Michigan Math. J. 51 (2003)

On Parameter Spaces for Artin Level Algebras

J. V. Chipalkatti & A. V. Geramita

Let $R = k[x_1, \dots, x_n]$ denote a polynomial ring and let $\underline{h}\colon \mathbb{N} \to \mathbb{N}$ be a numerical function. Consider the set of all graded Artin level quotients $A = R/I$ having Hilbert function $\underline{h}$. This set (if nonempty) is naturally in bijection with the closed points of a quasiprojective scheme $\mathcal{L}^\circ(\underline{h})$. The object of this note is to prove some specific geometric properties of these schemes, especially for $n = 2$. The case of Gorenstein Hilbert functions (i.e., where A has type 1) has been extensively studied, and several qualitative and quantitative results are known (see [17]). Our results should be seen as generalizing some of them to the non-Gorenstein case.

After establishing notation, we summarize the results in the next section. See [12; 17] as general references for most of the constructions used here.

1. Notation and Preliminaries

The base field k will be algebraically closed and of characteristic 0 (but see Remark 4.11). Let V be an n-dimensional k-vector space, and let

$$R = \bigoplus_{i \geq 0} \operatorname{Sym}^i V^*, \qquad S = \bigoplus_{i \geq 0} \operatorname{Sym}^i V.$$

Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be dual bases of V^* and V (respectively), leading to identifications $R = k[x_1, \dots, x_n]$ and $S = k[y_1, \dots, y_n]$. There are internal products (see [11, p. 476])

$$\operatorname{Sym}^j V^* \otimes \operatorname{Sym}^i V \to \operatorname{Sym}^{i-j} V, \qquad u \otimes F \to u \cdot F,$$

making S into a graded R-module. This action may be seen as partial differentiation; if $u(\underline{x}) \in R$ and $F(\underline{y}) \in S$, then

$$u \cdot F = u(\partial/\partial y_1, \dots, \partial/\partial y_n) F.$$

If $I \subseteq R$ is a homogeneous ideal, then I^{-1} is the R-submodule of S defined as $\{F \in S : u \cdot F = 0 \text{ for all } u \in I\}$. This module (called Macaulay's inverse system for I) inherits a grading from S, so $I^{-1} = \bigoplus_i (I^{-1})_i$. Reciprocally, if $M \subseteq S$ is a graded submodule, then $\operatorname{ann}(M) = \{u : u \cdot F = 0 \text{ for all } F \in M\}$ is a homogeneous ideal in R. In classical terminology, if $u \cdot F = 0$ and $\deg u \leq \deg F$, then u and F are said to be *apolar* to each other.

Received December 14, 2001. Revision received March 19, 2002.

For any i, we have the Hilbert function $H(R/I, i) = \dim_k(R/I)_i = \dim_k(I^{-1})_i$. The following theorem is fundamental.

THEOREM 1.1 (Macaulay–Matlis duality). *We have a bijective correspondence*

$$\{\textit{homogeneous ideals } I < R\} \rightleftharpoons \{\textit{graded } R\textit{-submodules of } S\};$$

$$I \to I^{-1}, \qquad \operatorname{ann}(M) \leftarrow M.$$

Moreover, I^{-1} is a finitely generated R-module if and only if R/I is Artin.

Let $R/I = A$ be Artin with graded decomposition

$$A = k \oplus A_1 \oplus \cdots \oplus A_d, \quad A_d \neq 0, \ A_i = 0 \text{ for } i > d.$$

Recall that

$$\operatorname{socle}(A) = \{u \in A : ux_i = 0 \text{ for every } i\}.$$

Then $A_d \subseteq \operatorname{socle}(A)$, and A is said to be *level* if equality holds. This is true iff I^{-1} is generated as an R-module by exactly $t := \dim A_d$ elements in S_d. When A is level, the number t is called the *type* of A, and it coincides with its Cohen–Macaulay type. Thus A is Gorenstein iff $t = 1$. The number d is the *socle degree* of A. Altogether we have a bijection

$$\{A : A = R/I \text{ Artin level of type } t \text{ and socle degree } d\} \rightleftharpoons G(t, S_d);$$

$$A \to (I^{-1})_d, \qquad R/\operatorname{ann}(\Lambda) \leftarrow \Lambda.$$

Here $G(t, S_d)$ denotes the Grassmannian of t-dimensional vector subspaces of S_d. Notice the canonical isomorphism

$$G(t, S_d) \simeq G(\dim R_d - t, R_d) \tag{1}$$

taking Λ to I_d. We sometimes write Λ_i for $(I^{-1})_i$ and Λ for $\Lambda_d = (I^{-1})_d$.

REMARK 1.2. The algebra A is level iff Λ_d generates I^{-1} as an R-module—that is, iff the internal product map

$$\alpha_i \colon R_{d-i} \otimes \Lambda_d \to \Lambda_i$$

is surjective for all $i \leq d$. This is so iff the dual map

$$\beta_i \colon (R/I)_i \to S_{d-i} \otimes (R/I)_d$$

is injective for all i. This map can be written as

$$\beta_i \colon u \to \sum_M \underline{y}^M \otimes u\underline{x}^M, \tag{2}$$

the sum quantified over all monomials $\underline{x}^M$ of degree $d - i$.

As a consequence, if R/I is level then the graded piece I_d determines I by the following recipe: $I_i = \{u \in R_i : u \cdot R_{d-i} \subseteq I_d\}$ for $i \leq d$, and $I_i = R_i$ for $i > d$. In the terminology of [12] (a related terminology was originally introduced by A. Iarrobino), I is the *ancestor ideal* of the vector space I_d.

REMARK 1.3. We can detect whether R/I is level from the last syzygy module in its minimal resolution. Indeed, let

$$0 \to P_n \to \cdots \to P_0 \to R/I \to 0 \qquad (\dagger)$$

(where $P_0 = R$) be the graded minimal free resolution of R/I, and let

$$0 \to R(-n) \to R(-n+1)^n \to \cdots \wedge^i(R(-1)^n) \to \cdots \to R \to k \to 0 \quad (\dagger\dagger)$$

be the Koszul resolution of k. We will calculate the graded R-module $N = \mathrm{Tor}_n^R(R/I, k)$ in two ways. If we tensor (†) with k, then all differentials are zero and hence $N = P_n \otimes k$. When we tensor (††) with R/I, the kernel in the left-most place is $N = \mathrm{socle}(A)(-n)$. Hence A is level of socle degree d and type t iff $\mathrm{socle}(A) \simeq k(-d)^t$, iff $P_n \simeq R(-d-n)^t$.

Henceforth $A = R/I$ always denotes a level algebra of type t and socle degree d, loosely said to be of type (t, d). Let $\mathcal{B} \subseteq S_d \otimes \mathcal{O}_G$ denote the tautological bundle on $G(t, S_d)$; thus its fiber over a point $\Lambda \in G$ is the subspace Λ. The internal products give vector bundle maps

$$\varphi_i\colon R_{d-i} \otimes \mathcal{B} \to S_i \otimes \mathcal{O}_G, \quad 1 \le i \le d. \qquad (3)$$

Dually, there are maps

$$\varphi_i^*\colon R_i \otimes \mathcal{O}_G \to S_{d-i} \otimes \mathcal{B}^*.$$

Now $\mathcal{B}^*$ is the universal quotient bundle of $G(\dim R_d - t, \dim R_d)$ via (1), so its fiber over the point I_d is the subspace R_d/I_d.

1.1. Definition of Level Subschemes

We fix (t, d) and let $G = G(t, S_d)$. The Hilbert function of A is given by

$$H(A, i) = \dim(R_i/I_i) = \dim \Lambda_i.$$

This motivates the following definition.

For integers i and r, let $\mathcal{L}(i, r)$ be the closed subscheme of G defined by the condition $\{\mathrm{rank}(\varphi_i) \le r\}$. (Locally it is defined by the vanishing of $(r+1)$-minors of the matrix representing φ_i.) Let $\mathcal{L}^\circ(i, r)$ be the locally closed subscheme $\mathcal{L}(i, r) \setminus \mathcal{L}(i, r-1)$. Thus A represents a closed point of $\mathcal{L}(i, r)$ (resp. $\mathcal{L}^\circ(i, r)$) whenever $H(A, i) \le r$ (resp. $H(A, i) = r$).

Let $\underline{h} = (h_0, h_1, h_2, \ldots)$ be a sequence of nonnegative integers such that $h_0 = 1$, $h_d = t$, and $h_i = 0$ for $i > d$. (It is a useful convention that $h_i = 0$ for $i < 0$.) Define scheme-theoretic intersections

$$\mathcal{L}(\underline{h}) = \bigcap_{i=1}^{d-1} \mathcal{L}(i, h_i), \qquad \mathcal{L}^\circ(\underline{h}) = \bigcap_{i=1}^{d-1} \mathcal{L}^\circ(i, h_i).$$

These are, respectively, closed and locally closed subschemes of $G(t, S_d)$. Via the identification in (1), we will occasionally think of them as subschemes of $G(\dim R_d - t, R_d)$. The point $A = R/I$ lies in $\mathcal{L}(\underline{h})$ (resp. $\mathcal{L}^\circ(\underline{h})$) iff $\dim_k A_i \le h_i$ (resp. $\dim_k A_i = h_i$) for all i.

Of course either of the schemes may be empty, and it is in general an open problem to characterize those $\underline{h}$ for which they are not. For $n = 2$, such a characterization is given in Theorem 3.1. If $\mathcal{L}^\circ(\underline{h})$ is nonempty then we will say that $\underline{h}$ is a *level Hilbert function*.

1.2. The Structure of $\mathcal{L}(\underline{h})$ for $t = 1$, $n = 2$

The structure of the parameter spaces for Gorenstein quotients of $R = k[x_1, x_2]$ is rather well understood and provides a useful paradigm for our study of Artin level quotients of R having type > 1. An outline of this story is given below; see [12] for details.

It is easy to show that $A = R/I$ is a graded Gorenstein Artin algebra iff I is a complete intersection. Thus $I = (u_1, u_2)$, where u_1, u_2 are homogeneous and $\deg u_1 = a \le b = \deg u_2$. In this case $d = a + b - 2$, and the Hilbert function of A is

$$H(i) = \begin{cases} i+1 & \text{for } 0 \le i \le a-1, \\ a & \text{for } a \le i \le b-1, \\ a+b-(i+1) & \text{for } b \le i \le d; \end{cases}$$

in particular, it is centrally symmetric. We will denote this function by $\underline{h}_a$. It follows that, for socle degree d, there are precisely

$$\ell = \begin{cases} (d+2)/2 & \text{if } d \text{ is even,} \\ (d+1)/2 & \text{if } d \text{ is odd,} \end{cases}$$

possible Hilbert functions for Gorenstein Artin quotients of R. The collection $\{\underline{h}_a\}$ is totally ordered, that is, $h_a(j) \le h_{a+1}(j)$ for all $0 \le j \le d$ and $1 \le a \le \ell$. For brevity, let $\mathcal{L}_a$ denote the scheme $\mathcal{L}(\underline{h}_a) \subseteq \mathbf{P}S_d$.

In fact, $\mathcal{L}_a^\circ$ is the locus of *power sums* of length a; that is,

$$\mathcal{L}_a^\circ = \{F \in \mathbf{P}S_d : F = L_1^d + \cdots + L_a^d \text{ for some } L_i \text{ in } S_1\},$$

with $\mathcal{L}_a$ its Zariski closure. Thus $\mathcal{L}_1$ can be identified with the rational normal curve in $\mathbf{P}S_d$, and $\mathcal{L}_a$ is the union of (possibly degenerate) secant $(a-1)$-planes to $\mathcal{L}_1$. In particular, $\dim \mathcal{L}_a = 2a - 1$.

Let $z_i = x_1^{d-i} x_2^i$; then $\mathrm{Sym}^\bullet R_d = k[z_0, \dots, z_d]$ is the coordinate ring of $\mathbf{P}S_d$. Consider the Hankel matrix

$$\mathcal{C}_a := \begin{bmatrix} z_0 & z_1 & \cdots & z_{d-a} \\ z_1 & z_2 & \cdots & z_{d-a+1} \\ \vdots & \vdots & & \vdots \\ z_a & z_{\ell+1} & \cdots & z_d \end{bmatrix},$$

and let $\wp_a$ denote the ideal of its maximal minors. Then it is a theorem of Gruson and Peskine (see [14]) that $\wp_a$ is perfect, prime, and equal to the ideal of $\mathcal{L}_a$ in $k[z_0, \dots, z_d]$. Now the Eagon–Northcott theorem implies that $\mathcal{L}_a \subseteq \mathbf{P}S_d$ is an arithmetically Cohen–Macaulay variety.

1.3. Summary of Results

In the next section, we derive an expression for the tangent space to a point of $\mathcal{L}^\circ(\underline{h})$. This is a direct generalization of [17, Thm. 3.9] to the non-Gorenstein case. For Sections 3 and 4, we assume $n = 2$. In Section 3, we give a geometric description of a point of $\mathcal{L}^\circ(\underline{h})$ in terms of secant planes to the rational normal curve, which generalizes the one just given for $t = 1$. We relate this description to Waring's problem for systems of algebraic forms and solve the problem for $n = 2$. In the last section we prove a projective normality theorem for a class of schemes $\mathcal{L}(i, r)$ using spectral sequence techniques. The results in the following three sections are largely independent of each other, so they may be read separately.

We thank the referee for several helpful suggestions, and specifically for contributing Corollary 3.3. We owe the result of Theorem 3.1 to G. Valla. We also acknowledge the help of John Stembridge's Maple package "SF" for some calculations in Section 3.4.

2. Tangent Spaces to Level Subschemes

Let $R = k[x_1, \ldots, x_n]$ and let $A = R/I$ be an Artin level quotient of type (t, d). Given a degree-0 morphism $\psi : I \to R/I$ of graded R-modules, we have induced maps of k-vector spaces $\psi_i : I_i \to (R/I)_i$. We claim that ψ_d entirely determines ψ. Indeed, let $u \in I_i$ and $\underline{x}^M$ be a monomial of degree $d - i$. Then $\psi_d(u\underline{x}^M) = \psi_i(u)\underline{x}^M$. But then $\beta_i(\psi_i(u)) = \sum_M \underline{y}^M \otimes \psi_d(u\underline{x}^M)$. Since β_i is injective, this determines $\psi_i(u)$ uniquely. Thus we have an inclusion

$$\mathrm{Hom}_R(I, R/I)_0 \hookrightarrow \mathrm{Hom}_k(I_d, R_d/I_d), \qquad \psi \to \psi_d. \tag{4}$$

We also have a parallel inclusion

$$\mathrm{Hom}_R(I^{-1}, S/I^{-1})_0 \hookrightarrow \mathrm{Hom}_k(\Lambda, S_d/\Lambda). \tag{5}$$

Recall that if U is a vector space and W an m-dimensional subspace, then the tangent space to $G(m, U)$ at W (denoted $T_{G,W}$) is canonically isomorphic to $\mathrm{Hom}(W, U/W)$. Thus

$$T_{G(t,S_d),\Lambda} = \mathrm{Hom}_k(\Lambda, S_d/\Lambda), \qquad T_{G(\dim R_d - t, R_d), I_d} = \mathrm{Hom}_k(I_d, R_d/I_d).$$

THEOREM 2.1. *Let $A = R/I$ be as before with Hilbert function $\underline{h}$ and inclusions* (4) *and* (5).

(A) *Regarding $\mathcal{L}^\circ(\underline{h})$ as a subscheme of $G(t, S_d)$, we have a canonical isomorphism*

$$T_{\mathcal{L}^\circ(\underline{h}),\Lambda} = \mathrm{Hom}_R(I^{-1}, S/I^{-1})_0 = \mathrm{Hom}_R(I^{-1}, (I^2)^{-1}/I^{-1})_0.$$

(B) *Regarding $\mathcal{L}^\circ(\underline{h})$ as a subscheme of $G(\dim R_d - t, R_d)$, we have a canonical isomorphism*

$$T_{\mathcal{L}^\circ(\underline{h}), I_d} = \mathrm{Hom}_R(I, R/I)_0 = \mathrm{Hom}_R(I/I^2, R/I)_0.$$

Proof. We begin by recalling the relevant result about the tangent space to a generic determinantal variety (see [1, Chap. 2]).

Let $M = M(p, q)$ denote the space of all $p \times q$ matrices over $\mathbf{C}$ or (equivalently) the space of vector space maps $\mathbf{C}^p \to \mathbf{C}^q$. Since M is an affine space, it follows for any $X \in M$ that the tangent space $T_{M,X}$ can be canonically identified with M. Fix an integer $r \le \min\{p, q\}$ and let M_r be the subvariety of matrices with rank $\le r$. If $X \in M_r \setminus M_{r-1}$, then X is a smooth point of M_r and

$$T_{M_r,X} = \{Y \in M : Y(\ker X) \subseteq \text{image } X\}.$$

Now if $\Lambda \in \mathcal{L}^\circ(i, r)$, then φ_i is represented in a neighborhood $U \subseteq G(t, S_d)$ of Λ by a matrix of size $t(d - i + 1) \times (i + 1)$, whose entries are regular functions on U. Writing $M = M(t(d - i + 1), i + 1)$, these functions define a morphism $f : U \to M$. Thus the following is a fiber square:

$$\begin{tikzcd} U \cap \mathcal{L}^\circ(i, r) \arrow[r] \arrow[d] & U \arrow[d, "f"] \\ M_r \arrow[r, "i"'] & M . \end{tikzcd}$$

Hence

$$T_{\mathcal{L}^\circ(i,r),\Lambda} = \{\tau \in T_{U,\Lambda} = \mathrm{Hom}(\Lambda, S_d/\Lambda) : df(\tau) \in T_{M_r, f(\Lambda)}\}.$$

(Here df denotes the induced map on tangent spaces.)

This expression may be translated into the statement that $T_{\mathcal{L}^\circ(i,r),\Lambda}$ consists of all $\tau \in \mathrm{Hom}_k(\Lambda, S_d/\Lambda)$ such that the broken arrow in the following diagram is zero:

$$\begin{tikzcd} \ker\alpha_i \arrow[r] \arrow[rrd, dashed] & R_{d-i} \otimes \Lambda \arrow[r, "\mathrm{id}\otimes\tau"] & R_{d-i} \otimes S_d/\Lambda \arrow[d, "\mu"] \\ & & S_i/\Lambda_i . \end{tikzcd}$$

The map μ comes from the internal product in an obvious way. This implies that $\tau \in T_{\mathcal{L}^\circ(i,r),\Lambda}$ iff the composite $\mu \circ (\mathrm{id} \otimes \tau)$ factors through image $\alpha_i = \Lambda_i$. Let $\tau_i : \Lambda_i \to S_i/\Lambda_i$ denote the induced map. Now

$$T_{\mathcal{L}^\circ(\underline{h}),\Lambda} = \bigcap_i T_{\mathcal{L}^\circ(i,h_i),\Lambda},$$

hence $\tau \in T_{\mathcal{L}^\circ(\underline{h}),\Lambda}$ iff it defines a sequence (τ_i) as just described that glues to give an R-module map $I^{-1} \to S/I^{-1}$. This proves (A).

For (B), a parallel argument leads to this: an element $\omega \in \mathrm{Hom}_k(I_d, R_d/I_d)$ belongs to $T_{\mathcal{L}^\circ(i,r),I_d}$ iff, in the following diagram, the broken arrow can be filled in:

$$\begin{array}{ccc} I_d \otimes S_{d-i} & \xrightarrow{\omega\otimes\mathrm{id}} & R_d/I_d \otimes S_{d-i} \\ \uparrow & & \uparrow \\ I_i & \dashrightarrow & R_i/I_i\,. \end{array}$$

Here both vertical maps are given by formula (2); in particular, they are injective. Hence the broken arrow is unique if it exists, which we then denote by ω_i. Thus $\omega \in T_{\mathcal{L}^\circ(\underline{h}),I_d}$ iff it defines a sequence (ω_i) as before that glues to give an R-module map $I \longrightarrow R/I$. This proves the theorem. □

REMARK 2.2. The scheme GradAlg($\underline{h}$) (defined by Kleppe [18]) parametrizes graded quotients of R (level or not) with Hilbert function $\underline{h}$. Its tangent space at the point R/I is also canonically isomorphic to $\mathrm{Hom}(I, R/I)_0$. See Remarks 3.10 and 4.3 in [17] for a more detailed comparison of these two spaces (in the Gorenstein case).

3. Level Algebras in Codimension 2

In this section (and the next) we consider quotients of $R = k[x_1, x_2]$.

3.1. Preliminaries

Let $A = R/I$ be an Artin level algebra with Hilbert function H, type t, and socle degree d. By Remark 1.3, we have a resolution

$$0 \to R^t(-d-2) \to \bigoplus_{\ell=1}^{d+1} R^{e_\ell}(-\ell) \to R \to R/I \to 0. \quad (6)$$

Here e_ℓ is the number of minimal generators of I in degree ℓ, and $\sum e_\ell = t+1$. Hence

$$H(A,i) = (i+1) - \sum_{\ell=1}^{i} e_\ell(i-\ell+1) \quad \text{for all } \ i \le d+1. \quad (7)$$

With a little manipulation, this implies

$$e_{i+1} = 2H(A,i) - H(A,i-1) - H(A,i+1) \quad \text{for } \ 0 \le i \le d. \quad (8)$$

Hence the sequence (e_i) can be recovered from the Hilbert function. Applying the functor $\mathrm{Hom}_R(-, R/I)$ to the resolution of I, we have an exact sequence

$$0 \to \mathrm{Hom}(I, R/I) \to \bigoplus_{\ell=1}^{d+1} (R/I)^{e_\ell}(\ell) \to (R/I)^t(d+2)$$

and so

$$\dim_k \mathrm{Hom}_R(I, R/I)_0 = \sum_{\ell=1}^{d} e_\ell H(A,\ell). \quad (9)$$

The next result characterizes the level Hilbert functions of type (t, d) in codimension 2. It is due to G. Valla, who had kindly communicated its proof to the second author several years ago. A more general version (that covers codimension two nonlevel algebras) is stated by A. Iarrobino in [16, Thm. 4.6A].

THEOREM 3.1 (Iarrobino, Valla). *Let $\underline{h} = (h_0, h_1, \dots)$ be a sequence of nonnegative integers satisfying $h_0 = 1$, $h_d = t$, and $h_i = 0$ for $i > d$. Then $\mathcal{L}^\circ(\underline{h})$ is nonempty if and only if*

$$2h_i \geq h_{i-1} + h_{i+1} \quad \textit{for all } \ 0 \leq i \leq d.$$

(*By convention, $h_i = 0$ for $i < 0$.*)

Proof. The "only if" part follows from (8). Assume that $\underline{h}$ satisfies the hypotheses. Then we inductively deduce $h_i \leq i + 1$. Define $e_i = 2h_{i-1} - (h_{i-2} + h_i)$ for $1 \leq i \leq d + 1$ and $e_i = 0$ elsewhere. Then $\sum_{i=1}^{d+1} e_i = h_d + h_0 = t + 1$. Define a sequence of integers

$$\underline{q} : q_1 \leq q_2 \leq \cdots \leq q_{t+1}$$

such that, for $1 \leq i \leq d + 1$, the integer i occurs e_i times. An easy calculation shows that $\sum q_i = \sum i \cdot e_i = t(d + 2)$.

Let M be the $t \times (t + 1)$ matrix whose only nonzero entries are $M_{i,i} = x_1^{d+2-q_i}$ and $M_{i,i+1} = x_2^{d+2-q_{i+1}}$ for $1 \leq i \leq t$, and let I be the ideal of its maximal minors. Since I is (x_1, x_2)-primary, it has depth 2. The $t + 1$ maximal minors of M are nonzero, and they have degrees $q_1, \dots, q_{t+1}$. By the Hilbert–Burch theorem, R/I has a resolution with Betti numbers as in (6). Then, by Remark 1.3, the point $A = R/I$ lies in $\mathcal{L}^\circ(\underline{h})$. □

EXAMPLE 3.2. Let $(t, d) = (3, 7)$ and $\underline{h} = (1, 2, 3, 4, 5, 5, 4, 3, 0)$. Then $e_5 = e_6 = 1$, $e_8 = 2$, and $\underline{q} = (5, 6, 8, 8)$. Hence

$$M = \begin{bmatrix} x_1^4 & x_2^3 & 0 & 0 \\ 0 & x_1^3 & x_2 & 0 \\ 0 & 0 & x_1 & x_2 \end{bmatrix}$$

and $I = (x_2^5, x_1^4 x_2^2, x_1^7 x_2, x_1^8)$.

We owe the following observation to the referee.

COROLLARY 3.3. *The level Hilbert functions $\underline{h}$ of type (t, d) are in bijection with partitions of $d - t + 1$ with no part exceeding $t + 1$.*

Proof. Given $\underline{h}$, define $\mu_i = h_i - h_{i+1} + 1$ for $0 \leq i \leq d - 1$. Then $\underline{\mu} = (\mu_{d-1}, \dots, \mu_1, \mu_0)$ is a partition as described in the corollary. Conversely, given such a partition, we append zeros to make its length equal to d and then determine h_i recursively. □

REMARK 3.4. If $\underline{h}$ is a level Hilbert function, then $\mathcal{L}^\circ(\underline{h})$ is an irreducible and smooth variety. Indeed, the scheme GradAlg$(\underline{h})$ is irreducible and smooth by a result of Iarrobino [15, Thm. 2.9], and $\mathcal{L}^\circ(\underline{h}) \subseteq \mathrm{GradAlg}(\underline{h})$ is a dense open subset. Hence, from (9),

$$\dim \mathcal{L}^\circ(\underline{h}) = \sum_{i=1}^{d} e_i h_i = \sum_{i=1}^{d} h_i(2h_{i-1} - h_i - h_{i-2}). \tag{10}$$

For instance, we have $\dim \mathcal{L}^\circ(\underline{h}) = 9$ in Example 3.2.

3.2. Geometric Description of Points in $\mathcal{L}^\circ(\underline{h})$

We start with an example to illustrate the description we have in mind. We need the following classical lemma (see [17, pp. 23–25; 19]).

Lemma 3.5 (Jordan). *Let $u \in R_m$ be a form factoring as*

$$\prod_i (a_i x_1 + b_i x_2)^{\mu_i}, \quad \text{so that} \quad \sum \mu_i = m.$$

If $n \geq m$, then $(u)_n^{-1}$ (the subspace of forms in S_n that are apolar to u) equals

$$\sum_i S_{\mu_i - 1}(b_i y_1 - a_i y_2)^{n-\mu_i+1} = \left\{ \sum f_i (b_i y_1 - a_i y_2)^{n-\mu_i+1} : f_i \in S_{\mu_i-1} \right\}.$$

In particular, this is an m-dimensional vector space.

Example 3.6. Let $(t, d) = (2, 6)$ and consider the level Hilbert function

$$\underline{h} = (1, 2, 3, 4, 4, 3, 2, 0).$$

Then $\Lambda \in \mathcal{L}^\circ(\underline{h})$ defines a line $\mathbf{P}\Lambda$ in $\mathbf{P}^6$ $(= \mathbf{P}S_6)$. We identify the subset $C_6 = \{[L^6] \in \mathbf{P}S_6 : L \in S_1\}$ as the rational normal sextic in $\mathbf{P}S_6$.

By formula (8), $I = \mathrm{ann}(\Lambda)$ has one minimal generator each in degrees 4, 5, 7. Let $u_4 \in R_4$ be the first generator, factoring as $u_4 = \prod_{i=1}^4 (a_i x_1 + b_i x_2)$. For simplicity, assume that $[a_1, b_1], \ldots, [a_4, b_4]$ are distinct points in $\mathbf{P}^1$. Then, by Jordan's lemma, the subspace $(u_4)_6^{-1} \subseteq S_6$ is the span of $(b_i y_1 - a_i y_2)^6$ for $1 \leq i \leq 4$. Let Π_4 denote the projectivization $\mathbf{P}(u_4)_6^{-1}$, which is the secant 3-plane to C_6 spanned by the four points $[b_i, -a_i]$. Consider generators u_5, u_7 and define Π_5, Π_7 analogously. (Of course, $\Pi_7 = \mathbf{P}S_6$.) Now

$$(I^{-1})_6 = ((u_4, u_5, u_7)^{-1})_6 \implies \mathbf{P}\Lambda = \Pi_4 \cap \Pi_5 \cap \Pi_7.$$

Thus (the line corresponding to) every element $\Lambda \in \mathcal{L}^\circ(\underline{h})$ is representable as an intersection of secant planes to the rational normal curve, in a way that depends only on the combinatorics of $\underline{h}$. If (say) u_4 has multiple roots, then Π_4 is tangent to the curve at one or more points and so must be counted as a degenerate secant plane.

Definition 3.7. Let $C_d = \{[L^d] : L \in S_1\}$ be the rational normal curve in $\mathbf{P}S_d$. A linear subspace $\Pi \subseteq \mathbf{P}S_d$ of (projective) dimension s will be called a *secant s-plane to C_d* if the scheme-theoretic intersection $C_d \cap \Pi$ has length $\geq s + 1$. (Then the length must equal $s + 1$, essentially by Jordan's lemma.)

Now, for an arbitrary level algebra in codimension 2, we have the following description.

PROPOSITION 3.8. *Let $\underline{h}$ be a level Hilbert function and $\Lambda \in \mathcal{L}^\circ(\underline{h})$. Define a sequence $\underline{q}$ as in the proof of Theorem 3.1. Then there exist secant $(q_i - 1)$-planes Π_{q_i} such that*

$$\mathbf{P}\Lambda = \Pi_{q_1} \cap \cdots \cap \Pi_{q_{t+1}}. \tag{11}$$

Proof. The essential point already occurs in Example 3.6. Let q be one of the q_i and $u \in I_q$ a generator. Write

$$u = \prod_i (a_i x_1 + b_i x_2)^{\mu_i}.$$

Let Φ_i be the osculating $(\mu_i - 1)$-plane to C_d at the point $(b_i y_1 - a_i y_2)^d$. (This is the point itself if $\mu_i = 1$.) Algebraically, Φ_i is the projectivization $\mathbf{P}(S_{\mu_i - 1}(b_i y_1 - a_i y_2)^{d-\mu_i+1})$.

Let Π_q be the linear span of all the Φ_i. Then $\dim \Pi_q = q - 1$ and by Jordan's lemma, Π_q is the locus of forms $F \in S_d$ that are apolar to u. Construct such a plane Π_{q_i} for each q_i. A form F lies in Λ iff it is apolar to each generator of I, iff it belongs to $\bigcap \Pi_{q_i}$. The proposition is proved. □

REMARK 3.9. The argument depends heavily on the fact that any zero-dimensional subscheme of C_d is in linearly general position. This property characterizes rational normal curves (see [13, p. 270]).

The preceding proposition admits a converse. Consider the following example.

EXAMPLE 3.10. Let $\mathbf{P}\Lambda \subseteq \mathbf{P}S_{11}$ $(= \mathbf{P}^{11})$ be a line appearing as an intersection

$$\mathbf{P}\Lambda = \Pi_8 \cap \Pi'_8 \cap \Pi_{10},$$

where each Π_q is a secant $(q-1)$-plane to C_{11}. Note that the planes intersect properly (i.e., in the expected codimension). We claim that Λ belongs to $\mathcal{L}^\circ(\underline{h})$ for

$$\underline{h} = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 7\ 6\ 4\ 2\ 0.$$

Let $W_8 \subseteq S_{11}$ be the subspace such that $\Pi_8 = \mathbf{P}W_8, \ldots$. Assume Π_8 intersects C_{11} at points $(b_i y_1 - a_i y_2)^{11}$ with respective multiplicities μ_i $\big($so that $\sum \mu_i = 8\big)$. Let $u_8 \in R_8$ be the element $\prod(a_i x_1 + b_i x_2)^{\mu_i}$; then $\operatorname{ann}(W_8)$ is the principal ideal (u_8). Define elements u'_8, u_{10} similarly. Now

$$I = \operatorname{ann}(\Lambda) = \operatorname{ann}(W_8) + \operatorname{ann}(W'_8) + \operatorname{ann}(W_{10}) = (u_8, u'_8, u_{10}).$$

Since (by construction) R/I is a level algebra of type two, it follows that I has three minimal generators. Hence their degrees must be $(8, 8, 10)$ and then the Hilbert function of R/I is determined by formula (8). This completes the argument.

PROPOSITION 3.11. *Let $\mathbf{P}\Lambda \subseteq \mathbf{P}S_d$ be a $(t-1)$-dimensional subspace that is expressible as an intersection*

$$\mathbf{P}\Lambda = \Pi_{q_1} \cap \cdots \cap \Pi_{q_{t+1}},$$

where:

(i) Π_{q_i} *is a secant* $(q_i - 1)$*-plane to* C_d; *and*
(ii) *the intersection is proper—that is,*

$$\sum_i \operatorname{codim}(\Pi_{q_i}, \mathbf{P}^d) = \operatorname{codim}(\mathbf{P}\Lambda, \mathbf{P}^d) = d - t + 1.$$

Then $R/\operatorname{ann}(\Lambda) \in \mathcal{L}^\circ(\underline{h})$, *where* $h_0 = 1$ *and* $h_i = 2h_{i-1} - h_{i-2} - e_i$ *for* $1 \le i \le d$. *The* e_i *are defined by arranging the* q_i *as in Theorem 3.1.*

Proof. The proof is left to the reader. □

Propositions 3.8 and 3.11 are natural generalizations of the description of $\mathcal{L}(\underline{h})$ in the Gorenstein case.

EXAMPLE 3.12. The minimal Hilbert function of type (t, d) is $\underline{h} = (1, 2, \ldots, t-1, t, \ldots, t, 0)$, and then $\mathcal{L}^\circ(\underline{h}) = \mathcal{L}(\underline{h})$ is the variety of secant $(t-1)$-planes to C_d. Abstractly, $\mathcal{L}(\underline{h}) \simeq \operatorname{Sym}^t \mathbf{P}^1 \simeq \mathbf{P}^t$. See [2] for a description of the minimal level Hilbert function when $n > 2$.

The maximal function is $h_i = \min\{i+1, (d-i+1)t\}$ for $1 \le i \le d$. If we let

$$s_0 = \left\lceil \frac{t(d+1)}{t+1} \right\rceil,$$

then $h_{s_0} < s_0 + 1$. Hence every $\Lambda \in G(t, S_d)$ has an apolar form of degree $\le s_0$.

Using this formulation, the so-called Waring's problem can be solved completely for binary forms.

3.3. *Waring's Problem for Several Binary Forms*

We will start with an informal account. Given binary forms $F_1, \ldots, F_t$ of degree d, we would like to find linear forms $L_1, \ldots, L_s$ such that

$$F_i = c_{i1} L_1^d + \cdots + c_{is} L_s^d \tag{12}$$

for some $c_{ij} \in k$. This is always possible for $s = d+1$; indeed, if we choose $L_1, \ldots, L_{d+1}$ generally then the L_i^d span S_d.

The "simultaneous" Waring's problem (in one of its versions) is to find the smallest s that suffices for a *general* choice of F_i. (See Bronowski [3] for a discussion of n-ary forms.) Here we consider a more general version: we fix s and consider the locus Σ of forms $\{F_1, \ldots, F_t\}$ that admit such a representation. In practice, we allow not only representations as here but also generalized additive decompositions (GADs) in the sense of [17, Def. 1.30].

DEFINITION 3.13. A finite collection $\underline{L} = \{L_i\} \subseteq S_1 \setminus \{0\}$ of linear forms will be called *admissible* if any two L_i, L_j are nonproportional.

DEFINITION 3.14. Let $\underline{L}$ be an admissible collection as just defined and let $F \in S_d$. A *GAD for* F *with respect to* $\underline{L}$ is a collection $\{p_i\}$ of forms such that

$F = \sum_i p_i L_i^{d-\alpha_i}$ with $\alpha_i = \deg p_i < d$. (We set $\deg 0 = -1$ by convention.) The integer $\ell_{\underline{\alpha}} = \sum_i (\alpha_i + 1)$ is called the *length* of the GAD.

If all $\alpha_i = 0$, then this reduces to expression (12). Given sequences $\underline{\alpha}$ and $\underline{L}$ such that $-1 \le \alpha_i < d$, define a subspace

$$W(\underline{\alpha}, \underline{L}) = \sum S_{\alpha_i} L^{d-\alpha_i} \subseteq S_d.$$

(By convention, $S_{-1} = 0$.) Write $L_i = b_i y_1 - a_i y_2$; then, by hypothesis, $[a_i, b_i]$ represent distinct points of $\mathbf{P}^1$.

LEMMA 3.15 [17, Sec. 1.3]. *With notation as above*:

(a) *A form lies in* $W(\underline{\alpha}, \underline{L})$ *iff it is apolar to* $\prod_i (a_i x_1 + b_i x_2)^{\alpha_i + 1}$. *In particular,* $\dim W(\underline{\alpha}, \underline{L}) = \sum_i (\alpha_i + 1)$.
(b) *Let* Φ_{i,α_i} *be the osculating* α_i*-plane* (*empty, by convention, if* $\alpha_i = -1$) *to* C_d *at the point* $L_i^d \in C_d$. *Then the linear span of all* $\{\Phi_{i,\alpha_i}\}_i$ *is the projectivization* $\mathbf{P}W(\underline{\alpha}, \underline{L})$.

This is merely a rephrasing of Jordan's lemma. Thus the subspaces $W(\underline{\alpha}, \underline{L})$ exactly correspond to secant $(\ell_{\underline{\alpha}} - 1)$-planes to C_d. Now let (d, t) be as before and fix an integer s such that $d + 1 > s \ge t$. Define

$$\Sigma_s = \{\Lambda \in G(t, S_d) : \mathbf{P}\Lambda \text{ lies on some secant } (s-1)\text{-plane of } C_d\}. \tag{13}$$

Algebraically, $\Lambda \in \Sigma_s$ iff there exists an admissible collection $\{L_i\}$ such that each $F \in \Lambda$ has a GAD of length $\le s$ with respect to $\{L_i\}$. There is no loss of generality in assuming that $\{L_i\}$ has cardinality s.

Now Waring's problem can be interpreted as one of calculating $\dim \Sigma_s$. Evidently, it is bounded by $\dim G(t, S_d) = t(d - t + 1)$. Let $U_s \subseteq \mathrm{Sym}^s(\mathbf{P}S_1)$ be the open subset of admissible collections $\underline{L} = \{L_1, \dots, L_s\}$, and consider the incidence correspondence

$$\tilde{\Sigma}_s = \{(\Lambda, \underline{L}) \in G(t, S_d) \times U_s : \text{each element of } \Lambda \text{ admits a GAD of length} \le s \text{ w.r.t. } \underline{L}\}.$$

Then Σ_s is the image of the projection $\pi_1 : \tilde{\Sigma}_s \to G(t, S_d)$.

LEMMA 3.16. *Each fibre of the projection* $\pi_2 : \tilde{\Sigma}_s \to U_s$ *is of dimension* $t(s-t)$, *so* $\dim \tilde{\Sigma}_s = s + t(s - t)$.

Proof. Each $\Lambda \in \pi_2^{-1}(\underline{L})$ is a subspace of the (finite) union

$$\bigcup_{\ell_{\underline{\alpha}} \le s} W(\underline{\alpha}, \underline{L}),$$

hence it is a subspace of one of the W. Now use the fact that $\dim G(t, W)$ equals $t(\dim W - t)$. $\square$

Thus we have a naïve estimate,

$$\dim \Sigma_s \le \min\{\underbrace{s + t(s-t)}_{N_1},\ \underbrace{t(d-t+1)}_{N_2}\}. \tag{14}$$

THEOREM 3.17. *In* (14), *we have an equality.*

Proof. Assume $N_1 < N_2$. We will exhibit a level Hilbert function $\underline{h}$ of type (t, d) such that $h_s = s$ and $\dim \mathcal{L}^\circ(\underline{h}) = N_1$. Then, by construction, for each $\Lambda \in \mathcal{L}^\circ(\underline{h})$ there exists a nonzero form in R_s that is apolar Λ. This form defines a secant $(s-1)$-plane containing $\mathbf{P}\Lambda$. Hence $\mathcal{L}^\circ(\underline{h}) \subseteq \Sigma_s$, which forces $\dim \Sigma_s = N_1$.

Let m be the unique integer such that $(m+1)t > s \ge mt$. Then $N_1 < N_2$ forces $s \le d - m$. Define a sequence $\underline{h}$ by

$$h_i = \begin{cases} i+1 & \text{for } 0 \le i \le s-1, \\ s & \text{for } s \le i \le d-m, \\ (d-i+1)t & \text{for } d-m+1 \le i \le d, \\ 0 & \text{for } i > d. \end{cases}$$

It is an immediate verification that $\underline{h}$ is level. Now

$$e_s = 1, \quad e_{d-m+1} = s - mt, \quad e_{d-m+2} = mt + t - s,$$

and $e_i = 0$ elsewhere, hence $\dim \mathcal{L}^\circ(\underline{h}) = N_1$.

For example, for $(t, d, s) = (3, 11, 7)$ we have

$$\underline{h} = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 7\ 7\ 7\ 6\ 3\ 0.$$

If $N_1 \ge N_2$, then $s \ge s_0$ (defined as in Example 3.12). But since every $\mathbf{P}\Lambda$ lies on an $(s_0 - 1)$-secant, $\Sigma_{s_0} = G(t, S_d)$ and we are done. $\square$

The theorem implies that, given *general* binary forms $F_1, \dots, F_t$, a reduction to the expression (12) is always possible provided we have a sufficient number of constants implicitly available on the right-hand side of (12). This is no longer so for $n \ge 3$, and the corresponding reduction problem is open. See [4] for one approach, where Theorem 3.17 is proved using a different method.

A length-s subscheme of C_d is called a polar s-hedron of Λ if Λ lies on the corresponding $(s-1)$-secant plane. We have shown that the variety of polar s-hedra of Λ is the projective space $\mathbf{P}(R_s/\mathrm{ann}(\Lambda)_s)$. For $n \ge 3$, the geometry of this variety is rather more mysterious—see [7; 22].

3.4. An Analogue of the Catalecticant

An interesting special case occurs when $N_1 = N_2 - 1$, that is, when Σ_s is a hypersurface in $G(t, S_d)$. This is possible iff

$$(t+1) \mid (d+2) \quad \text{and} \quad s = d + 1 - \frac{d+2}{t+1}.$$

Then Σ_s is set-theoretically equal to $\mathcal{L}(s,s) = \{\text{rank}(\mathcal{B} \otimes R_{d-s} \to S_s) \le s\}$. Now $\mathcal{L}(s,s)$ is the zero scheme of a global section of the line bundle

$$\wedge^{s+1}(\mathcal{B}^* \otimes S_{d-s}) = \mathcal{O}_G(d-s+1).$$

Hence the fundamental class $[\mathcal{L}(s,s)]$ equals $(d-s+1)c_1(\mathcal{B}^*) \in H^2(G,\mathbb{Z})$.

The preceding rank condition can be written as a determinant. For instance, let $t = 2$, $d = 7$, and $s = 5$, and let

$$F_1 = \sum_{i=0}^{7} a_i y_0^{7-i} y_1^i, \qquad F_2 = \sum_{i=0}^{7} b_i y_0^{7-i} y_1^i$$

be linearly independent forms. Then the pencil $\mathbf{P}\Lambda = \mathbf{P}(\text{span}(F_1, F_2))$ lies on a secant 4-plane to C_7 iff

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{vmatrix} = 0.$$

This is the analogue of the catalecticant for systems of binary forms.

EXAMPLE 3.18. Even if Σ_s is not a hypersurface, such determinantal conditions can always be written down. For example, let $t = 2$, $d = 5$, and $\dim G(2, S_5) = 8$. The possible level Hilbert functions are

$$\begin{aligned} \underline{h}_1 &: 1\ \ 2\ \ 2\ \ 2\ \ 2\ \ 2\ \ 0, \\ \underline{h}_2 &: 1\ \ 2\ \ 3\ \ 3\ \ 3\ \ 2\ \ 0, \\ \underline{h}_3 &: 1\ \ 2\ \ 3\ \ 4\ \ 3\ \ 2\ \ 0, \\ \underline{h}_4 &: 1\ \ 2\ \ 3\ \ 4\ \ 4\ \ 2\ \ 0, \end{aligned}$$

with $\dim \mathcal{L}^\circ(\underline{h}) = 2, 5, 6, 8$ respectively. Now $\Sigma_2 = \mathcal{L}(\underline{h}_1)$, which is set-theoretically equal to any of the schemes $\mathcal{L}(i, 2)$, $i = 2, 3, 4$ (as defined in Section 1.1). Similarly $\Sigma_3 = \mathcal{L}(\underline{h}_2)$, which is set-theoretically $\mathcal{L}(3,3)$. It is clear that we can write the condition for $\Lambda \in G$ to lie in Σ_2 (or Σ_3) as the vanishing of certain minors. Indeed, in general this can be done in more than one way.

We can calculate the classes of these schemes in $G(t, S_d)$ by the Porteous formula. We explain this briefly; see [10, Sec. 14.4] for the details.

Let $E \xrightarrow{\alpha} F$ be a morphism of vector bundles on a smooth projective variety. Assume that E and F have ranks e and f respectively and that the locus $X_r = \{\text{rank}\,\alpha \le r\}$ is of pure codimension $(e-r)(f-r)$ in the ambient variety. Then the fundamental class of X_r (which we denote by $[X_r]$) equals the $(e-r)\times(e-r)$ determinant whose (i, j)th entry equals the $(f-r+j-i)$th Chern class of the virtual bundle $F - E$. By the Whitney product formula, the total Chern class $c_t(F - E) = c_t(F)/c_t(E)$.

We will follow the conventions of [10, Sec. 14.7] for Schubert calculus. Thus the ith Chern class of the tautological bundle $\mathcal{B}$ is $(-1)^i\{1,\dots,1\}$, where 1 occurs i times. For $G(2, S_5)$, we have $c_t(\mathcal{B}) = 1 - \{1\} + \{1, 1\}$. A straightforward calculation (using the Maple package "SF") shows that

$$[\mathcal{L}(2,2)] = [\mathcal{L}(4,2)] = 10\{3,3\} + 6\{4,2\}.$$

The formula does not apply to $\mathcal{L}(3, 2)$, since it fails to satisfy the codimension hypothesis. By a similar calculation, $\mathcal{L}(3, 3) = 8\{2, 1\}$.

For any $I_5 \in \mathcal{L}(\underline{h}_1)$, a map in $\mathrm{Hom}_R(I, R/I)$ is entirely determined by the image of the unique generator in I_2. Now the proof of Theorem 2.1 shows that the space $T_{\mathcal{L}(2,2), I_5}$ must be 2-dimensional, which implies $\mathcal{L}(2, 2) = \mathcal{L}(\underline{h}_1)$. By the Littlewood–Richardson rule,

$$[\mathcal{L}(\underline{h}_1)] \cdot \{1,1\} = 10\{4,4\};$$

that is, a general hyperplane $\Omega \subseteq \mathbf{P}S_5$ contains ten secant lines of the rational normal quintic C_5. Of course, these are the pairwise joins of the five points $\Omega \cap C_5$. Similarly,

$$[\mathcal{L}(\underline{h}_1)] \cdot \{2,0\} = 6\{4,4\};$$

that is, there are six secant lines to C_5 touching a general 2-plane Ψ. This can be seen differently: the projection from Ψ maps C_5 onto a rational nodal quintic in $\mathbf{P}^2$, and the six secants give rise to the six nodes of the image.

4. Free Resolutions of Level Subschemes

We continue to assume $n = 2$. Since the scheme $\mathcal{L}(i, r)$ is a degeneracy locus in the sense of [1, Chap. 2], we can describe its minimal resolution following Lascoux [20] provided it has the "correct" codimension in $G(t, S_d)$.

This granted, in the presence of an additional numerical hypothesis (to be explained shortly) we can deduce that it is arithmetically Cohen–Macaulay in the Plücker embedding. In particular, we obtain another proof of the known fact that $\mathcal{L}(i, r)$ is always ACM in the Gorenstein case. In the sequel, we need the Borel–Weil–Bott theorem for calculating the cohomology of homogeneous vector bundles on Grassmannians. We refer to [5] for an explanation of the combinatorics involved, but see also [21, p. 687].

Recall the definition of $\mathcal{L}(i, r)$ given in Section 1.1. To avoid trivialities, we assume $r < i + 1$ throughout the section. From [1, Chap. 2] we have the following estimate: if c is the codimension of any component of $\mathcal{L}(i, r)$ in $G(t, S_d)$, then

$$\begin{aligned} c &\le (\mathrm{rank}(\mathcal{B} \otimes R_{d-i}) - r)(\mathrm{rank}\, S_i - r) \\ &= (t(d-i+1) - r)(i+1-r). \end{aligned} \tag{15}$$

Consider the following conditions:

(C1) the scheme $\mathcal{L}(i, r)$ is equidimensional and equality holds in (15) for each component;

(C2) $r - (d-i)(i-r) \ge t$.

We will impose conditions (C1) and (C2) on the data (t, d, i, r). The next result shows the rationale behind (C2) as well as its scope of validity.

LEMMA 4.1. *The condition* (C2) *holds iff there is a level algebra A of type* (t, d) *such that $H(A, i-1) = i$ and $H(A, i) \le r$.*

Proof. Given the existence of A, we have

$$r - t = H(A, i) - H(A, d) = \sum_{j=i}^{d-1} H(A, j) - H(A, j+1)$$
$$\ge (d-i)(H(A, i-1) - H(A, i)) \ge (d-i)(i-r),$$

where the first inequality follows from Theorem 3.1. Conversely, assume (C2) and define

$$h_j = \begin{cases} j+1 & \text{for } 0 \le j \le i-1, \\ \min\{r - (j-i)(i-r), t(d-j+1)\} & \text{for } i \le j \le d, \\ 0 & \text{for } j > d. \end{cases}$$

Then $\underline{h}$ satisfies the hypotheses of Theorem 3.1, hence it is the Hilbert function of a level algebra A. Evidently, $h_{i-1} = i$ and $h_i \le r$. □

Now assume that $\mathcal{L} = \mathcal{L}(i, r)$ satisfies (C1) (but not necessarily (C2)) and let $c = \operatorname{codim} \mathcal{L}$. Then, by the central result of [20], we have a locally free resolution:

$$0 \to \mathcal{E}^{-c} \to \cdots \to \mathcal{E}^p \to \mathcal{E}^{p+1}$$
$$\to \cdots \to \mathcal{E}^0 \to \mathcal{O}_{\mathcal{L}} \to 0 \quad \text{for } -c \le p \le 0; \tag{16}$$

where $\mathcal{E}^0 = \mathcal{O}_G$ and

$$\mathcal{E}^p = \bigoplus_{\nu(\lambda') - |\lambda| = p} \mathbb{S}_\lambda(\mathcal{B} \otimes R_{d-i}) \otimes H^{\nu(\lambda')}(G', \mathbb{S}_{\lambda'} Q^*_{G'}). \tag{17}$$

This is to be read as follows: G' denotes the Grassmannian $G(r, S_i)$, and $Q_{G'}$ its universal quotient bundle. The λ denote partitions and $\mathbb{S}_\lambda$ the corresponding Schur functors (where we follow the indexing conventions of [11, Chap. 6]).

Given a partition λ, the Borel–Weil–Bott theorem implies that the bundle $\mathbb{S}_{\lambda'} Q^*_{G'}$ (resident on G') has nonzero cohomology in at most one dimension. This number (if it exists) is labeled $\nu(\lambda')$. The direct sum is quantified over all λ such that $\nu(\lambda')$ is defined. Since λ has at most $t(d-i+1)$ rows and $i-r+1$ columns, the sum is finite.

EXAMPLE 4.2. Let $(t, d, i, r) = (2, 7, 5, 4)$ and consider the level Hilbert functions

$$\underline{h}_1 : 1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\ 0,$$
$$\underline{h}_2 : 1\ 2\ 3\ 4\ 4\ 4\ 4\ 2\ 0.$$

Then $\mathcal{L} = \mathcal{L}(5, 4)$ is a union of two components $\mathcal{L}(\underline{h}_1)$, $\mathcal{L}(\underline{h}_2)$, each of codimension 4. Hence (C1) is satisfied and we have a resolution

$$0 \to \mathcal{E}^{-4} \to \cdots \to \mathcal{E}^{-1} \to \mathcal{O}_{G(2,S_7)} \to \mathcal{O}_{\mathcal{L}} \to 0,$$

where

$$\begin{aligned}
\mathcal{E}^{-1} &= \wedge^5(\mathcal{B} \otimes R_2) \otimes S_5,\\
\mathcal{E}^{-2} &= \wedge^6(\mathcal{B} \otimes R_2) \otimes \mathbb{S}_{(21111)}(S_5) \oplus \mathbb{S}_{(21111)}(B \otimes R_2),\\
\mathcal{E}^{-3} &= (\mathcal{B} \otimes R_2) \otimes \wedge^6(\mathcal{B} \otimes R_2) \otimes \wedge^5 S_5,\\
\mathcal{E}^{-4} &= [\wedge^6(\mathcal{B} \otimes R_2)]^{\otimes 2}.
\end{aligned}$$

The ranks of $\mathcal{E}^0, \ldots, \mathcal{E}^{-4}$ are $1, 36, 70, 36, 1$, hence $\mathcal{L}$ is a Gorenstein scheme. This resolution is equivariant with respect to the action of SL_2 on the embedding $\mathcal{L} \subseteq G$.

The term $\mathbb{S}_\lambda(\mathcal{B} \otimes R_{d-i})$ decomposes as a direct sum

$$\bigoplus_{\rho,\mu} (\mathbb{S}_\rho \mathcal{B} \otimes \mathbb{S}_\mu R_{d-i})^{C_{\lambda\rho\mu}}, \tag{18}$$

quantified over all partitions ρ, μ of $|\lambda|$. The coefficients $C_{\lambda\rho\mu}$ come from the Kronecker product of characters of the symmetric group. We explain this briefly; see [11, p. 61] for details. Also see [6] for a tabulation of $C_{\lambda\rho\mu}$ for small values of $|\lambda|$.

Let λ, ρ, μ be partitions of an integer a, and let R_λ, R_ρ, R_μ denote the corresponding irreducible representations of the symmetric group on a letters (in characteristic 0). Then $C_{\lambda\rho\mu}$ is the number of trivial representations in the tensor product $R_\lambda \otimes R_\rho \otimes R_\mu$. In particular, this number is symmetric in the three partitions involved. The main combinatorial result that we need is a direct corollary of [8, Thm. 1.6].

Theorem 4.3 (Dvir). *With notation as before, assume $C_{\lambda\rho\mu} \neq 0$. Then*

$$\rho_1 \ (\textit{the largest part in } \rho) \leq (\textit{number of parts in } \lambda) \cdot (\textit{number of parts in } \mu).$$

Now we come to the main theorem of this section. Recall that a closed subscheme $X \subseteq \mathbf{P}^N$ is said to be *projectively normal* if the map $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}}(m)) \to H^0(X, \mathcal{O}_X(m))$ is surjective for $m \geq 0$.

We will regard $\mathcal{L}(i, r)$ as a closed subscheme of $\mathbf{P}\big(\bigwedge^t S_d\big)$ via the Plücker embedding of $G(t, S_d)$.

Theorem 4.4. *Assume that the data (t, d, i, r) satisfy* (C1).

(a) *If* (C2) *holds, then $\mathcal{L}(i, r)$ is projectively normal.*
(b) *Moreover, if either $t = 1$ or* (C2) *is a strict inequality, then $\mathcal{L}(i, r)$ is arithmetically Cohen–Macaulay.*

Proof. We will use the following criterion (see [9, p. 467]): An equidimensional closed subscheme $X \subseteq \mathbf{P}^N$ (of dim > 0) is arithmetically Cohen–Macaulay

(ACM) iff it is projectively normal and $H^j(X, \mathcal{O}_X(m)) = 0$ for all $m \in \mathbf{Z}$ and $0 < j < \dim X$.

Since the Grassmannian is projectively normal in the Plücker embedding (see e.g. [1]), part (a) will follow if the map $H^0(\mathcal{O}_G(m)) \to H^0(\mathcal{O}_{\mathcal{L}}(m))$ is shown to be surjective for $m \geq 0$.

For $m \in \mathbf{Z}$, we have a hypercohomology spectral sequence coming from the resolution (16):

$$\begin{aligned} E_1^{p,q} &= H^q(G(t, S_d), \mathcal{E}^p(m)), \quad d_r^{p,q} \to d_r^{p+r,q-r+1}, \\ E_\infty^{p,q} &\Rightarrow H^{p+q}(\mathcal{O}_{\mathcal{L}}(m)). \end{aligned} \tag{19}$$

The terms live in the second quadrant, specifically in the range

$$-c \leq p \leq 0, \quad 0 \leq q \leq t(d - t + 1).$$

Now the theorem will follow from the next lemma.

LEMMA 4.5.

(1) *Assume that* (C2) *holds. Then* $E_1^{p,q} = 0$ *for* $m \geq 0$ *and* $q \neq 0$.
(2) *Assume that either* $t = 1$ *or* (C2) *is strict. Then* $E_1^{p,q} = 0$ *for* $m < 0$ *and* $q \neq \dim G(t, S_d)$.

Let us show that the lemma implies the theorem. First assume $m \geq 0$ and (C2) holds. Then the only nonzero term on the diagonal $p + q = 0$ is at $p = q = 0$. Hence $E_\infty^{0,0} = H^0(\mathcal{O}_{\mathcal{L}}(m))$ is a quotient of $E_1^{0,0} = H^0(\mathcal{O}_G(m))$, which proves (a).

Now assume m arbitrary and that either (C2) is strict or $t = 1$. Let (p, q) be such that $0 < p + q < \dim \mathcal{L}$. Then $E_1^{p,q} = 0$, which implies $H^j(\mathcal{O}_{\mathcal{L}}(m)) = 0$ for $j \neq 0, \dim \mathcal{L}$. This proves (b). □

Proof of Lemma 4.5. Let p, q be such that $E_1^{p,q} \neq 0$. By hypothesis, $\mathcal{E}^p$ has a summand

$$\mathcal{A} = \mathbb{S}_\rho \mathcal{B} \otimes \mathbb{S}_\mu R_{d-i} \otimes H^{\nu(\lambda')}(G', \mathbb{S}_{\lambda'} Q_{G'}^*)$$

such that $H^q(G(t, S_d), \mathcal{A}(m)) \neq 0$. Now:

(i) $\mathbb{S}_\mu R_{d-i} \neq 0$ implies that μ has at most $d - i + 1$ rows;
(ii) $\mathbb{S}_{\lambda'} Q_{G'}^* \neq 0$ implies that λ' has at most $i - r + 1$ rows—that is, λ has at most $i - r + 1$ columns.

But then, by Dvir's theorem, $C_{\lambda\rho\mu} \neq 0$ implies $\rho_1 \leq (d - i + 1)(i - r + 1)$. The next step is to use the Borel–Weil–Bott theorem on $\mathbb{S}_\rho \mathcal{B} \otimes \mathcal{O}_G(m)$. Let γ be the sequence $(\underbrace{m, \dots, m}_{d-t+1}; \rho_1, \dots, \rho_t)$. Since $H^q(\mathbb{S}_\rho \mathcal{B} \otimes \mathcal{O}_G(m)) \neq 0$, we have $q = l_\gamma$ (in the notation of [5]). But now

$$\rho_1 \leq (d - i + 1)(i - r + 1) \leq d - t + 1,$$

so it is immediate that $l_\gamma = 0$ if $m \geq 0$. If $m < 0$, then l_γ can only be a multiple of $d - t + 1$. If $t = 1$, then necessarily $l_\gamma = d + 1 = \dim G$. If $t > 1$, then $l_\gamma < \dim G$ is possible only if $\rho_1 = d - t + 1$, that is, only if (C2) is an equality. The lemma is proved, and the proof of the main theorem is complete. □

EXAMPLE 4.6. (1) The data $(t, d, i, r) = (3, 16, 13, 11)$ satisfy (C1) and strict (C2). Set-theoretically, $\mathcal{L}(13, 11) = \mathcal{L}(\underline{h})$ for

$$\underline{h} = 1\ \ 2\ \ \ldots\ \ 11\ \ 12\ \ 13\ \ 11\ \ 9\ \ 6\ \ 3\ \ 0.$$

Thus $\mathcal{L}(13, 11) \subseteq \mathbf{P}\big(\bigwedge^3 S_{16}\big)$ is ACM.

(2) Similarly the data $(t, d, i, r) = (5, 32, 28, 24)$ satisfy (C1) and strict (C2). In this case $\mathcal{L}(28, 24) = \mathcal{L}(\underline{h})$ for

$$\underline{h} = 1\ \ 2\ \ \ldots\ \ 27\ \ 28\ \ 24\ \ 20\ \ 15\ \ 10\ \ 5\ \ 0.$$

EXAMPLE 4.7. Choose an integer s such that $t \leq s < \frac{t(d+1)}{(t+1)}$. It then follows that the data (t, d, s, s) satisfy both (C1) and (C2); in fact, (C2) is strict unless $s = t$. Set-theoretically, $\mathcal{L}(s, s) = \mathcal{L}(\underline{h})$, where $\underline{h}$ is defined as in the proof of Theorem 3.17.

If $t = 1$, then this is the function $\underline{h}_s$ defined in Section 1.2. By the Gruson–Peskine theorem, we then know that $\mathcal{L}(s, s) = \mathcal{L}(\underline{h}_s)$ as schemes. We do not know if this remains true for $t > 1$.

EXAMPLE 4.8. Let $t = 2$ and choose integers i, d such that $i \geq 5$ and $3i = 2d + 1$. Then $(2, d, i, i - 1)$ satisfy (C1) and (C2), with the latter being strict iff $i > 5$. (We recover Example 4.2 for $i = 5$.) In this case, $\mathcal{L}(i, i - 1)$ is reducible with two components of dimension $3i - 7$ (i.e., codimension 4) each.

For instance, let $(i, d) = (9, 13)$; then $\mathcal{L}(9, 8) = \mathcal{L}(\underline{h}_1) \cup \mathcal{L}(\underline{h}_2)$, where

$$\underline{h}_1 = 1\ \ 2\ \ \ldots\ \ 7\ \ 8\ \ 9\ \ 8\ \ 7\ \ 6\ \ 4\ \ 2\ \ 0,$$

$$\underline{h}_2 = 1\ \ 2\ \ \ldots\ \ 7\ \ 8\ \ 8\ \ 8\ \ 8\ \ 6\ \ 4\ \ 2\ \ 0.$$

EXAMPLE 4.9. The data $(t, d, i, r) = (3, 14, 11, 9)$ satisfy (C2) but not (C1). Indeed, $\mathcal{L}(11, 9) = \mathcal{L}(\underline{h}_1) \cup \mathcal{L}(\underline{h}_2) \cup \mathcal{L}(\underline{h}_3)$, where

$$\underline{h}_1 = 1\ \ 2\ \ \ldots\ \ 8\ \ 9\ \ 10\ \ 11\ \ 9\ \ 7\ \ 5\ \ 3\ \ 0,$$

$$\underline{h}_2 = 1\ \ 2\ \ \ldots\ \ 8\ \ 9\ \ \ 9\ \ \ 9\ \ 9\ \ 9\ \ 6\ \ 3\ \ 0,$$

$$\underline{h}_3 = 1\ \ 2\ \ \ldots\ \ 8\ \ 9\ \ 10\ \ 10\ \ 9\ \ 8\ \ 6\ \ 3\ \ 0.$$

The components $\mathcal{L}(\underline{h}_1)$ and $\mathcal{L}(\underline{h}_2)$ have the expected dimension 27, but $\mathcal{L}(\underline{h}_3)$ is 28-dimensional.

REMARK 4.10. By Lemma 4.1, it is easy to produce examples where (C2) holds. In contrast, (C1) is rather restrictive. (Although for small values of (t, d) it is satisfied more often than not.) It would be worthwhile to characterize all sequences $\underline{h}$ such that $\mathcal{L}(\underline{h})$ is ACM (or projectively normal), but it is unlikely that the technique used here can be pushed any further.

REMARK 4.11. Some of the results proved here can be extended to char > 0 with appropriate care. Replacing S by the divided power algebra (see [17, Apx. A]), all results until the beginning of Section 3.2 remain valid in arbitrary characteristic. (The reference to partial differentiation should be ignored.)

All results in Sections 3.2–3.4 are valid for char $> d$. In Section 4 we need to assume char $= 0$, since (inter alia) Lascoux's result and the Borel–Weil–Bott theorem fail to hold in positive characteristic.

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves,* vol. I, Grundlehren Math. Wiss., 267, Springer-Verlag, New York, 1985.

[2] A. Bigatti and A. V. Geramita, *Level algebras, lex segments and minimal Hilbert functions,* preprint, 2001.

[3] J. Bronowski, *The sums of powers as simultaneous canonical expressions,* Math. Proc. Cambridge Philos. Soc. 29 (1933), 245–256.

[4] E. Carlini and J. Chipalkatti, *On Waring's problem for several algebraic forms,* preprint, 2001 (Algebraic Geometry archives http://xxx.lanl.gov).

[5] J. Chipalkatti, *Notes on Grassmannians and Schubert varieties,* Queen's Papers in Pure and Appl. Math., 13, Queen's Univ., Kingston, ON, 2000.

[6] A. J. Coleman, *Induced representations with applications to S_n and* $\mathrm{GL}(n)$, Queen's Papers in Pure and Appl. Math., 4, Queen's Univ., Kingston, ON, 1966.

[7] I. Dolgachev and V. Kanev, *Polar covariants of plane cubics and quartics,* Adv. Math. 98 (1993), 216–301.

[8] Y. Dvir, *On the Kronecker product of S_n characters,* J. Algebra 154 (1993), 125–140.

[9] D. Eisenbud, *Commutative algebra, with a view toward algebraic geometry,* Grad. Texts in Math., 150, Springer-Verlag, New York, 1995.

[10] W. Fulton, *Intersection theory,* 2nd ed., Ergeb. Math. Grenzgeb. (3), Springer-Verlag, Berlin, 1998.

[11] W. Fulton and J. Harris, *Representation theory, a first course,* Grad. Texts in Math., Springer-Verlag, New York, 1991.

[12] A. V. Geramita, *Inverse systems of fat points,* The curves seminar at Queen's, vol. X (Kingston, ON, 1995), pp. 2–114, Queen's University, 1995.

[13] P. A. Griffiths and J. Harris, *Principles of algebraic geometry,* Wiley, New York, 1978.

[14] L. Gruson and C. Peskine, *Courbes de l'espace projectif: variétés de sécants,* Enumerative geometry and classical algebraic geometry (Nice, 1981), pp. 1–31, Birkhäuser, Boston, 1982.

[15] A. Iarrobino, *Punctual Hilbert schemes,* Mem. Amer. Math. Soc. 10 (1977).

[16] ———, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc. 285 (1984), 337–378.

[17] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras and determinantal loci*, Lecture Notes in Math., 1721, Springer-Verlag, Berlin, 1999.

[18] J. O. Kleppe, *Deformations of graded algebras,* Math. Scand. 45 (1979), 205–231.

[19] J. P. S. Kung and G.-C. Rota, *The invariant theory of binary forms,* Bull. Amer. Math. Soc. (N.S.) 10 (1984), 27–85.

[20] A. Lascoux, *Syzygies des variétés déterminantales*, Adv. Math. 30 (1978), 202–237.
[21] O. Porras, *Rank varieties and their resolutions*, J. Algebra 186 (1996), 677–723.
[22] K. Ranestad and F.-O. Schreyer, *Varieties of sums of powers*, J. Reine Angew. Math. 525 (2000), 147–181.

J. V. Chipalkatti
Department of Mathematics
University of British Columbia
Vancouver, BC V6T 1Z2
Canada

jaydeep@math.ubc.ca

A. V. Geramita
Department of Mathematics
and Statistics
Queen's University
Kingston, Ontario K7L 3N6
Canada

tony@mast.queensu.ca

Dipartimento di Matematica
Universitá di Genova
Genova, Italia

geramita@dima.unige.it