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On the Distribution of the Farey Sequence with Odd Denominators

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1. Introduction and Statement of Results

Given a positive integer Q, we denote by $\mathcal{F}_Q$ the set of irreducible rational fractions in (0, 1] whose denominators do not exceed Q. That is,

$$\mathcal{F}_Q = \{a/q : 1 \le a \le q \le Q,\ \gcd(a,q) = 1\}.$$

Problems concerning the distribution of Farey fractions were studied in the 1920s by Franel [6] and Landau [15] and more recently in [1; 2; 3; 4; 7; 8; 9; 10; 11; 13; 14].

It is well known that

$$N_Q = \#\mathcal{F}_Q = 6Q^2/\pi^2 + O(Q\log Q).$$

We denote by $\mathcal{F}_Q^<$ the set of pairs (γ,γ') of consecutive elements in $\mathcal{F}_Q$.

In this paper we are concerned with the set

$$\mathcal{F}_{Q,\mathrm{odd}} = \{a/q \in \mathcal{F}_Q : q \text{ odd}\}$$

of Farey fractions of order Q with odd denominators. For instance,

$$\mathcal{F}_8 = \left\{\tfrac{1}{8}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, \tfrac{7}{8}, 1\right\},$$
$$\mathcal{F}_{8,\mathrm{odd}} = \left\{\tfrac{1}{7}, \tfrac{1}{5}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{4}{5}, \tfrac{6}{7}, 1\right\}.$$

The set of pairs (γ,γ') of consecutive elements in $\mathcal{F}_{Q,\mathrm{odd}}$ is denoted by $\mathcal{F}_{Q,\mathrm{odd}}^<$. It is not hard to prove (see [11]) that

$$N_{Q,\mathrm{odd}} = \#\mathcal{F}_{Q,\mathrm{odd}} = 2Q^2/\pi^2 + O(Q\log Q). \tag{1.1}$$

It is well known that $\Delta(\gamma,\gamma') := a'q - aq' = 1$ whenever $\gamma = a/q < a'/q' = \gamma'$ are consecutive elements in $\mathcal{F}_Q$. This certainly fails when $\gamma < \gamma'$ are consecutive in $\mathcal{F}_{Q,\mathrm{odd}}$. A first step in the study of the distribution of the values of $\Delta(\gamma,\gamma')$ for pairs (γ,γ') of consecutive fractions in $\mathcal{F}_{Q,\mathrm{odd}}$ was undertaken by Haynes in [11]. He proved that if one denotes

$$N_{Q,\mathrm{odd}}(k) = \#\{\gamma < \gamma' \text{ successive in } \mathcal{F}_{Q,\mathrm{odd}} : \Delta(\gamma,\gamma') = k\},$$

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then the asymptotic frequency

$$\rho_{\mathrm{odd}}(k) = \lim_{Q\to\infty} \frac{N_{Q,\mathrm{odd}}(k)}{N_{Q,\mathrm{odd}}}$$

exists and is expressed as

$$\rho_{\mathrm{odd}}(k) = \frac{4}{k(k+1)(k+2)}, \quad k \in \mathbb{N}^*.$$

This can be written as

$$\rho_{\mathrm{odd}}(k) = \begin{cases} \mathrm{Area}(\mathcal{T}_k) & \text{if } k \ge 2, \\ \frac{1}{2} + \mathrm{Area}(\mathcal{T}_1) & \text{if } k = 1, \end{cases} \tag{1.2}$$

where (as in [3]) we denote $\mathcal{T}_k = \bigl\{(x,y) \in \mathcal{T} : \bigl[\frac{1+x}{y}\bigr] = k\bigr\}$ for $k \in \mathbb{N}^*$ and $\mathcal{T} = \{(x,y) \in [0,1] : x + y > 1\}$.

In this note we study, for fixed $h \ge 1$, the distribution of consecutive elements $\gamma_i < \gamma_{i+1} < \cdots < \gamma_{i+h}$ in $\mathcal{F}_{Q,\mathrm{odd}}$ and then compute the probability that such an $(h+1)$-tuple satisfies $\Delta(\gamma_i, \gamma_{i+1}) = \Delta_1, \dots, \Delta(\gamma_{i+h-1}, \gamma_{i+h}) = \Delta_h$. More precisely, we prove that if one denotes

$$N_{Q,\mathrm{odd}}(\Delta_1, \dots, \Delta_h) = \#\{i : \gamma_i < \gamma_{i+1} < \cdots < \gamma_{i+h} \text{ consecutive in } \mathcal{F}_{Q,\mathrm{odd}} \\ \Delta(\gamma_{i+j-1}, \gamma_{i+j}) = \Delta_j \ (j = 1, \dots, h)\},$$

then

$$\rho_{\mathrm{odd}}(\Delta_1, \dots, \Delta_h) = \lim_{Q\to\infty} \frac{N_{Q,\mathrm{odd}}(\Delta_1, \dots, \Delta_h)}{N_{Q,\mathrm{odd}}}$$

exists for all $h \ge 2$, and we give an explicit formula for it.

To state the main result, we shall employ the area-preserving transformation T of $\mathcal{T}$, introduced in [3] and defined by

$$T(x,y) = \left(y, \left[\frac{1+x}{y}\right] y - x\right). \tag{1.3}$$

We denote

$$\mathcal{T}_{k_1,\dots,k_h} = \mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_{k_2} \cap \cdots \cap T^{-h+1}\mathcal{T}_{k_h}.$$

We notice that if $\gamma = a/q < \gamma' = a'/q' < \gamma'' = a''/q''$ are consecutive elements in $\mathcal{F}_Q$, then $T\bigl(\frac{q}{Q}, \frac{q'}{Q}\bigr) = \bigl(\frac{q'}{Q}, \frac{q''}{Q}\bigr)$. Moreover, if we set $\kappa(x,y) = \bigl[\frac{1+x}{y}\bigr]$, then the positive integer $\kappa\bigl(\frac{q}{Q}, \frac{q'}{Q}\bigr) = \bigl[\frac{Q+q}{q'}\bigr]$ coincides with the index $\nu_Q(\gamma)$ of the Farey fraction γ in $\mathcal{F}_Q$ considered in [9].

It will be worthwhile to consider the tree $\mathfrak{T}_h$ defined by the following properties:

(a) vertices are labeled by O and E;
(b) the starting vertex $\star$ is labeled by O;
(c) there is exactly one edge starting from an E vertex, and such an edge always ends into an O vertex;
(d) there are exactly two edges starting from an O vertex, and they end (respectively) into an E vertex and into an O vertex;

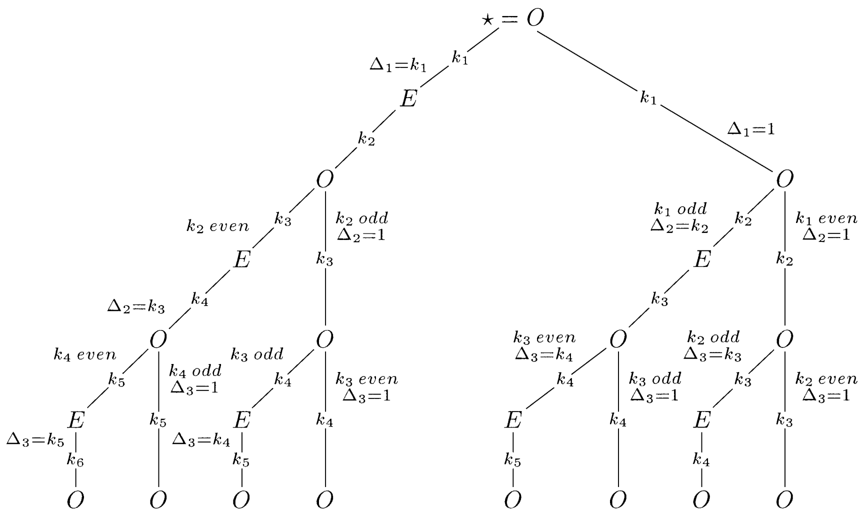

Figure 1 The tree $\mathfrak{T}_3$

(e) the number of O vertices (besides $\star$) on any path that originates at $\star$ is equal to h.

See Figure 1.

We also consider the set $\mathfrak{L}_h$ of labeled paths

$$w = \left(\star = O \xrightarrow{\;k_1\;} v_1 \xrightarrow{\;k_2\;} v_2 \xrightarrow{\;k_3\;} \cdots \xrightarrow{\;k_{|w|}\;} v_{|w|}\right), \quad k_j \in \mathbb{N}^*,$$

on the tree $\mathfrak{T}_h$ that start at $\star$ and pass through $h+1$ vertices labeled by O (including $\star$). That is, $\#\{j : v_j = O\} = h$. We set $o(O) = \text{odd}$ and $o(E) = \text{even}$.

For each labeled path $w \in \mathfrak{L}_h$ and each h-tuple $\Delta = (\Delta_1, \dots, \Delta_h) \in (\mathbb{N}^*)^h$, we define $c_{OE}(w)$ and $c_\Delta(w)$ by induction as follows:

$$c_{OE}\left(\star = O \xrightarrow{\;k_1\;} E \xrightarrow{\;k_2\;} O\right) = k_1, \quad c_{OE}\left(\star = O \xrightarrow{\;k_1\;} O\right) = \emptyset;$$

$$c_{\Delta_1}\left(\star = O \xrightarrow{\;k_1\;} E \xrightarrow{\;k_2\;} O\right) = \Delta_1, \quad c_{\Delta_1}\left(\star = O \xrightarrow{\;k_1\;} O\right) = \emptyset.$$

For $w = w'w'' \in \mathfrak{L}_{h+1}$ with $w' \in \mathfrak{L}_h$ and $w'' = O \xrightarrow{\;k\;} E \xrightarrow{\;l\;} O$ or $w'' = O \xrightarrow{\;k\;} O$, we have

$$c_{OE}(w) = \begin{cases} (c_{OE}(w'), k) & \text{if } w'' = O \xrightarrow{\;k\;} E \xrightarrow{\;l\;} O, \\ c_{OE}(w') & \text{if } w'' = O \xrightarrow{\;k\;} O; \end{cases}$$

$$c_{(\Delta_1,\dots,\Delta_{h+1})}(w) = \begin{cases} (c_\Delta(w'), \Delta_{h+1}) & \text{if } w'' = O \xrightarrow{\;k\;} E \xrightarrow{\;l\;} O, \\ c_\Delta(w') & \text{if } w'' = O \xrightarrow{\;k\;} O. \end{cases}$$

For instance, if w is the labeled path

$$\star = O \overset{k_1}{\text{———}} E \overset{k_2}{\text{———}} O \overset{k_3}{\text{———}} O \overset{k_4}{\text{———}} E \overset{k_5}{\text{———}} O \overset{k_6}{\text{———}} O \overset{k_7}{\text{———}} E \overset{k_8}{\text{———}} O$$

in $\mathfrak{L}_5$, then

$$c_{OE}(w) = (k_1, k_4, k_7) \quad \text{and} \quad c_{(\Delta_1,\dots,\Delta_5)}(w) = (\Delta_1, \Delta_3, \Delta_5).$$

We also denote by $\mathfrak{S}_\Delta$ the set of labeled paths

$$v_0 = \star = O \overset{k_1}{\text{———}} v_1 \overset{k_2}{\text{———}} \cdots \overset{k_{|w|}}{\text{———}} v_{|w|}$$

such that $c_{OE}(w) = c_\Delta(w)$ and such that k_j is even whenever it occurs as $E \overset{k_j}{\text{———}} O \text{———} E$ or as $O \overset{k_j}{\text{———}} O \text{———} O$ and (respectively) odd whenever it occurs as $E \overset{k_j}{\text{———}} O \text{———} O$ or as $O \overset{k_j}{\text{———}} O \text{———} E$.

Having established this notation, we may state our main result.

Theorem 1.1. *Let $h \ge 1$, and let $\Delta = (\Delta_1, \dots, \Delta_h) \in (\mathbb{N}^*)^h$. Then*

$$\rho_{Q,\mathrm{odd}}(\Delta) := \frac{N_{Q,\mathrm{odd}}(\Delta_1,\dots,\Delta_h)}{N_{Q,\mathrm{odd}}} = \rho_{\mathrm{odd}}(\Delta) + O_h\Big(\frac{\log^2 Q}{Q}\Big)$$

as $Q \to \infty$, where

$$\rho_{\mathrm{odd}}(\Delta) = \sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} \mathrm{Area}(\mathcal{T}_{k_1,\dots,k_{|w|-1}}). \tag{1.4}$$

For $h = 1$, this gives

$$\rho_{\mathrm{odd}}(\Delta_1) = \begin{cases} \sum_{k_1} \mathrm{Area}(\mathcal{T}_{k_1}) + \mathrm{Area}(\mathcal{T}_1) = \frac{1}{2} + \mathrm{Area}(\mathcal{T}_1) & \text{if } \Delta_1 = 1, \\ \sum_{k_2} \mathrm{Area}(\mathcal{T}_{\Delta_1} \cap T^{-1}\mathcal{T}_{k_2}) = \mathrm{Area}(\mathcal{T}_{\Delta_1}) & \text{if } \Delta_1 \ge 2; \end{cases}$$

this is the aforementioned result of Haynes [11].

For $h = 2$, we obtain the following.

Corollary 1.2. *$\rho_{Q,\mathrm{odd}}(\Delta_1, \Delta_2)$ tends to $\rho_{\mathrm{odd}}(\Delta_1, \Delta_2)$ for any $\Delta_1, \Delta_2 \in \mathbb{N}^*$ as $Q \to \infty$. Moreover, we have*:

(i) $$\rho_{\mathrm{odd}}(1,1) = \sum_{k_1 \text{ even}} \mathrm{Area}(\mathcal{T}_{k_1}) + \sum_{k_1 \text{ odd}} \mathrm{Area}(\mathcal{T}_{k_1,1}) + \sum_{k_2 \text{ odd}} \mathrm{Area}(\mathcal{T}_{1,k_2}) + \sum_{k_2 \text{ even}} \mathrm{Area}(\mathcal{T}_{1,k_2,1});$$

(ii) *if $\Delta_2 \ge 2$, then*

$$\rho_{\mathrm{odd}}(1,\Delta_2) = \sum_{k_1 \text{ odd}} \mathrm{Area}(\mathcal{T}_{k_1,\Delta_2}) + \sum_{k_2 \text{ even}} \mathrm{Area}(\mathcal{T}_{1,k_2,\Delta_2});$$

(iii) *if* $\Delta_1 \geq 2$, *then*

$$\rho_{\text{odd}}(\Delta_1, 1) = \sum_{k_2 \text{ odd}} \text{Area}(\mathcal{T}_{\Delta_1,k_2}) + \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_{\Delta_1,k_2,1});$$

(iv) *if* $\min(\Delta_1, \Delta_2) \geq 2$, *then*

$$\rho_{\text{odd}}(\Delta_1, \Delta_2) = \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_{\Delta_1,k_2,\Delta_2}).$$

Actually, it follows from Lemma 3.4 and Remark 3.5 that all sums in (ii), (iii), and (iv) are finite.

In this kind of situation, one can give a short-interval version of Theorem 1.1. For each interval $I \subseteq [0, 1]$ and for each $\Delta = (\Delta_1, \dots, \Delta_h) \in (\mathbb{N}^*)^h$, let

$$\begin{aligned} N^I_{Q,\text{odd}} &= \#\{\gamma_0 < \cdots < \gamma_h \text{ consecutive in } \mathcal{F}_{Q,\text{odd}} : \gamma_0 \in I\} \\ &= 2|I|Q^2/\pi^2 + O(Q \log Q), \end{aligned}$$

$$\begin{aligned} N^I_{Q,\text{odd}}(\Delta) = \#\{i : \gamma_i \in I,\ \gamma_i < \gamma_{i+1} < \cdots < \gamma_{i+h} \text{ consecutive in } \mathcal{F}_{Q,\text{odd}} \\ \Delta(\gamma_{i+j-1}, \gamma_{i+j}) = \Delta_j\ (j = 1, \dots, h)\}. \end{aligned}$$

Then the following result holds.

THEOREM 1.3. *Let* $h \geq 1$, *and assume that* $\Delta = (\Delta_1, \dots, \Delta_h) \in (\mathbb{N}^*)^h$ *is such that only finitely many nonvanishing terms appear on the right-hand side of* (1.4). *Then, for any interval* $I \subseteq [0, 1]$, *we have*

$$\rho^I_{Q,\text{odd}}(\Delta) := \frac{N^I_{Q,\text{odd}}(\Delta)}{N^I_{Q,\text{odd}}} = \rho_{\text{odd}}(\Delta) + O_{h,\varepsilon}(Q^{-1/2+\varepsilon})$$

for every $\varepsilon > 0$.

The main techniques of a proof involve the basic properties of Farey fractions, the transformation T from (1.3), and estimates of Weil type for Kloosterman sums (see [5; 12; 16]).

2. Reduction of $N_{Q,\text{odd}}(\Delta_1, \dots, \Delta_h)$

We set throughout

$$\mathbb{Z}^2_{\text{pr}} = \{(a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1\}$$

and, for any subset Ω of $\mathbb{R}^2$ and for $Q \in \mathbb{N}^*$, denote

$$\partial\Omega = \text{the boundary of } \Omega, \qquad Q\Omega = \{(Qx, Qy) : (x, y) \in \Omega\};$$

$$M(\Omega) = \#(\Omega \cap \mathbb{Z}^2),$$

$$M_{\text{odd}}(\Omega) = \#\{(x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ odd}\},$$

$$M_{\text{even}}(\Omega) = \{(x, y) \in \Omega \cap \mathbb{Z}^2 : x \text{ even}\} = M(\Omega) - M_{\text{odd}}(\Omega);$$

$$
\begin{aligned}
N(\Omega) &= \#(\Omega \cap \mathbb{Z}^2_{\mathrm{pr}}),\\
N_{\mathrm{odd}}(\Omega) &= \#\{(x,y)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} : x \text{ odd}\},\\
N_{\mathrm{even}}(\Omega) &= N(\Omega) - N_{\mathrm{odd}}(\Omega) = \{(x,y)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} : x \text{ even}\},\\
N_{\mathrm{odd,\,odd}}(\Omega) &= \{(x,y)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} : x \text{ odd, } y \text{ odd}\},\\
N_{\mathrm{odd,\,even}}(\Omega) &= \{(x,y)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} : x \text{ odd, } y \text{ even}\},\\
N_{\mathrm{even,\,odd}}(\Omega) &= \{(x,y)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} : x \text{ even, } y \text{ odd}\}.
\end{aligned}
$$

If $\gamma_{i_0} = a_{i_0}/q_{i_0} < \gamma_{i_0+1} = a_{i_0+1}/q_{i_0+1} < \cdots < \gamma_{i_0+h} = a_{i_0+h}/q_{i_0+h}$ are consecutive in $\mathcal{F}_Q$, then (cf. [3])

$$\left(\frac{q_{i_0+r}}{Q}, \frac{q_{i_0+r+1}}{Q}\right) = T^r\left(\frac{q_{i_0}}{Q}, \frac{q_{i_0+1}}{Q}\right).$$

There is a one-to-one correspondence between $\mathbb{Z}^2_{\mathrm{pr}} \cap Q\mathcal{T}_{k_1,\dots,k_r}$ and the set $\mathcal{F}_{Q,k_1,\dots,k_r}$ of consecutive elements $\gamma_0 < \gamma_1 < \cdots < \gamma_r$ in $\mathcal{F}_Q$, with $\nu_Q(\gamma_{j-1}) = k_j$ $(j = 1,\dots,r)$, that is given by

$$(q_0, q_1) \mapsto (\gamma_0, \gamma_1, \dots, \gamma_r),$$

where (γ_0, γ_1) is the unique pair in $\mathcal{F}_Q^<$ with denominators q_0 and q_1 and where (γ_j, γ_{j+1}) is the unique pair in $\mathcal{F}_Q^<$ with denominators $QT^j\big(\frac{q_0}{Q}, \frac{q_1}{Q}\big)$, $j = 1,\dots,r$. This also shows that the set

$$\mathcal{F}^{\mathrm{odd,\,odd/even}}_{Q,k_1,\dots,k_r} = \{(\gamma_0,\dots,\gamma_r) \in \mathcal{F}_{Q,k_1,\dots,k_r} : q_0 \text{ odd}, q_1 \text{ odd/even}\}$$

has cardinality $N_{\mathrm{odd,\,odd/even}}(Q\mathcal{T}_{k_1,\dots,k_r})$.

Suppose that $\gamma = a/q < a''/q'' = \gamma''$ are two consecutive elements in $\mathcal{F}_{Q,\mathrm{odd}}$ and that $\Delta(\gamma,\gamma'') = a''q - aq'' > 1$. Since two fractions with even denominators cannot occur as consecutive elements in $\mathcal{F}_Q$, it follows that there is precisely one fraction $\gamma' = a'/q'$ in $\mathcal{F}_Q$ such that $\gamma < \gamma' < \gamma''$ are consecutive in $\mathcal{F}_Q$. One readily finds (see e.g. [11, p. 4]) that

$$\Delta(\gamma,\gamma'') = \nu_Q(\gamma) = \left[\frac{Q+q}{q'}\right] = \kappa\left(\frac{q}{Q}, \frac{q'}{Q}\right). \tag{2.1}$$

To summarize, suppose that $\gamma < \gamma' < \gamma'' < \gamma''' < \gamma^{IV}$ are consecutive in $\mathcal{F}_Q$ and that q is odd. Denote by $q, q', \dots, q^{IV}$ (respectively) the denominators of $\gamma, \gamma', \dots, \gamma^{IV}$. Denote also

$$k_j = k_j(q,q') = \kappa\left(T^{j-1}\left(\frac{q}{Q}, \frac{q'}{Q}\right)\right), \quad j \ge 1.$$

Then $q'' = k_1 q' - q$, $q''' = k_2 q'' - q'$, and so forth. The following situations may occur.

(O) q' is odd and thus $\Delta(\gamma,\gamma') = 1$. Next, it could be *either* that

- (OO) q'' is odd (if k_1 is even), in which case $(\gamma',\gamma'') \in \mathcal{F}^<_{Q,\mathrm{odd}}$ and $\Delta(\gamma',\gamma'') = 1$, *or* that
- (OEO) q'' is even (if k_1 is odd), in which case $q''' = k_2 q'' - q'$ is odd, $(\gamma',\gamma''') \in \mathcal{F}^<_{Q,\mathrm{odd}}$, and $\Delta(\gamma',\gamma''') = k_2$.

(E) q' is even and thus q'' is odd, $(\gamma, \gamma'') \in \mathcal{F}^{<}_{Q,\mathrm{odd}}$, and $\Delta(\gamma, \gamma'') = k_1$. Next, we have *either* that

(EOO) q''' is odd (if k_2 is odd), in which case $(\gamma'', \gamma''') \in \mathcal{F}^{<}_{Q,\mathrm{odd}}$ and $\Delta(\gamma'', \gamma''') = 1$, *or* that

(EOEO) q''' is even (if k_2 is even), in which case $q^{IV} = k_3 q''' - q''$ will also be odd, $(\gamma'', \gamma^{IV}) \in \mathcal{F}^{<}_{Q,\mathrm{odd}}$, and $\Delta(\gamma'', \gamma^{IV}) = k_3$.

This suggests that one may express $N_{Q,\mathrm{odd}}(\Delta_1, \ldots, \Delta_h)$ for any $h \geq 1$ by an inductive procedure. Note first that

$$N_{Q,\mathrm{odd}}(\Delta_1) = \begin{cases} \sum_{k_1} N_{\mathrm{odd,odd}}(Q\mathcal{T}_{k_1}) + \sum_{k_2} N_{\mathrm{odd,even}}(Q\mathcal{T}_{1,k_2}) & \\ \qquad = N_{\mathrm{odd,odd}}(Q\mathcal{T}) + N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1}) & \text{if } \Delta_1 = 1, \\ \sum_{k_2} N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1,k_2}) = N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1}) & \text{if } \Delta_1 \geq 2. \end{cases}$$

One may also express $N_{Q,\mathrm{odd}}(\Delta_1, \Delta_2)$ as

$$\begin{cases} \sum_{k_1\,\mathrm{even}} N_{\mathrm{odd,odd}}(Q\mathcal{T}_{k_1}) + \sum_{k_1\,\mathrm{odd}} N_{\mathrm{odd,odd}}(Q\mathcal{T}_{k_1,1}) & \\ \qquad + \sum_{k_2\,\mathrm{odd}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{1,k_2}) & \\ \qquad + \sum_{k_2\,\mathrm{even}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{1,k_2,1}) & \text{if } \Delta_1 = \Delta_2 = 1, \\ \sum_{k_1\,\mathrm{odd}} N_{\mathrm{odd,odd}}(Q\mathcal{T}_{k_1,\Delta_2}) & \\ \qquad + \sum_{k_2\,\mathrm{even}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{1,k_2,\Delta_2}) & \text{if } \Delta_1 = 1 \text{ and } \Delta_2 \geq 2, \\ \sum_{k_2\,\mathrm{odd}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1,k_2}) & \\ \qquad + \sum_{k_2\,\mathrm{even}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1,k_2,1}) & \text{if } \Delta_1 \geq 2 \text{ and } \Delta_2 = 1, \\ \sum_{k_2\,\mathrm{even}} N_{\mathrm{odd,even}}(Q\mathcal{T}_{\Delta_1,k_2,\Delta_2}) & \text{if } \Delta_1, \Delta_2 \geq 2. \end{cases}$$

For $h \geq 2$, we may express $\rho_{Q,\mathrm{odd}}(\Delta_1, \ldots, \Delta_h)$ as in the following proposition.

PROPOSITION 2.1. *Assume that $h \geq 2$ and $\Delta = (\Delta_1, \ldots, \Delta_h) \in (\mathbb{N}^*)^h$. Then*

$$\rho_{Q,\mathrm{odd}}(\Delta) = \frac{1}{N_{Q,\mathrm{odd}}} \sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} N_{\mathrm{odd},o(v_1)}(Q\mathcal{T}_{k_1,\ldots,k_{|w|-1}}). \tag{2.2}$$

3. Estimating $N_{\mathrm{odd,odd}}(\Omega)$ and $N_{\mathrm{odd,even}}(\Omega)$

For a bounded region Ω in $\mathbb{R}^2$ with rectifiable boundary and for a function f defined on Ω, we set

$$S_f(\Omega) = \sum_{(a,b)\in\Omega\cap\mathbb{Z}^2} f(a,b), \qquad S'_f(\Omega) = \sum_{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}} f(a,b),$$

$$S_{f,\mathrm{odd/even}}(\Omega) = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2 \\ a\ \mathrm{odd/even}}} f(a,b), \qquad S'_{f,\mathrm{odd/even}} = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} \\ a\ \mathrm{odd/even}}} f(a,b),$$

$$S'_{f,\,\mathrm{odd},\,\mathrm{odd/even}}(\Omega) = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}} \\ a \text{ odd},\, b \text{ odd/even}}} f(a,b),$$

$$\|Df\|_{L^\infty(\Omega)} = \sup_{(x,y)\in\Omega}\left(\left|\frac{\partial f}{\partial x}(x,y)\right| + \left|\frac{\partial f}{\partial y}(x,y)\right|\right).$$

LEMMA 3.1. *Let $R_1, R_2 > 0$, and let $R \ge \min(R_1, R_2)$. Then, for any region $\Omega \subseteq [0, R_1] \times [0, R_2]$ and any function f that is C^1 on Ω, we have*

(i) $$S'_{f,\,\mathrm{odd}}(\Omega) = \frac{4}{\pi^2} \iint_\Omega f(x,y)\,dx\,dy + O(A_{f,R,\Omega}).$$

(ii) $$S'_{f,\,\mathrm{odd},\,\mathrm{odd/even}}(\Omega) = \frac{2}{\pi^2} \iint_\Omega f(x,y)\,dx\,dy + O(A_{f,R,\Omega}).$$

(iii) $$S'_{f,\,\mathrm{even},\,\mathrm{odd}}(\Omega) = \frac{2}{\pi^2} \iint_\Omega f(x,y)\,dx\,dy + O(A_{f,R,\Omega}),$$

where

$$A_{f,R,\Omega} = \frac{\|f\|_{L^1(\Omega)}}{R} + \|Df\|_{L^\infty(\Omega)}\,\mathrm{Area}(\Omega)\log R + \|f\|_{L^\infty(\Omega)}(R + \mathrm{length}(\partial\Omega)\log R).$$

Proof. (i) It is well known (see e.g. [3, Lemma 1]) that

$$S_f(\Omega) = \iint_\Omega f(x,y)\,dx\,dy + O(B_{f,\Omega}),$$

where

$$B_{f,\Omega} = \|Df\|_{L^\infty(\Omega)}\,\mathrm{Area}(\Omega) + \|f\|_{L^\infty(\Omega)}(1 + \mathrm{length}(\partial\Omega)).$$

Denoting $\Omega' = \{(x/2, y) : (x,y) \in \Omega\}$, we have that $S_{f,\,\mathrm{even}}(\Omega)$—and eventually $S_{f,\,\mathrm{odd}}(\Omega)$—can be expressed as

$$\begin{aligned}\sum_{(a,b)\in\Omega'\cap\mathbb{Z}^2} f(2a,b) &= \iint_{\Omega'} f(2x,y)\,dx\,dy + O(B_{f,\Omega'}) \\ &= \frac{1}{2}\iint_\Omega f(x,y)\,dx\,dy + O(B_{f,\Omega}). \qquad (3.1)\end{aligned}$$

We now proceed to estimate $S'_{f,\,\mathrm{odd}}(\Omega)$, which is written as

$$\begin{aligned}&\sum_{\substack{(a,b)\in\Omega \\ a \text{ odd}}} f(a,b) - \sum_{\substack{(a,b)\in\Omega/3 \\ a \text{ odd}}} f(a,b) - \sum_{\substack{(a,b)\in\Omega/5 \\ a \text{ odd}}} f(a,b) - \cdots \\ &\qquad = \sum_{\substack{1\le n\le R \\ n \text{ odd}}} \mu(n) \sum_{\substack{(a,b)\in\Omega/n \\ a \text{ odd}}} f(na,nb). \qquad (3.2)\end{aligned}$$

The inner sum in (3.2) is expressed by means of (3.1) as

$$\frac{1}{2}\iint\limits_{\Omega/n} f(nx, ny)\,dx\,dy + O\biggl(\frac{\|Df\|_{L^\infty(\Omega)}\,\mathrm{Area}(\Omega)}{n} + \|f\|_{L^\infty(\Omega)}\biggl(1+\frac{\mathrm{length}(\partial\Omega)}{n}\biggr)\biggr). \tag{3.3}$$

Changing (nx, ny) to (x, y) in the double integral and summing over n, we infer from (3.2) and (3.3) that

$$S'_{f,\,\mathrm{odd}}(\Omega) = \frac{1}{2}\sum_{\substack{1\le n\le R\\ n\ \mathrm{odd}}}\frac{\mu(n)}{n^2}\iint\limits_{\Omega} f(x, y)\,dx\,dy + O(\|Df\|_{L^\infty(\Omega)}\,\mathrm{Area}(\Omega)\log R) + O\bigl(\|f\|_{L^\infty(\Omega)}(R + \mathrm{length}(\partial\Omega)\log R)\bigr).$$

The equality (i) now follows from

$$\sum_{\substack{1\le n\le R\\ n\ \mathrm{odd}}}\frac{\mu(n)}{n^2} = \frac{8}{\pi^2} + O\biggl(\frac{1}{R}\biggr).$$

The equality (ii) follows by combining (i) with

$$S'_{f,\,\mathrm{odd,\,even}}(\Omega) = \sum_{\substack{(a,b)\in\Omega''\cap\mathbb{Z}^2_{\mathrm{pr}}\\ a\ \mathrm{odd}}} f(a, 2b),$$

where we set $\Omega'' = \{(x, y/2) : (x, y)\in\Omega\}$, and then using

$$\iint\limits_{\Omega''} f(x, 2y)\,dx\,dy = \frac{1}{2}\iint\limits_{\Omega} f(x, y)\,dx\,dy.$$

The equality (iii) now follows from symmetry. □

We need the following improvement of Lemma 1 in [11].

COROLLARY 3.2. *Let* $R_1, R_2 > 0$, *and let* $R \ge \min(R_1, R_2)$. *Then, for any region* $\Omega \subseteq [0, R_1]\times[0, R_2]$ *with rectifiable boundary, we have*

(i) $N_{\mathrm{odd}}(\Omega) = 4\,\mathrm{Area}(\Omega)/\pi^2 + O(C_{R,\Omega})$,
(ii) $N_{\mathrm{even}}(\Omega) = 2\,\mathrm{Area}(\Omega)/\pi^2 + O(C_{R,\Omega})$,
(iii) $N_{\mathrm{odd,\,even}}(\Omega) = 2\,\mathrm{Area}(\Omega)/\pi^2 + O(C_{R,\Omega})$,
(iv) $N_{\mathrm{odd,\,odd}}(\Omega) = 2\,\mathrm{Area}(\Omega)/\pi^2 + O(C_{R,\Omega})$, *and*
(v) $N_{\mathrm{even,\,odd}}(\Omega) = 2\,\mathrm{Area}(\Omega)/\pi^2 + O(C_{R,\Omega})$,

where

$$C_{R,\Omega} = \mathrm{Area}(\Omega)/R + R + \mathrm{length}(\partial\Omega)\log R.$$

The following lemma is contained in [3]. We include the proof for the reader's convenience.

LEMMA 3.3. *For any integers* $k_1, \dots, k_r \ge 1$, *the set* $\mathcal{T}_{k_1,\dots,k_r}$ *is a convex polygon.*

Proof. If for $(x, y) \in \mathbb{R}^2$ we define $L_0(x, y) = x$, $L_1(x, y) = y$, and $L_{i+1}(x, y) = k_i L_i(x, y) - L_{i-1}(x, y)$ for $i \geq 1$, then $\mathcal{T}_{k_1,\dots,k_r}$ is defined by the following inequalities:

$$1 \geq L_0(x, y), L_1(x, y), \dots, L_{r+1}(x, y) > 0,$$

$$L_0(x, y) + L_1(x, y), L_1(x, y) + L_2(x, y), \dots, L_r(x, y) + L_{r+1}(x, y) > 1.$$

Because $L_0, L_1, \dots, L_{r+1}$ are linear functions, the set $\mathcal{T}_{k_1,\dots,k_r}$ is the intersection of finitely many convex polygons. □

LEMMA 3.4. (i) *Let $r \geq 1$. Then, for any $m \geq c_r = 4r + 2$, we have that all sets $T^{-i}\mathcal{T}_m$ $(i = 0, 1, \dots, r)$ are convex. Moreover,*

$$T^{-1}\mathcal{T}_m \subset \mathcal{T}_1, \qquad \bigcup_{i=2}^{r} T^{-i}\mathcal{T}_m \subset \mathcal{T}_2,$$

and, for all $(x, y) \in \mathcal{T}_m$ and $i \in \{1, 2, \dots, r\}$,

$$T^{-i}(x, y) = (x - iy, x - (i - 1)y).$$

(ii) *For any $m \geq c_r$,*

$$T\mathcal{T}_m \subset \mathcal{T}_1, \qquad \bigcup_{i=2}^{r} T^{i}\mathcal{T}_m \subset \mathcal{T}_2,$$

and, for all $(x, y) \in \mathcal{T}_m$ and $i \in \{2, \dots, r\}$,

$$T^{i}(x, y) = ((m + 2 - i)y - x, (m + 1 - i)y - x).$$

(iii) *Let $j \in \{1, \dots, r\}$. Then*

$$\operatorname{length}(\partial T^{j-1}\mathcal{T}_{k_1,\dots,k_r}) \ll_r \frac{1}{k_j}$$

uniformly in $k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r$ as $k_j \to \infty$.

Proof. (i) In the beginning we follow closely the proof of Lemma 5 in [3]. The inverse of the transformation T is given by

$$T^{-1}(x, y) = \left(\left[\frac{1 + y}{x}\right]x - y, x\right), \quad (x, y) \in \mathcal{T}. \tag{3.4}$$

Since $0 \leq 1 - y < x$, we also have $\left[\frac{1-y}{x}\right] = 0$ and thus, for all $(x, y) \in \mathcal{T}$,

$$\kappa(T^{-1}(x, y)) = \left[\frac{1 + \left[\frac{1+y}{x}\right]x - y}{x}\right] = \left[\frac{1 + y}{x}\right] + \left[\frac{1 - y}{x}\right] = \left[\frac{1 + y}{x}\right]. \tag{3.5}$$

Consider next a fixed element $(x, y) \in \mathcal{T}_m$ with $m \geq c_r$. Since $m \geq 5$, we have

$$m \leq \frac{1 + x}{y} < m + 1 \quad \text{and} \quad x > \frac{m - 1}{m + 1}.$$

This leads to

$$1 < \frac{1+y}{x} \le \frac{1+\frac{1+x}{m}}{x} = \frac{x+m+1}{mx} < \frac{\frac{m-1}{m+1}+m+1}{m \cdot \frac{m-1}{m+1}} = \frac{m+3}{m-1} \le 2,$$

showing that $\kappa(T^{-1}(x,y)) = 1$ and—using also (3.4)—that

$$T^{-1}(x,y) = (x-y,x) \in \mathcal{T}_1. \tag{3.6}$$

Next, the inequality $m \ge c_r$ gives

$$1 + \frac{2(2r-1)}{m} \le 1 + \frac{m-4}{m} \le 1 + \frac{m-3}{m+1} = \frac{2(m-1)}{m+1}. \tag{3.7}$$

Hence the inequalities $x > \frac{m-1}{m+1}$ and $y \le \frac{2}{m}$, fulfilled by $(x,y) \in \mathcal{T}_m$ (see [3, Figure 1]), imply in conjunction with (3.7) that $2x > 1 + (2i-1)y$ for all $(x,y) \in \mathcal{T}_m$ and $i \in \{2,\dots,r\}$, or equivalently that

$$\frac{1+y}{x-(i-1)y} < 2, \quad i \in \{2,\dots,r\}.$$

At the same time, it is clear that $\frac{1+y}{x-(i-1)y} > 1$, so that

$$\left[\frac{1+x-(i-2)y}{x-(i-1)y}\right] = 1 + \left[\frac{1+y}{x-(i-1)y}\right] = 2, \quad i \in \{2,\dots,r\}. \tag{3.8}$$

For $i = 2$, equalities (3.5), (3.7), and (3.8) give

$$\kappa(T^{-2}(x,y)) = \left[\frac{1+x}{x-y}\right] = 2,$$

$$T^{-2}(x,y) = (2(x-y)-x, x-y) = (x-2y, x-y);$$

thus, by (3.5) and by (3.8) with $i = 3$ we have

$$\kappa(T^{-3}(x,y)) = \left[\frac{1+x-y}{x-2y}\right] = 2,$$

$$T^{-3}(x,y) = (2(x-2y)-x+y, x-2y) = (x-3y, x-2y).$$

Arguing by induction, it follows at once that, for all $i \in \{2,\dots,r\}$,

$$\kappa(T^{-i}(x,y)) = \left[\frac{1+x-(i-2)y}{x-(i-1)y}\right] = 2,$$

$$T^{-i}(x,y) = (x-iy, x-(i-1)y).$$

As a consequence, $T^{-i}\mathcal{T}_m$ is the quadrangle with vertices at $\left(1-\frac{2i}{m}, 1-\frac{2(i-1)}{m}\right)$, $\left(1-\frac{2i}{m+1}, 1-\frac{2(i-1)}{m+1}\right)$, $\left(1-\frac{2(i+1)}{m+2}, 1-\frac{2i}{m+2}\right)$, and $\left(1-\frac{2(i+1)}{m+1}, 1-\frac{2i}{m+1}\right)$. This quadrangle is obviously contained in $\mathcal{T}_2$.

(ii) Let $(x,y) \in \mathcal{T}_m$. Then $T(x,y) = (y, my-x)$ and so

$$\kappa(T(x,y)) = \left[\frac{1+y}{my-x}\right] \ge 1.$$

Since $m \le \frac{1+x}{y} < m+1$ and $y \le \frac{2}{m} \le \frac{1}{3}$, it follows that $(2m-1)y \ge 1+(2m+2)y-2 > 1+2x$. This leads to $\frac{1+y}{my-x} < 2$, and so we obtain $\kappa(T(x,y)) = 1$. Therefore,

$$T^2(x,y) = (my-x, (m-1)y-x).$$

On the other hand, $y \le \frac{1+x}{m} < \frac{1+x}{m-i}$; whence

$$2 \le \left[\frac{1+(m+2-i)y-x}{(m+1-i)y-x}\right] = 1 + \left[\frac{1+y}{(m+1-i)y-x}\right], \quad i \ge 1. \tag{3.9}$$

The inequality $m \ge 4r+2$ leads to $m-2r \ge (2r+1)x$, which is equivalent to $(m+1)(1+2x) \le (2m+1-2r)(1+x)$. Since $1+x < (m+1)y$, we infer that $1+2x < (2m+1-2r)y \le (2m+1-2i)y$. That is,

$$\frac{1+y}{(m+1-i)y-x} < 2, \quad i \in \{1,\dots,r\}. \tag{3.10}$$

By (3.9) and (3.10), we gather that

$$\left[\frac{1+(m+2-i)y-x}{(m+1-i)y-x}\right] = 2, \quad i \in \{2,\dots,r\}. \tag{3.11}$$

Now we infer inductively that $T^i(x,y) \in \mathcal{T}_2$ and that

$$T^i(x,y) = ((m+2-i)y-x, (m+1-i)y-x), \quad i \in \{2,\dots,r\}.$$

(iii) We use the fact that if Ω_1 and Ω_2 are convex polygons with $\Omega_1 \subseteq \Omega_2$, then $\text{length}(\partial\Omega_1) \le \text{length}(\partial\Omega_2)$. For $k_j > c_r$ this yields, in conjunction with (i) and (ii),

$$\text{length}(\partial T^{j-1}\mathcal{T}_{k_1,\dots,k_r}) \le \text{length}(\partial\mathcal{T}_{k_j}) \ll_r \frac{1}{k_j}$$

uniformly in $k_1,\dots,k_{j-1},k_{j+1},\dots,k_r$. □

REMARK 3.5. Suppose that $(x,y) \in \mathcal{T}_m$ with $m \ge 3$. Then $\frac{1+x}{y} < m+1$ and $y \le \frac{2}{m}$; hence $\frac{1+y}{my-x} < \frac{1+2/m}{1-y} \le \frac{1+2/m}{1-2/m} = \frac{m+2}{m-2}$ and thus

$$\bigcup_{m\ge 6} T\mathcal{T}_m \subset \mathcal{T}_1, \quad T(\mathcal{T}_4 \cup \mathcal{T}_5) \subset \mathcal{T}_1 \cup \mathcal{T}_2, \quad \text{and} \quad T\mathcal{T}_3 \subset \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4.$$

If $(x,y) \in \mathcal{T}_2$, then $y > \frac{1+x}{3} \ge \frac{x}{2} + \frac{1}{6} \ge \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ and so

$$T\mathcal{T}_2 \subset \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3.$$

On the other hand, if $(x,y) \in \mathcal{T}_m$ $(m \ge 2)$ then it follows from the proof of Lemma 3.4(i) that $\kappa(T^{-1}(x,y)) < \frac{m+3}{m-1}$. Therefore,

$$\bigcup_{m\ge 5} T\mathcal{T}_m \subset \mathcal{T}_1, \quad T(\mathcal{T}_3 \cup \mathcal{T}_4) \subset \mathcal{T}_1 \cup \mathcal{T}_2, \quad \text{and} \quad T\mathcal{T}_2 \subset \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4.$$

Owing to the presence of the term R in $C_{R,\Omega}$, we need one more fact, which was noticed already (in a different form) in [4].

LEMMA 3.6. *Let $k \in \mathbb{N}^*$ and let $\mathcal{D}$ be a subset of $\mathcal{T}$. Then the following equalities hold.*

(i) *For k even*:

$$N_{\text{odd, even}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{even, odd}}(QT(\mathcal{T}_k \cap \mathcal{D})),$$
$$N_{\text{even, odd}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{odd, even}}(QT(\mathcal{T}_k \cap \mathcal{D})),$$
$$N_{\text{odd, odd}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{odd, odd}}(QT(\mathcal{T}_k \cap \mathcal{D})).$$

(ii) *For k odd*:

$$N_{\text{odd, even}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{even, odd}}(QT(\mathcal{T}_k \cap \mathcal{D})),$$
$$N_{\text{even, odd}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{odd, odd}}(QT(\mathcal{T}_k \cap \mathcal{D})),$$
$$N_{\text{odd, odd}}(Q(\mathcal{T}_k \cap \mathcal{D})) = N_{\text{odd, even}}(QT(\mathcal{T}_k \cap \mathcal{D})).$$

Proof. We denote by T_k the linear transformation defined on $\mathbb{R}^2$ by $T_k(x, y) = (y, ky - x)$. Assume that k is even and let $(a, b) \in Q(\mathcal{T}_k \cap \mathcal{D})$. Then $T\left(\frac{a}{Q}, \frac{b}{Q}\right) = \left(\frac{b}{Q}, \frac{kb}{Q} - \frac{a}{Q}\right)$, so

$$QT(\mathcal{T}_k \cap \mathcal{D}) = \{(b, kb - a) : (a, b) \in Q\mathcal{T}_k \cap Q\mathcal{D}\} = T_k(Q(\mathcal{T}_k \cap \mathcal{D})).$$

Moreover, since the matrix that defines T_k is unimodular, the elements of $\mathbb{Z}^2_{\text{pr}} \cap Q(\mathcal{T}_k \cap \mathcal{D})$ are in 1–1 correspondence with the elements of $\mathbb{Z}^2_{\text{pr}} \cap T_k(Q(\mathcal{T}_k \cap \mathcal{D})) = \mathbb{Z}^2_{\text{pr}} \cap Q(T(\mathcal{T}_k \cap \mathcal{D}))$. Besides, we see that a is odd and b is even if and only if b is even and $kb - a$ is odd, implying that

$$\#\{(a, b) \in \mathbb{Z}^2_{\text{pr}} \cap Q(\mathcal{T}_k \cap \mathcal{D}) : a \text{ odd},\ b \text{ even}\}$$
$$= \#\{(c, d) \in \mathbb{Z}^2_{\text{pr}} \cap QT(\mathcal{T}_k \cap \mathcal{D}) : c \text{ even},\ d \text{ odd}\}.$$

The other five equalities follow in a similar way. □

Proof of Theorem 1.1. We wish to apply Corollary 3.2 to $\Omega = Q\mathcal{T}_{k_1,\dots,k_r}$. Note first that, since T is area-preserving, we have

$$\text{Area}(\mathcal{T}_{k_1,\dots,k_r}) \le \text{Area}(T^{-j+1}\mathcal{T}_{k_j}) = \text{Area}(\mathcal{T}_{k_j}) \ll \frac{1}{k_j^3}, \quad j \in \{1, \dots, r\}.$$

We claim that (for every $j \in \{1, \dots, r\}$) all the numbers $N_{\text{odd, odd}}(Q\mathcal{T}_{k_1,\dots,k_r})$, $N_{\text{odd, even}}(Q\mathcal{T}_{k_1,\dots,k_r})$, and $N_{\text{even, odd}}(Q\mathcal{T}_{k_1,\dots,k_r})$ can be expressed as

$$\frac{2Q^2}{\pi^2}\,\text{Area}(\mathcal{T}_{k_1,\dots,k_r}) + O_r\left(\frac{Q}{k_j}\log Q\right) \tag{3.12}$$

uniformly in $k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r$ as $Q \to \infty$.

If $j \ge 2$, we apply Lemma 3.6 successively $j-1$ times: to k_1 and $\mathcal{D} = T^{-1}\mathcal{T}_{k_2,\dots,k_r}$; to k_2 and $\mathcal{D} = T\mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_{k_3,\dots,k_r}$; …; and to k_{j-1} and $\mathcal{D} = T^{j-2}\mathcal{T}_{k_1,\dots,k_{j-2}} \cap T^{-1}\mathcal{T}_{k_j,\dots,k_r}$. This yields

$$N_{\text{odd, odd}}(Q\mathcal{T}_{k_1,\dots,k_r}) = N_{\delta_1,\delta_2}(QT^{j-1}\mathcal{T}_{k_1,\dots,k_r})$$

for some pair $(\delta_1, \delta_2) \in \{(\text{odd}, \text{odd}), (\text{odd}, \text{even}), (\text{even}, \text{odd})\}$ that depends on $k_1, \dots, k_{j-1}$. We may now apply Corollary 3.2 to $\Omega = QT^{j-1}\mathcal{T}_{k_1,\dots,k_r} \subseteq Q\mathcal{T}_{k_j} \subset [0, Q] \times [0, 2Q/k_j]$, with $R \asymp Q/k_j$, $\text{Area}(\Omega) \le \text{Area}(Q\mathcal{T}_{k_j}) \ll Q^2/k_j^3$, and (according to Lemma 3.4) $\text{length}(\partial\Omega) \ll_r Q/k_j$. Therefore, we gather that $N_{\text{odd},\text{odd}}(Q\mathcal{T}_{k_1,\dots,k_r})$ is indeed given by (3.12). The same estimates are proved for $N_{\text{odd},\text{even}}(Q\mathcal{T}_{k_1,\dots,k_r})$ and $N_{\text{even},\text{odd}}(Q\mathcal{T}_{k_1,\dots,k_r})$ in a similar fashion.

We may now complete the proof of Theorem 1.1. If $k_j \ge c_r$, then we infer from Lemma 3.4(i) that $\mathcal{T}_{k_1,\dots,k_r} = \emptyset$ unless $k_1 = \dots = k_{j-2} = k_{j+2} = \dots = k_r = 2$ and $k_{j-1} = k_{j+1} = 1$. On the other hand, we see from [4, Rem. 2.3] that $Q\mathcal{T}_{k_1,\dots,k_r} \cap \mathbb{Z}^2 = \emptyset$ unless $\max(k_1, \dots, k_r) \le 2Q$.

As a result, the only nonzero terms that may appear in the sum from (2.2) arise from paths w having all labels $k_j \le 2Q$ and at most one $> c_{2h-1}$. Taking now also into account (3.12), the sum $\sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} N_{\text{odd},o(v_1)}(Q\mathcal{T}_{k_1,\dots,k_{|w|-1}})$ can be expressed as

$$\frac{2Q^2}{\pi^2} \sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} \text{Area}(\mathcal{T}_{k_1,\dots,k_{|w|-1}}) + O_h\biggl(\sum_{k=1}^{2Q} \frac{Q \log Q}{k}\biggr)$$
$$= \frac{2Q^2}{\pi^2} \sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} \text{Area}(\mathcal{T}_{k_1,\dots,k_{|w|-1}}) + O_h(Q \log^2 Q). \quad (3.13)$$

The statement in Theorem 1.1 now follows from Proposition 2.1, (3.13), and equation (1.1). □

4. Consecutive Farey Fractions with Odd Denominators in Short Intervals

For each interval $I \subseteq [0, 1]$ and each subset $\Omega \subseteq \mathbb{R}^2$, we set

$$\Omega^I = \{(a, b) \in \Omega \cap \mathbb{Z}^2_{\text{pr}} : \bar{b} \in I_a\},$$

where $\bar{b}$ denotes the unique number in $\{1, \dots, a-1\}$ for which $b\bar{b} = 1 \pmod a$. If $I = [\alpha, \beta]$, then we also set $I_a = [a(1-\beta), a(1-\alpha)]$.

For any function f defined on Ω, denote

$$S^I_{f,\text{odd},\text{odd/even}}(\Omega) = \sum_{\substack{(a,b) \in \Omega \cap \mathbb{Z}^2_{\text{pr}} \\ a \text{ odd}, \, b \text{ odd/even} \\ \bar{b} \in I_a}} f(a, b).$$

The following analogue of Proposition 2.1 holds and is similarly proved.

Proposition 4.1. *Let $h \ge 1$, and let $\Delta = (\Delta_1, \dots, \Delta_h) \in (\mathbb{N}^*)^h$. Then, for any interval $I \subseteq [0, 1]$,*

$$N^I_{Q,\text{odd}}(\Delta) = \sum_{w \in \mathfrak{L}_h \cap \mathfrak{S}_\Delta} N_{\text{odd},o(v_1)}((Q\mathcal{T}_{k_1,\dots,k_{|w|-1}})^I).$$

Proposition 4.2. *Assume that $\Omega \subseteq [0, R_1] \times [0, R_2]$ is a convex region and that f is a C^1 function on Ω. Then $S^I_{f,\text{odd},\text{odd/even}}(\Omega)$ is given by*

$$|I|S'_{f,\,\mathrm{odd},\,\mathrm{odd/even}}(\Omega) + O_\varepsilon\big(\|f\|_{L^\infty(\Omega)}(R_2\log R_1 + m_f R_1^{1/2+\varepsilon}(R_1+R_2))\big)$$

for every $\varepsilon > 0$, *where* m_f *is an upper limit for the number of intervals of monotonicity of the functions* $y \mapsto f(x,y)$.

Proof. The proof is similar to that of Lemma 8 in [3]. As in [3, (65)], we write

$$S^I_{f,\,\mathrm{odd},\,\mathrm{odd/even}}(\Omega) = S_1 + S_2, \tag{4.1}$$

where

$$\begin{aligned} S_1 &= \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}\\ a\ \mathrm{odd},\, b\ \mathrm{odd/even}}} f(a,b)\sum_{x\in I_a}\frac{1}{a} \\ &= \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}\\ a\ \mathrm{odd},\, b\ \mathrm{odd/even}}} f(a,b)\frac{1}{a}(|I_a| + O(1)) \\ &= |I|S'_{f,\,\mathrm{odd},\,\mathrm{odd/even}}(\Omega) + O(\|f\|_{L^\infty(\Omega)}R_2\log R_1) \end{aligned} \tag{4.2}$$

and

$$S_2 = \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2_{\mathrm{pr}}\\ a\ \mathrm{odd},\, b\ \mathrm{odd/even}}} f(a,b)\sum_{x\in I_a}\frac{1}{a}\sum_{l=1}^{a-1} e\left(\frac{l(\bar{b}-x)}{a}\right).$$

As in [3, (67)], we write

$$S_2 = \sum_{\substack{a\in\mathrm{pr}_1(\Omega)\\ a\ \mathrm{odd}}} \frac{1}{a}\sum_{l=1}^{a-1}\left(\sum_{x\in I_a} e\left(-\frac{lx}{a}\right)\right) S_{f,\,\mathrm{odd/even},\,I'_a}(l,a), \tag{4.3}$$

where $I'_a = \{b : (a,b)\in\Omega\}$ is an interval for every a in the projection $\mathrm{pr}_1(\Omega)$ of Ω on the first coordinate. Here, for any interval J we denote

$$S_{f,\,\mathrm{odd/even},\,J}(l,a) = \sum_{\substack{b\in J\\ b\ \mathrm{odd/even}\\ \gcd(a,b)=1}} f(a,b)e\left(\frac{l\bar{b}}{a}\right) \tag{4.4}$$

and

$$S_{f,J}(l,a) = \sum_{\substack{b\in J\\ \gcd(a,b)=1}} f(a,b)e\left(\frac{l\bar{b}}{a}\right).$$

By [3, Lemma 9] we have

$$|S_{f,J}(l,a)| \ll_\varepsilon R_{\Omega,f,J,l,a,\varepsilon}, \tag{4.5}$$

where

$$R_{\Omega,f,J,l,a,\varepsilon} = m_f\|f\|_{L^\infty(\Omega)}(|J|a^{-1/2+\varepsilon} + a^{1/2+\varepsilon})\gcd(l,a)^{1/2}.$$

Writing now

$$S_{f,\mathrm{even},J}(l,a) = \sum_{\substack{c\in J/2\\ \gcd(a,c)=1}} f(a,2c)e\left(\frac{\bar{2}l\bar{c}}{a}\right) = S_{f_2,J/2}(\bar{2}l,a),$$

where $f_2(x,y) = f(x,2y)$, and then using (4.5) and $S_{f,\mathrm{odd},J}(l,a) = S_{f,J}(l,a) - S_{f,\mathrm{even},J}(l,a)$, we infer that

$$\max(|S_{f,\mathrm{even},J}(l,a)|, |S_{f,\mathrm{odd},J}(l,a)|) \ll_\varepsilon R_{\Omega,f,J,l,a,\varepsilon}. \tag{4.6}$$

As in [3, (67)–(69)], we infer—from (4.3), (4.4), (4.6), the fact that the inner sum in (4.3) is a geometric progression $\ll \left(\frac{a}{l}, \frac{a}{a-l}\right)$, and $|I'_a| \le R_2$—that

$$\begin{aligned}|S_2| &\ll \sum_{a=1}^{R_1} \frac{1}{a} \sum_{l=1}^{a-1} \frac{a}{l} |S_{f,\mathrm{odd/even},I'_a}(l,a)| \\ &\ll_\varepsilon m_f \|f\|_{L^\infty(\Omega)} R_1^{1/2+\varepsilon}(R_1 + R_2).\end{aligned} \tag{4.7}$$

The desired conclusion now follows from (4.1), (4.2), and (4.7). □

COROLLARY 4.3.

$$N_{\mathrm{odd,odd/even}}((Q\mathcal{T}_{k_1,\dots,k_r})^I) = |I| N_{\mathrm{odd,odd/even}}(Q\mathcal{T}_{k_1,\dots,k_r}) + O_\varepsilon(Q^{3/2+\varepsilon}).$$

Theorem 1.3 is now a consequence of Proposition 4.1 and Corollary 4.3.

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