge n^{1/\rho}. \end{cases}$$

So $m_n(r) \le C''(n^2 \log r + nr^\rho)$ for all $r \ge 1$, where C'' is a constant. □

We remark that, by Theorem 5.1 and Corollary 2.6, if $f(z) = e^{p(z)}$ then for every $n \ge 1$ and $r \ge 1$ we have that

$$\frac{\log r}{2} \le \frac{m_n(r)}{n^2} \le C'' \log r + \frac{C'' r^\rho}{n},$$

$$\frac{\log^+(r/3)}{2\log r + 32} \le \frac{Z_n(r)}{n^2} \le 2C'' \log(3r) + \frac{C''' r^\rho}{n}.$$

6. An Entire Gap Series of Order 1

We consider the entire function

$$f(z) = \sum_{k=1}^{\infty} \left(\frac{z}{n_k}\right)^{n_k}, \tag{7}$$

where n_k is an increasing sequence of positive integers such that

$$n_1 \ge 10, \qquad n_{k+1} \ge n_k^2 \quad \text{for } k \ge 1. \tag{8}$$

The function f has order of growth $\rho = 1$ and type $\mu = 1/e$. By Stirling's formula $n! \le e(n/e)^n \sqrt{n}$ (see [R]) it follows that $(e/n)^n \le e/(n-1)!$, so we obtain

$$M_f(r) < M_g(r) \le re^{r/e} < e^r, \quad r \ge 0,$$

where $g(z) = \sum_{n=1}^{\infty} (z/n)^n$. Let $[x]$ denote the greatest integer in x. We have the following theorem.

THEOREM 6.1. *Let f be defined by* (7), *where $\{n_k\}$ satisfies* (8). *There exists a constant $C > 0$ such that, if $k \ge 1$ and $n_k \le n < n_{k+1}$, then*

$$m_n(n) - n^2 \le e_n(f) \le 2nn_{k+1} \log n_{k+1} + Cnn_{k+1},$$

$$m_n(r) \ge (\max\{n^2/2, [n/n_k]n_{k+1}\}) \log r, \quad r \ge 1,$$

$$m_n(r) \le \left(2nn_{k+1} + \frac{Cnn_{k+1}}{\log n_{k+1}}\right) \log r, \quad 1 \le r \le n_{k+1}.$$

COROLLARY 6.2. *Let f be defined by* (7), *where $n_1 \ge 10$, $n_{k+1} = n_k^{\alpha-1}$, and $\alpha \ge 3$. There exists a constant $C > 0$ such that, if $k \ge 1$, then*

$$\frac{m_n(r)}{n^\alpha} \le \left(2 + \frac{C}{\log n}\right) \log r, \quad 1 \le r \le n,$$

$$\frac{m_n(r)}{n^{\alpha-1}} \ge \frac{\log r}{2^{\alpha-1}}, \quad n_k \le n \le 2n_k, \ r \ge 1,$$

$$\frac{\log r}{2} \le \frac{m_n(r)}{n^2} \le 4 \log r + \frac{C \log r}{\log n}, \quad \frac{n_k}{2} \le n < n_k, \ 1 \le r \le n.$$

We note that, by the results of Section 2, this corollary shows the following interesting fact: The maximal number of intersection points (z, w) between the graph of f and algebraic curves of degree n that lie over a disk of fixed radius $\{|z| \le r_0\}$ grows like n^2 if $n_k/2 \le n < n_k$ and at least like $n^{\alpha-1}$ if $n_k \le n \le 2n_k$. This is in contrast to the case of the function $f(z) = e^{p(z)}$ from Section 5.

The proof of Theorem 6.1 uses the techniques from Section 4 and relies on the study of the roots of the equation $f(z) = e^{i\theta}$.

THEOREM 6.3. *If f and $\{n_k\}$ are as in Theorem 6.1 and if $\theta \in [0, 2\pi]$, then all roots of the equation $f(z) = e^{i\theta}$ are simple and are located as follows: each annulus $\{n_k - 1 < |z| < 2n_k\}$, $k \ge 1$, contains exactly $n_k - n_{k-1}$ roots, where $n_0 = 0$. Moreover, if $f(z) = f(\zeta) = e^{i\theta}$ and $z \ne \zeta$ then $|z - \zeta| > 5$.*

In the proof of Theorem 6.3 we need the following lemmas.

LEMMA 6.4. *If $|z| = R \ge 1$ and $n \ge 1$, then*

$$\left|\sum_{j=n}^{\infty}\left(\frac{z}{j}\right)^j\right| \le e^R\left(\frac{R}{n}\right)^n, \quad \left|\sum_{j=1}^{n}\left(\frac{z}{j}\right)^j\right| \le eR^n.$$

Proof. If $g(z) = \sum_{n=1}^{\infty}(z/n)^n$, then

$$\begin{aligned}\left|\sum_{j=n}^{\infty}\left(\frac{z}{j}\right)^j\right| &\le \sum_{j=n}^{\infty}\left(\frac{R}{j}\right)^j = \frac{R^n}{n^n}\left(1 + \sum_{k=1}^{\infty}\frac{R^k n^n}{(n+k)^{n+k}}\right)\\ &\le \left(\frac{R}{n}\right)^n(1 + g(R)) \le e^R\left(\frac{R}{n}\right)^n,\\ \left|\sum_{j=1}^{n}\left(\frac{z}{j}\right)^j\right| &\le \sum_{j=1}^{n}\left(\frac{R}{j}\right)^j \le R^n g(1) < eR^n. \qquad \square\end{aligned}$$

LEMMA 6.5. *Let $n \ge 2$. If $|z - z_0| \le |z_0|/(2n)$, then*

$$\frac{3n|z_0|^{n-1}|z - z_0|}{2} \ge |z^n - z_0^n| \ge \frac{n|z_0|^{n-1}|z - z_0|}{2}.$$

Proof. Since

$$h(z) - h(0) = z\int_0^1 h'(rz)\,dr = z\int_0^1 h'(0)\,dr + z^2\int_0^1 r\int_0^1 h''(trz)\,dt\,dr,$$

we can use $h(z) = (1+z)^n$ to obtain

$$|(1+z)^n - 1| \ge n|z|\left|1 - \frac{(n-1)|z|(1+|z|)^{n-2}}{2}\right|.$$

If $|z| \le 1/(2n)$, then

$$|(1+z)^n - 1| \ge n|z|\left(1 - \frac{(1+1/(2n))^n}{4}\right) \ge n|z|\left(1 - \frac{\sqrt{e}}{4}\right) \ge \frac{n|z|}{2}.$$

Thus
$$|z^n - z_0^n| = |z_0|^n \left| \left(1 + \frac{z - z_0}{z_0}\right)^n - 1 \right| \geq \frac{n|z_0|^{n-1}|z - z_0|}{2}.$$

The proof of the other inequality is similar. □

Proof of Theorem 6.3. We fix $\theta \in [0, 2\pi]$, and for $k \geq 1$ we let:
$$A_k = \{z \in \mathbb{C} : n_k - 1 < |z| < 2n_k\};$$
$$T_k(z) = \sum_{j=1}^{k} \left(\frac{z}{n_j}\right)^{n_j}, \quad Q_k(z) = f(z) - T_k(z);$$
$$R_1 = n_1, \quad R_k = \left(\frac{n_k^{n_k}}{n_{k-1}^{n_{k-1}}}\right)^{1/(n_k - n_{k-1})} \text{ for } k \geq 2.$$

Since $n_{k-1} \leq \sqrt{n_k}$, $n_1 \geq 10$, and $n_2 \geq 100$, we have for $k \geq 2$ that
$$n_k < R_k = n_k \left(\frac{n_k}{n_{k-1}}\right)^{n_{k-1}/(n_k - n_{k-1})} < n_k n_k^{1/(\sqrt{n_k} - 1)} \leq 10^{2/9} n_k. \tag{9}$$

Let
$$P_1(z) = \left(\frac{z}{n_1}\right)^{n_1} - e^{i\theta} = \frac{1}{n_1^{n_1}}(z^{n_1} - R_1^{n_1} e^{i\theta})$$
and
$$P_k(z) = \left(\frac{z}{n_{k-1}}\right)^{n_{k-1}} + \left(\frac{z}{n_k}\right)^{n_k} = \frac{z^{n_{k-1}}}{n_k^{n_k}}(z^{n_k - n_{k-1}} + R_k^{n_k - n_{k-1}})$$
if $k \geq 2$.

We show first that, for each $k \geq 1$, the equation $f(z) = e^{i\theta}$ has $n_k - n_{k-1}$ simple solutions ζ_{kj} ($j = 1, \ldots, n_k - n_{k-1}$) in the annulus A_k, located as follows. If z_{kj} are the roots of P_k with $|z_{kj}| = R_k$, then there is a root ζ_{kj} with $|\zeta_{kj} - z_{kj}| < 1/2$. Assuming this, we notice that by (9) the disks $|z - z_{kj}| \leq 1/2$ are contained in A_k. Using (9), we have
$$|z_{kj} - z_{kl}| \geq 2R_k \sin \frac{\pi}{n_k - n_{k-1}} \geq 2n_k \sin \frac{\pi}{n_k} > 6$$
for $j \neq l$. It follows that
$$|\zeta_{kj} - \zeta_{kl}| \geq |z_{kj} - z_{kl}| - |\zeta_{kj} - z_{kj}| - |\zeta_{kl} - z_{kl}| > 5.$$

To prove this claim, we fix a number z_{kj} with $P_k(z_{kj}) = 0$ and $|z_{kj}| = R_k$. We write the equation $f(z) = e^{i\theta}$ in the following form:
$$P_1(z) + Q_1(z) = 0 \quad \text{for } k = 1; \tag{10}$$
$$P_k(z) + (T_{k-2}(z) + Q_k(z) - e^{i\theta}) = 0 \quad \text{for } k \geq 2, \tag{11}$$
where $T_0 = 0$. We will apply Rouché's theorem on each disk $|z - z_{kj}| \leq 1/2$. By (9) we have
$$\frac{|z_{kj}|}{2(n_k - n_{k-1})} \geq \frac{n_k}{2(n_k - n_{k-1})} \geq \frac{1}{2}.$$

Applying Lemma 6.5 with $|z - z_{kj}| = 1/2$ and $n = n_k - n_{k-1}$ then yields

$$|P_k(z)| \geq \frac{n|z|^{n_{k-1}}}{4n_k^{n_k}} R_k^{n-1} > \frac{n_k(n_k-1)^{n_{k-1}}}{4n_k^{n_k}} \left(1 - \frac{n_{k-1}}{n_k}\right) R_k^{n-1}$$
$$= \frac{1}{4}\left(1 - \frac{1}{n_k}\right)^{n_{k-1}} \left(1 - \frac{n_{k-1}}{n_k}\right) \frac{n_k}{R_k} \left(\frac{R_k}{n_k}\right)^{n_k - n_{k-1}}.$$

Using (9) and (8), we obtain $n_k/R_k > 1/2$, $1 - n_{k-1}/n_k \geq 0.9$, and

$$(1 - 1/n_k)^{n_{k-1}} > (1/4)^{n_{k-1}/n_k} > 1/2.$$

Therefore,

$$|P_k(z)| > \frac{1}{20}\left(\frac{R_k}{n_k}\right)^{n_k - n_{k-1}} \quad \text{for } |z - z_{kj}| = 1/2.$$

By Lemma 6.4 and (8), it follows that for $|z - z_{kj}| = 1/2$ we have

$$|Q_k(z)| \leq \left(\frac{|z|}{n_{k+1}}\right)^{n_{k+1}} e^{|z|} < \left(\frac{2n_k}{n_{k+1}}\right)^{n_{k+1}} e^{2n_k}$$
$$\leq \left(2n_{k+1}^{-1/2}\right)^{n_{k+1}} e^{2\sqrt{n_{k+1}}}.$$

As $n_{k+1} \geq n_1^2 \geq 100$, we get

$$|Q_k(z)| < 5^{-100} e^{20} < 5^{-80} \quad \text{for } |z - z_{kj}| = 1/2.$$

Thus for $k = 1$ we obtain $|P_1(z)| > 1/20 > |Q_1(z)|$ for $|z - z_{1j}| = 1/2$. Hence, by Rouché's theorem, equation (10) has exactly one solution in each disk $|z - z_{1j}| < 1/2$.

For $k \geq 2$ and $|z - z_{kj}| = 1/2$, by Lemma 6.4 we have

$$|T_{k-2}(z)| \leq e|z|^{n_{k-2}} < e(2n_k)^{\sqrt{n_{k-1}}},$$

so

$$|T_{k-2}(z) + Q_k(z) - e^{i\theta}| < \alpha_k := e(2n_k)^{\sqrt{n_{k-1}}} + 2.$$

Using (8) and (9), it follows for $|z - z_{kj}| = 1/2$ that

$$|P_k(z)| > \frac{1}{20}\left(\frac{R_k}{n_k}\right)^{n_k - n_{k-1}} = \frac{1}{20}\left(\frac{n_k}{n_{k-1}}\right)^{n_{k-1}} \geq \beta_k := \frac{1}{20} n_k^{n_{k-1}/2}.$$

Since $\sqrt{n_{k-1}} - n_{k-1}/2 < 0$, $n_k \geq n_{k-1}^2$, and $n_{k-1} \geq 10$, we have

$$\alpha_k/\beta_k = 20\left(e2^{\sqrt{n_{k-1}}} n_k^{\sqrt{n_{k-1}} - n_{k-1}/2} + 2n_k^{-n_{k-1}/2}\right)$$
$$\leq 20\left(e2^{\sqrt{n_{k-1}}} n_{k-1}^{2\sqrt{n_{k-1}} - n_{k-1}} + 2n_{k-1}^{-n_{k-1}}\right)$$
$$\leq 20\left(e2^{\sqrt{10}} 10^{2\sqrt{10} - 10} + 2 \cdot 10^{-10}\right) < 1.$$

Hence, by Rouché's theorem, equation (11) has exactly one solution in each disk $|z - z_{kj}| < 1/2$.

To complete the proof, we show that ζ_{kj} are the only roots of the equation $f(z) = e^{i\theta}$ by proving that it has exactly n_k solutions in the disk $|z| < 3n_k$ $(k \geq 1)$. We write the equation in the form

$$(z/n_k)^{n_k} + (T_{k-1}(z) + Q_k(z) - e^{i\theta}) = 0.$$

If $|z| = 3n_k$, then $|z/n_k|^{n_k} = 3^{n_k}$ and by Lemma 6.4 and (8) we have

$$|Q_k(z)| \le \left(\frac{3n_k}{n_{k+1}}\right)^{n_{k+1}} e^{3n_k} \le \left(3n_{k+1}^{-1/2}\right)^{n_{k+1}} e^{3\sqrt{n_{k+1}}} \le 0.3^{100} e^{30},$$

$$|T_{k-1}(z)| \le e(3n_k)^{n_{k-1}} \le e(3n_k)^{\sqrt{n_k}} \quad \text{if } k \ge 2.$$

If $k = 1$ and $|z| = 3n_1$, then $|T_{k-1}(z) + Q_k(z) - e^{i\theta}| < 2 < 3^{n_1}$. If $k \ge 2$ and $|z| = 3n_k$ then

$$|T_{k-1}(z) + Q_k(z) - e^{i\theta}| \le e(3n_k)^{\sqrt{n_k}} + 2 < 3^{n_k},$$

since $n_k \ge 100$. The conclusion now follows by Rouché's theorem. □

Proof of Theorem 6.1. By Corollary 2.6 we have that $m_n(t) \ge (n^2/2)\log t$ for all $t \ge 1$. Following an idea of [BLMT], we let

$$P(z,w) = (w - T_k(z))^{[n/n_k]}, \quad T_k(z) = \sum_{j=1}^{k} \left(\frac{z}{n_j}\right)^{n_j}.$$

Then $\deg P \le n$ and P_f has a zero of order $N = [n/n_k]n_{k+1}$ at $z = 0$. By the maximum principle we obtain $M_{P_f}(r) \ge M_{P_f}(1)r^N$ for $r \ge 1$, which gives the lower bound on $m_n(r)$. By (3) we have $m_n(r) \le e_n + nm(r)$. Using $m(r) < r$ and letting $r = n$ then yields the lower bound on E_n.

To prove the upper bound on E_n, we let $r = 2n_{k+1}$ and $\delta = 2$. By Theorem 6.3 we have $A_{r\delta} = S$, where $A_{r\delta}$ is the set from Corollary 4.5. Using $m(er) < r + \log(er)$ and Corollary 4.5, we obtain

$$\begin{aligned} e_n &\le nm(er)\log r + n\log(er)\left(\log\frac{72er}{\delta} + 2\right) \\ &\le n(2n_{k+1} + C\log n_{k+1})\log(2n_{k+1}) + Cn(\log n_{k+1})^2 \\ &\le 2nn_{k+1}\log n_{k+1} + Cnn_{k+1}. \end{aligned}$$

Finally, using (5) with $r = n_{k+1}$ and the foregoing upper bound on E_n, we have for $1 \le t \le n_{k+1}$ that

$$m_n(t) \le \frac{e_n + nm(n_{k+1})}{\log n_{k+1}}\log t \le \left(2nn_{k+1} + \frac{Cnn_{k+1}}{\log n_{k+1}}\right)\log t. \qquad \square$$

Proof of Corollary 6.2. The first inequality follows because $n_{k+1} \le n^{\alpha-1}$ when $n_k \le n < n_{k+1}$. If $n_k \le n \le 2n_k$ then $[n/n_k] \ge 1$ and $n_{k+1} = n_k^{\alpha-1} \ge (n/2)^{\alpha-1}$, so by Theorem 6.1 we have

$$m_n(r) \ge \left[\frac{n}{n_k}\right] n_{k+1}\log r \ge \left(\frac{n}{2}\right)^{\alpha-1}\log r, \quad r \ge 1.$$

If $n_k/2 \le n < n_k$ then, by Theorem 6.1,

$$m_n(r) \le \left(2nn_k + \frac{Cnn_k}{\log n_k}\right)\log r \le \left(4n^2 + \frac{2Cn^2}{\log n}\right)\log r,$$

where $1 \le r \le n$. $\square$

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D. Coman
Department of Mathematics
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244

dcoman@syr.edu

E. Poletsky
Department of Mathematics
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244

eapolets@syr.edu