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Regularity of Continuous CR Maps in Arbitrary Dimension

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1. Introduction

In this article we come back to one of those questions of complex analysis that are at the same time natural, important for applications, easy to formulate, and yet quite intriguing in the sense that they have given rise to much research without having been solved completely. More specifically, we mean the following problem.

General Problem. Let $D, D' \subset\subset \mathbb{C}^n$ be domains and let $f: D \to D'$ be a proper holomorphic map. Suppose that ∂D and $\partial D'$ have a certain regularity property (e.g., $\mathcal{C}^k$-smooth for some $k = 1, 2, 3, \dots, \infty, \omega$). Does the map f extend automatically to a map $\hat{f}: \bar{D} \to \bar{D}'$ with some regularity depending on the regularity of the boundaries $\partial D, \partial D'$? (For instance, for which k do $\mathcal{C}^k$-smooth boundaries imply that $\hat{f}$ is $\mathcal{C}^k$?)

For $n = 1$ there is a quite precise understanding of these questions, whereas good answers for $n > 1$ are known only under additional hypotheses. The answers to the general questions are unknown even for $k = \infty$ and $k = \omega$.

The question of boundary regularity of proper holomorphic mappings is not only natural but also important as a tool for other questions. Namely, if any proper holomorphic map $f: D \to D'$ automatically has a sufficiently high boundary regularity, then this will imply that the local biholomorphic invariants of a real hypersurface are part of the geometry of the domains bounded by them. This can, for instance, be very useful for studying the existence of certain proper holomorphic maps and many other problems.

Extensive research has been done in the area of the General Problem. We cannot mention it in full detail. Instead, we refer the reader to existing survey articles on the subject (e.g. [8; 22]).

In this article we deal, more specifically, with the case $k = \omega$ of the General Problem. We want to know whether this implies that $\hat{f}$ necessarily is $\mathcal{C}^\omega$ or, in other terms, whether all proper holomorphic maps $f: D \to D'$ extend holomorphically to an open neighborhood of $\bar{D}$ if ∂D and $\partial D'$ are $\mathcal{C}^\omega$-smooth. We will show that the answer to this question is indeed "yes" if one knows already that f

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extends to a continuous map $\hat{f}: \bar{D} \to \bar{D}'$ (see Theorem 1.4). (The question of whether all proper holomorphic maps $f: D \to D'$ extend continuously up to ∂D remains open; it might require totally different methods.)

Our specific problem and the method we use also have a long history. It started with the articles by S. Pinchuk [28] and H. Lewy [25], both of whom developed the first generalization of the Schwarz reflection principle to strictly pseudoconvex domains in several complex variables. The first steps toward an extension of this method of a "geometric" reflection principle to the case of degenerate Levi forms were made by S. Webster [34] and by K. Diederich and S. Webster [21]. The technique was further developed step-by-step in [16], [17], [18], and [19]. This article builds on the work of all these previous papers.

From the viewpoint of results, the next major steps in the real-analytic case after [21] were done in [3], [5], and [16]. At this stage it had been completely proved that any proper holomorphic map $f: D \to D'$, ∂D and $\partial D'$ $\mathcal{C}^{\omega}$-smooth, extends holomorphically to a neighborhood of $\bar{D}$ if D is pseudoconvex (and hence also D'). A major advantage for showing this was that, owing to a long and different development ending with [15] and [7], it was already known that such f extend in a $\mathcal{C}^{\infty}$ way up to ∂D. This facilitated the application of the method of a geometric reflection principle and also made it possible to apply a more algebraic version of a reflection principle as used in [5]. After this, the situation where D is not necessarily pseudoconvex had to be dealt with. In [17] the holomorphic extendability of f to a neighborhood of $\bar{D}$ was shown for $n = 2$ and under the hypothesis that f is already known to be continuous up to ∂D. In [18] for $n = 2$ this additional continuity hypothesis was eliminated. (Now, for $n > 2$, we have to make it again.) The article [19] contains a result that (together with [30]) is a very important tool for the use of the reflection principle in this article.

The question posed in the General Problem is essentially global in nature, and in Section 6 we will give a proof of Theorem 1.4 that also is global at a crucial step. Of course, the problem immediately becomes a local one if continuity of f up to ∂D is supposed. We show here a strong local result in Theorem 1.1, where we simply assume continuity of the given CR map f. For $n = 2$ this result has been proved in [23] based on the ideas of a preliminary version of [18]. For $n > 2$ it is shown here for the first time even under the additional hypothesis that M and M' are pseudoconvex (see our Theorem 1.2, for which we will give a simpler proof than needed for the general Theorem 1.1; for a quite special recent result in the same direction, see [24]). In fact, all previous global results also give certain local versions (see e.g. [3; 5; 16; 17; 18]). But in the local situation of a continuous CR map $f: M \to M'$ from a (germ of a) real-analytic smooth hypersurface M to a (germ) M' of the same kind, many other questions—as variations of the original General Problem—can be asked. Namely, it now makes sense (a) to strengthen the hypothesis on f and ask for weaker hypotheses on M, M' that still imply holomorphic extendability of f and/or (b) to take into account algebraic hypersurfaces. Much research has been done in this direction. In particular, the following two cases have been studied in recent articles.

(1) The map f is a $\mathcal{C}^\infty$ CR map (see [2; 12]).
(2) The map f just is a formal CR map from the germ $(M, 0)$ to the germ $(M, 0')$ (without assuming that it is known to be continuous, but also in higher codimension) (see [1; 4; 26; 27]).

The cited articles all contain more detailed bibliographies in their area, to which we refer the reader interested in the respective specific question. Sometimes certain additional nondegeneracy hypotheses on f are made in such work. Our decision in this article is to avoid any further assumptions on f besides continuity.

The main results of this article are as follows.

THEOREM 1.1. *Let $M \subset W \subset \mathbb{C}^n$ (resp. $M' \subset W' \subset \mathbb{C}^n$) be real-analytic smooth closed real hypersurfaces of finite type in some open set W (resp. W') in $\mathbb{C}^n$ and let $f: M \to M'$ be a continuous CR map. Then f extends holomorphically to a neighborhood of M.*

Although it is obviously just a special case of Theorem 1.1, we formulate the following theorem separately because it treats a particularly important case that so far has not been known in general.

THEOREM 1.2. *Let $M \subset W \subset \mathbb{C}^n$ (resp. $M' \subset W' \subset \mathbb{C}^n$) be pseudoconvex real-analytic hypersurfaces of finite type in some open set W (resp. W') in $\mathbb{C}^n$ and let $f: M \to M'$ be a continuous CR map. Then f extends holomorphically to a neighborhood of M.*

REMARK 1.3. (a) Even the special case of this theorem in which f is, in addition, already known to be $\mathcal{C}^\infty$ has not been known before.

(b) We will give (in Section 6) a simpler proof for Theorem 1.2 than the one that shows Theorem 1.1.

The following natural global statement also follows from Theorem 1.1, which (as a local statement) is, of course, much stronger. However, we will also give a simpler proof for this global case.

THEOREM 1.4. *Let $D, D' \subset\subset \mathbb{C}^n$ be domains with real-analytic smooth boundaries, and let $f: D \to D'$ be a proper holomorphic map extending continuously to $\bar{D}$. Then f extends holomorphically to a neighborhood of $\bar{D}$.*

The structure of this article is as follows. In Section 2 we introduce our basic notation and concepts, also reminding the reader briefly of the machinery of the geometric reflection principle by the use of Segre varieties (for details, however, we must refer to our previous articles [18; 19]). Section 3 contains the proof that f in any case extends holomorphically past a dense open subset of M. This is needed to get the reflection principle started. Section 4 then contains a slightly changed variant of the method: how to obtain a candidate for an extending holomorphic correspondence for the map f by using Segre varieties. In addition, we introduce the important notion of a "pair of reflection", which is used in Section 5 to study and introduce a new extension technique (as a correspondence) along Segre varieties. This finally, leads to the construction of certain sequences of analytic sets σ_ν of dimension ≥ 1 that allow us to reduce the task of proving Theorems 1.1 and 1.4 to showing that the cluster set of the sequence (σ_ν) is not completely contained in M (resp. ∂D) (see Lemmas 5.9 and 5.10). In the general case, this seems to be difficult (see Conjecture 6.1). However, it turns out to be possible in the global situation of Theorem 1.4 and under the additional pseudoconvexity hypothesis of Theorem 1.2; for the details, see Section 6. In order to deal with this difficulty in the general situation, we are led to studying the question of convergence of families of analytic varieties. In Section 7 we prove a new criterion for this (see Theorem 7.4) which also might be of interest for other applications. In Section 8, using a recent result of A. Tumanov [33], we can verify Conjecture 6.1 when the dimension of the σ_ν is large enough by using a decisive generalization of this given in Proposition 8.3. In order to apply this to the proof of Theorem 1.1, we must study (in Section 9) intersections of Segre varieties. Finally, Section 10 contains the construction of the needed new complex-analytic sets σ_ν of higher dimension and the end of the proof of Theorem 1.1.

2. Notation and Preliminaries

Observe that Theorem 1.4 follows directly from Theorem 1.1 since, according to [14, Thm. 4], the boundaries ∂D and $\partial D'$ in Theorem 1.4 are smooth real-analytic hypersurfaces of finite type (not necessarily pseudoconvex). Hence, it suffices to prove Theorem 1.1 in this article. We will therefore consider, for a large part of the article, the following situation: M, M' are smooth real-analytic hypersurfaces of finite type as in Theorem 1.1, and $f: M \to M'$ is a continuous CR map. We will use the complete machinery of Segre varieties and mostly remain close to the notation used in our previous articles [18; 19]. We ask the reader to look there for further details.

We may assume that $0 \in M$, $0' \in M'$, and $f(0) = 0'$, and it suffices to work near 0 and $0'$. By $\rho(z, \bar{z})$ (resp. $\rho'(z', \bar{z}')$) we denote real-analytic defining functions of M (resp. M') near 0 (resp. $0'$). Wherever needed, we may assume that we have chosen normal coordinates z and z' such that

$$\rho(z, \bar{z}) = 2x_n + \sum_{\nu=0}^{\infty} \rho_\nu('z, '\bar{z})(2y_n)^\nu \tag{2.1}$$

with $\rho_\nu('z, '0) \equiv 0$ for all ν (and analogously for M', ρ', z'). By $U_2 \supset\supset U_1 \ni 0$ (resp. $U_2' \supset\supset U_1' \ni 0'$) we denote standard pairs of neighborhoods of 0 (resp. $0'$). For any $w \in U_1$, the Segre variety

$$Q_w := \{z \in U_2 : \rho(z, \bar{w}) = 0\}$$

is a well-defined closed smooth complex hypersurface in U_2. For any $w \in U_1 \setminus M$, we define the symmetric point ${}_s w$ as the unique intersection between the complex line through w normal to M and Q_w. For $w \in M$ we put ${}_s w := w$. We call

$$U_j^{\pm} := \{z \in U : \pm\rho(z, \bar{z}) > 0\}.$$

For $w \in U_1^+$, we define the canonical component Q_w^c of Q_w as the connected component of $Q_w \cap U_2^-$ containing the symmetric point ${}_s w$. For any point $\zeta \in Q_w$, we denote by ${}_\zeta Q_w$ the germ of Q_w at ζ.

We denote by M^+ the set of strictly pseudoconvex points on M and by M^- the set of strictly pseudoconcave points on M (in the sense that all eigenvalues of $\mathcal{L}_\rho$ on $T^{10}M$ are negative). By $M^{\pm}$ we mean the set of all points where $\mathcal{L}_\rho$ has eigenvalues of both signs on $T^{10}M$, and by M^0 we mean the set of points on M where $\mathcal{L}_\rho$ has at least one eigenvalue 0 on $T^{10}M$. Notice that M^0 is a closed real-analytic subset of M of real dimension at most $2n-2$. We have

$$M = M^+ \cup M^- \cup M^{\pm} \cup M^0.$$

By $\hat{D}$ we denote the envelope of holomorphy of D, and we call Σ the set of all points in M such that f extends holomorphically to a neighborhood of Σ (our goal, of course, is to show that $\Sigma = M$).

A fundamental fact for our proof is the result of A. M. Trépreau stating that, since M is minimal, every point $z \in M$ has a neighborhood V such that any continuous CR function g on M extends holomorphically either to $V^+ := \{z \in V : \rho(z, \bar{z}) > 0\}$ or $V^- := \{z \in V : \rho(z, \bar{z}) < 0\}$, where this side of extension does not depend on the choice of g.

We assume that the sign of ρ has been chosen so that the map f extends holomorphically to U_2^-, and we denote its extension again by f. We need only consider the case where 0 is not contained in the envelope of holomorphy of U_2^-. We may assume that $f(U_2^-) \subset\subset U_1'$.

As a warning to the reader we would like to point out that, for $n \geq 3$, it can happen that $0 \notin \hat{U}_2^-$ even though there are no strictly pseudoconvex points on $M \cap U_2$.

3. Extending the Map to a Dense Subset of *M*

In this section we want to show that Σ is dense in M. For this we may assume (throughout the section) that f is nonconstant unless otherwise stated. Note first that $(M^- \cup M^{\pm}) \cap U_2 \subset \hat{U}_2^-$ so that automatically $f \in \mathcal{O}(M^- \cup M^{\pm})$. Hence, it suffices to show that f extends holomorphically to a neighborhood of a dense subset of M^+. We may therefore assume that $0 \in M^+$. We will shrink the sets U_j and U_j' as convenient in the sequel without pointing it out each time. We consider two different cases.

(a) Assume that $0' = f(0) \in M'^+ \cup M'^-$. It then follows directly from the result of [29] that $0 \in \Sigma$.

(b) Assume next that $0' = f(0) \in M'^{\pm}$; in this case we claim that f is constant. For suppose that f is not constant. Then there exists a point $a \in U_1^-$ as close to M as we want such that $a' := f(a) \in U_1'^-$ or $\in U_2'^+$. This follows from the following lemma.

Lemma 3.1. *If $f(U_2^-) \subset M'$, then f is constant.*

Proof. By looking at a point $b \in U_2^-$ where the functional matrix of f has maximal rank, we see that $f(U_2^-)$ contains a positive-dimensional germ of a complex-analytic set if f is not constant. This, however, contradicts the fact that M' is of finite type. □

In order to continue case (b), it suffices to consider $a' \in U_1'^-$. We observe that, because of $0' \in M^\pm$, there is a closed complex-analytic subset $A_{a'} \subset U_1'$ of dimension 1 such that $a' \in A_{a'}$ and $A_{a'} \cap M' = \emptyset$. Now we have $a \in A_a := f^{-1}(A_{a'}) \subset U_2^-$. However, since $0 \in M^+$, the complex-analytic set A_a must have limit points on $M \cap U_2$. Hence, the same must be true for $A_{a'}$, a contradiction. Next we show the following lemma.

LEMMA 3.2. *Let $0' \in N' \subset M'$ be a real $\mathcal{C}^2$-smooth generic manifold, where $\dim_{\mathbb{R}} N' \leq 2n-2$, and let U be any neighborhood of $0 \in M^+$. Then it follows that $f(M \cap U) \nsubseteq N'$.*

Proof. Without loss of generality we may assume that $\dim_{\mathbb{R}} N' = 2n-2$. There exists a complex plane $L' \ni 0'$ with $\dim_{\mathbb{C}} L' = 2$ such that $L' \cap N'$ is a totally real manifold of real dimension 2 near $0'$. For $a' \in \mathbb{C}^n$ let $L_{a'}$ be the plane parallel to L' and passing through a'. For small enough U_j', all intersections $L_{a'}' \cap N' \cap U_2'$ are totally real and of real dimension 2. Let $\varphi_{a'}$ be a strongly plurisubharmonic function in U_2' such that:

(i) $\varphi_{a'} \geq 0$;
(ii) $\varphi_{a'} = 0$ on $S_{a'} := L_{a'}' \cap N' \cap U_2'$.

Notice that such a function $\varphi_{a'}$ can be defined by $\varphi_{a'} = \sum_{k=1}^{2n-2}(\rho_k')$, where $\{\rho_k'\}_{k=1}^{2n-2}$ is a defining system for $S_{a'}$. We may assume that $U_1 \cap M \subset M^+$ and $f(U_1^-) \subset U_1'$ and that

$$\rho(z,\bar{z}) = 2x_n + |'z|^2 + o(|z|^2). \tag{3.1}$$

For $a \in U_1^-$, let $\omega_a := \{z \in U_1^- : z_n = a_n\}$ and $A_a := \omega_a \cap f^{-1}(L_{f(a)}' \cap U_1')$. Notice that ω_a is a complex manifold of dimension $n-1$, $\omega_a \subset\subset U_1$, and $f(\partial\omega_a) \subset M'$. If Lemma 3.2 is false then we would have $f(\partial\omega_a) \subset N'$. Since $\dim_{\mathbb{C}} L_{f(a)}' = 2$, it follows that A_a is an analytic set in U_1^- of dimension ≥ 1 and $a \in A_a$. The function $\psi_a := \varphi_{f(a)} \circ f$ is plurisubharmonic and nonnegative on A_a, and $\psi_a|\partial A_a = 0$ because $f(\partial A_a) \subset N' \cap L_{f(a)}$. Hence $\psi_a|A_a \equiv 0$. But $\varphi_{f(a)}$ is strongly plurisubharmonic. Therefore, $f|A_a$ is constant with image in $N' \subset M'$. Since $a \in U_1^-$ was chosen arbitrarily, it follows that $f(U_1^-) \subset M'$, a contradiction to Lemma 3.1. This proves Lemma 3.2. □

As an immediate consequence we now have the following result.

COROLLARY 3.3. *The set Σ is dense in M.*

Proof. The real-analytic set M'^0 can be stratified by smooth generic submanifolds of dimension $\leq 2n-2$. By applying Lemma 3.2 to each of them, we see that

$f(M^+) \nsubseteq M'^0$. Hence there exist points on M^+ arbitrarily close to 0 that are mapped to $M'^+ \cup M'^- \cup M'^\pm$. According to case (b) described previously, such points cannot be mapped to $M'^\pm$ because f is assumed to be nonconstant. Therefore, according to case (a), the point must belong to Σ. $\square$

COROLLARY 3.4. *If f is not constant then* (i) *the Jacobi determinant J_f of f is not identically* 0 *and* (ii) *f is locally proper near all points $z \in M$ with $J_f(z) \neq 0$.*

Proof. We move to a point of Σ. Then it follows from a result of Baouendi and Rothschild [6] that $J_f \not\equiv 0$ and that f is locally proper near any such point. $\square$

4. An Important Brick for the Holomorphic Extension of the Map f

In this section, $0 \in M$ is arbitrary and $0' = f(0) \in M'$. We assume that the standard neighborhoods U_j and U_j' have been chosen in such a way that $f(U_2^-) \subset\subset U_1'$. Furthermore, we assume that f is not constant. The following set will be an important brick for the holomorphic extension of f to a neighborhood of 0:

$$F^+ := \{(w, w') \in U_1^+ \times U_1' : f(Q_w^c) \subset Q'_{w'}\}. \tag{4.1}$$

Define $\pi : F^+ \to U_1^+$ by $\pi(w, w') = w$ and $\pi' : F^+ \to U_1'$ by $\pi'(w, w') = w'$.

LEMMA 4.1. *The set F^+ is analytic in $U_1^+ \times U_1'$.*

Proof. According to (4.1), $(w, w') \in F^+$ iff $\rho'(f(z), \bar{w}') = 0$ for all $z \in Q_w^c$. Furthermore, $z \in Q_w$ iff $\rho(z, \bar{w}) = 0$. And this is the case if and only if $z_n = h('z, \bar{w})$, where h is a function holomorphic in $'z$ and antiholomorphic in w. So for any pair $(w^0, w'^0) \in U_1^+ \times U_1'$, the set F^+ is defined near (w^0, w'^0) by

$$\rho'(f('z, h('z, \bar{w})), \bar{w}') = 0 \quad \text{for } 'z \text{ close to } {}_s'w^0.$$

This is a family of (anti)holomorphic equations for w, w'. $\square$

We must now study the set

$$\alpha := \{w \in U_1^+ : J_f = 0 \text{ on } Q_w^c\}. \tag{4.2}$$

Since M is supposed to be of finite type, it also is essentially finite (see [21] and [3]). Hence it follows from Corollary 3.4 that α must be discrete.

We put $U^+ := \pi(F^+) \subset U_1^+$.

LEMMA 4.2. *The map $\pi|(F^+ \setminus \pi^{-1}(\alpha))$ is locally proper.*

Proof. We have to show that, for any $w^0 \in U^+ \setminus \alpha$, the set $\pi^{-1}(w^0) \cap F^+ = \{(w^0, w') \in F^+\}$ is discrete. For this we move to a point b on $Q_{w^0}^c$ where $J_f(b) \neq 0$. Then the image under f of ${}_bQ_{w^0}^c$ is a complex-analytic germ of dimension $n - 1$. Hence the inclusion $f(Q_{w^0}^c) \subset Q'_{w'}$ completely determines the Segre set $Q'_{w'}$. The lemma then follows from the fact that M' is essentially finite. $\square$

COROLLARY 4.3. *The dimension of F^+ is n.*

Proof. According to Lemma 4.2, F^+ has dimension $\le n$ at all points not lying on $\pi^{-1}(\alpha)$; since α is discrete, the dimension of F^+ must be $\le n$ everywhere. However, the dimension of F^+ must actually be n, since F^+ contains the graph of f near all points of Σ. □

We next modify the set F^+ by excluding from it all irreducible components of dimension $< n$. Furthermore, we choose U_1' so small that the Segre map $\lambda' \colon U_1' \to \lambda'(U_1') \subset \mathcal{S}'$ is proper.

LEMMA 4.4. *The map $\pi : F^+ \to U^+$ is proper and hence $U^+ \subset U_1^+$ is open.*

Proof. (a) We first show that $\pi \colon F^+ \setminus \pi^{-1}(\alpha) \to U^+ \setminus \alpha$ is proper. For this we need to show that $F^+ \setminus \pi^{-1}(\alpha)$ has no limit points on $(U^+ \setminus \alpha) \times \partial U_1'$. Let $(w^\nu, w'^\nu) \in F^+ \setminus \pi^{-1}(\alpha)$ be a sequence such that $w^\nu \to w^0 \in U^+ \setminus \alpha$ and $w'^\nu \to w'^0 \in \bar{U}_1'$. Since $w^0 \in U^+ \setminus \alpha$ there exists a point $(w^0, \tilde{w}'^0) \in F^+ \setminus \pi^{-1}(\alpha)$ with $\tilde{w}'^0 \in U_1'$ as well as a sequence $(w^\nu, \tilde{w}'^\nu) \in F^+ \setminus \pi^{-1}(\alpha)$ with $\tilde{w}'^\nu \to \tilde{w}'^0$; this follows because (by Lemma 4.2) the map π is locally proper away from $\pi^{-1}(\alpha)$. By (4.1), $f(Q^c_{w^\nu}) \subset Q'_{w'^\nu} \cap Q'_{\tilde{w}'^\nu}$, and since $w^\nu \notin \alpha$ we have $Q'_{w'^\nu} = Q'_{\tilde{w}'^\nu}$. Now the properness of λ' together with $\tilde{w}'^0 \in U_1'$ means that also $w'^0 \in U_1'$.

(b) Notice that, having eliminated low-dimensional components from F^+, we have $F^+ = \overline{F^+ \setminus \pi^{-1}(\alpha)}$ because any "vertical" component of F^+ over a point from α would have dimension $\le n-1$ and thus would also have been eliminated. Furthermore, by part (a) of the proof, the set F^+ is (over $U^+ \setminus \alpha$) contained in a set whose w'-coordinates are given by a system of monic polynomials in the w_k' ($k = 1, \dots, n$) with coefficients holomorphic in $w \in U^+ \setminus \alpha$. Since all these coefficients extend holomorphically across α and since F^+ is (over α) contained in the set given by these extended polynomials, the properness of $\pi : F^+ \to U^+$ follows. □

The next step is to show the following lemma.

LEMMA 4.5. *The map $\pi' \colon F^+ \to U_1'$ is locally proper.*

Proof. We need to show that any point $(w^0, w'^0) \in F^+$ is isolated in $\pi'^{-1}(w'^0) = \{(w, w'^0) \in U_1^+ \times U_1' : f(Q^c_w) \subset Q'_{w'^0}\}$. The set $\pi'^{-1}(w'^0)$ is an analytic set. If $\pi'^{-1}(w'^0)$ is not discrete, then $\dim_{\mathbb{C}} \pi'^{-1}(w'^0) \ge 1$ and hence the set $\bigcup_{(w, w'^0) \in \pi^{-1}(w'^0)} Q^c_w$ contains an open subset of U_1^-. This would imply that $f(U_1^-) \subset Q'_{w'^0}$, which contradicts the fact that $J_f \not\equiv 0$. □

Lemmas 4.4 and 4.5 together say that F^+ induces a holomorphic correspondence $\hat{F}^+ \colon U^+ \to U_1'$ defined by $\hat{F}^+ = \pi' \circ \pi^{-1}$. In other words,

$$\hat{F}^+(w) = \{w' \in U_1' : Q'_{w'} \supset f(Q^c_w)\} \quad \text{for } w \in U^+.$$

Define another holomorphic correspondence $\hat{F}^- \colon U_1^- \to U_1'$ by

$$\hat{F}^-(w) := \{w' \in U_1' : Q'_{w'} = Q'_{f(w)}\}.$$

We put $U := U_1^- \cup U^+ \cup (\Sigma \cap U_1)$. By the invariance property of Segre varieties, the correspondences $\hat{F}^+$ and $\hat{F}^-$ coincide near any point from $\Sigma \cap U_1$. Therefore, together they give a holomorphic correspondence $\hat{F}\colon U \to U_1'$ with $\hat{F}|U^+ = \hat{F}^+$ and $\hat{F}|U_1^- = \hat{F}^-$. Let

$$F := \{(w, w') \in U \times U_1' : w' \in \hat{F}(w)\}$$

be the "graph" of $\hat{F}$. It is an analytic set in $U \times U_1'$ of pure dimension n with proper projection $\pi : F \to U$, and $\hat{F} = \pi' \circ \pi^{-1}$.

By the definition of $\hat{F}$, all values $f^1(w), \ldots, f^m(w) \in \hat{F}(w)$ have the same Segre varieties. Therefore, $Q'_{\hat{F}(w)}$ is well-defined for all $w \in U$.

LEMMA 4.6. *If, for a $w \in U$, there exists a $w' \in U_1'$ such that $f^k(w) \in Q'_{w'}$ for some value $f^k(w) \in \hat{F}(w)$, then $f^l(w) \in Q'_{w'}$ for all other values $f^l(w) \in \hat{F}(w)$. Hence, in this case we can write $\hat{F}(w) \subset Q'_{w'}$.*

Proof. Notice that $f^k(w) \in Q'_{w'}$ iff $w' \in Q'_{f^k(w)}$. As just observed, we also have $Q'_{f^k(w)} = Q'_{f^l(w)}$. Together this is the case if and only if $f^l(w) \in Q'_{w'}$. $\square$

It will be convenient to introduce the following terminology.

DEFINITION 4.7. We say that a pair $(w^0, z^0) \in U \times (Q_{w^0} \cap U)$ is a *pair of reflection* if there are open neighborhoods $\Omega(w^0)$ of w^0 and $\Omega(z^0)$ of z^0 such that, for all $w \in \Omega(w^0)$,

$$\hat{F}(Q_w \cap \Omega(z^0)) \subset Q'_{\hat{F}(w)}.$$

REMARK 4.8. A typical example of a pair of reflection is the situation when $w^0 \in U^+$ and $z^0 \in Q^c_{w^0}$. Another simple example is the pair (w^0, w^0) for $w^0 \in \Sigma$. Notice, however, that a pair (w^0, z^0) is not necessarily a pair of reflection if we just have $w^0 \in U^+$ and $z^0 \in Q_{w^0} \cap U$, since $Q_{w^0} \cap U$ may (of course) be disconnected.

We have the following symmetry relation.

LEMMA 4.9. *If (w^0, z^0) is a pair of reflection, then also (z^0, w^0) is a pair of reflection.*

Proof. We take $z \in \Omega(z^0)$ and $w \in Q_z \cap \Omega(w^0)$. Then $z \in Q_w \cap \Omega(z^0)$ and hence $\hat{F}(z) \subset Q_{\hat{F}(w)}$. From this it follows (as already used) that $\hat{F}(w) \subset Q_{\hat{F}(z)}$ and hence $\hat{F}(Q_z \cap \Omega(w^0)) \subset Q'_{\hat{F}(z)}$. Therefore, (z^0, w^0) is a pair of reflection. $\square$

LEMMA 4.10. *We have*:

(i) $\mathrm{cl}_{\hat{F}}(w^0) \subset \partial U_1'$ *for any* $w^0 \in \partial U \cap U_1^+$;
(ii) $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$;
(iii) *if* $\mathrm{cl}_{\hat{F}}(0) = \{0'\}$ *then* $0 \in \Sigma$;
(iv) F *is a closed analytic subset of* $[U_1 \setminus (M \setminus \Sigma)] \times U_1'$.

Proof. (i) Let $(w^\nu, w'^\nu) \in F$ with $(w^\nu, w'^\nu) \to (w^0, w'^0) \in (\partial U \cap U_1^+) \times \bar{U}_1'$ as $\nu \to \infty$. For any $\nu = 1, 2, \dots$ we have $f(Q^c_{w^\nu}) \in Q'_{w'^\nu}$. If $w'^0 \in U_1'$, we can pass to the limit and so obtain

$$f(Q^c_{w^0}) \subset Q'_{w'^0}.$$

But this would mean that $(w^0, w'^0) \in F$ and hence that $w^0 \in U$, a contradiction. Hence we must have $w'^0 \in \partial U_1'$.

(ii) Let $w^\nu \in U$, $w^\nu \to 0$. It is enough to consider the following two cases:

(a) $w^\nu \in U_1^- \cup (\Sigma \cap U_1)$ for all ν;
(b) $w^\nu \in U^+$ for all ν.

Since f is continuous up to M, in the first case $f(w^\nu) \to 0$ and, for any $w'^\nu \in \hat{F}(w^\nu)$, we have $Q'_{w'^\nu} = Q'_{f(w^\nu)}$. Since we may assume that the equality $Q'_{w'} = Q'_{0'}$ holds only for $w' = 0$ $(w' \in U_1')$, this means that $w'^0 \to 0'$. Therefore, it only remains to consider the case when all $w'^\nu \in U^+$. We have $f(Q^c_{w^\nu}) \subset Q'_{w'^\nu}$ for any $w'^\nu \in \hat{F}(w^\nu)$. Suppose that $w'^\nu \to w'^0 \in U_1'$; then $Q'_{w'^\nu} \to Q'_{w'^0}$. Since $w^\nu \to 0$, also $\operatorname{dist}(Q^c_{w^\nu}, 0) \to 0$. Hence $\operatorname{dist}(Q'_{w'^\nu}, 0') \to 0$, implying $0 \in Q'_{w'^0}$, and therefore, $w'^0 \in Q'_{0'}$.

(iii) If $\mathrm{cl}_{\hat{F}}(0) = \{0\}$, then by (i) we have $0 \in U$ and hence $0 \in \Sigma$, or $\operatorname{dist}(0, \partial U \cap U_1^+) > 0$. Thus it remains to consider the case $\operatorname{dist}(0, \partial U \cap U_1^+) > 0$. We choose a small open neighborhood $\tilde{U}_1 \subset\subset U_1$ of 0 such that $\tilde{U}_1^+ \cap \partial U = \emptyset$. Then $\tilde{U}_1^+ \subset U$; hence $U \supset \tilde{U}_1 \setminus (M \setminus \Sigma)$. We now replace U_1 by $\tilde{U}_1$. The correspondence $F|U_1^+$ is a component of the zero set of a system of pseudopolynomials with bounded holomorphic functions on U_1^+ as coefficients. According to the theorem of Trépreau, all these coefficients extend holomorphically to U_1. The zero set of the extended system of pseudopolynomials contains a component that is an extension of $F|U_1^+$ to U_1. Since, however, $\Sigma \subset M$ is dense, this component must agree with F^+ over U_1^-, giving thereby an extension of F over all of U_1 that we will still call $\hat{F}$. The projection $\pi : F \to U_1$ is again proper. Hence, it follows from [19] that $0 \in \Sigma$.

(iv) This is a corollary of (i). □

5. Extension along Segre Varieties

We now want to study the possible extension of f along Segre varieties. For this we observe the following. For any $w^0 \in U$ we can find a neighborhood $\Omega = {}'\Omega \times \Omega_n \subset\subset U$ of w^0 and a neighborhood $V \subset U_1$ of $Q_{w^0} \cap U_1$ such that, for $z \in V$, the intersection $Q_z \cap \Omega$ is connected and nonempty. For such a pair (Ω, V) we define

$$\tilde{F} := \tilde{F}(w^0, \Omega, V) := \{(z, z') \in V \times U_1' : \hat{F}(Q_z \cap \Omega) \subset Q'_{z'}\}. \tag{5.1}$$

Such a construction also has been used in [31].

LEMMA 5.1. *The set $\tilde{F}$ is analytic in $V \times U_1'$, and* $\dim \tilde{F} \le n$.

Proof. Notice that $\zeta \in Q_z \cap \Omega$ iff $\rho(\zeta, \bar{z}) = 0$ and $'\zeta \in {}'\Omega$. This is again equivalent to $\zeta_n = h('\zeta, \bar{z})$ and $'\zeta \in {}'\Omega$. Furthermore, for $\hat{F}$ defined as in Section 4 (after Lemma 4.4) we have $\hat{F}(\zeta) \subset Q'_{z'}$ iff $\rho'(\zeta', \bar{z}') = 0$ for all $\zeta' \in \hat{F}(\zeta)$. Finally, $\zeta' \in \hat{F}(\zeta)$ iff $(\zeta, \zeta') \in F$. We put $t' := \bar{\zeta}'$ and consider

$$A := \{(z, z', t') \in V \times U_1' \times U_1'^* : \rho'(\bar{t}', \bar{z}') = 0,\ ('\zeta, h('\zeta, \bar{z}), \bar{t}') \in F\ \forall '\zeta \in {}'\Omega\}.$$

(Here $U_1'^*$ means the set of all conjugates of points in U_1'.) The set A is analytic because it is locally defined by a holomorphic family of equations (after conjugation). Let $j\colon (z, z', t') \to (z, z')$ be the natural projection. We obviously have $j(A) = \tilde{F}$. Since $\Omega \subset\subset U$, the projection $j\colon A \to V \times U_1'$ is proper. Namely, $\zeta \in \Omega$ implies $\zeta' \in \hat{F}(\Omega) \subset\subset U_1'$ and $t' \in \hat{F}(\Omega)^* \subset\subset U_1'^*$. Thus $\tilde{F} = j(A)$ is an analytic set in $V \times U_1'$, since it is obviously closed in $V \times U_1'$. We claim that $\dim \tilde{F} \le n$. Notice for this that the set

$$\alpha := \{z \in V : J_f | (Q_z \cap \Omega) \equiv 0\} \tag{5.2}$$

(defined similarly as in (4.2)) is discrete for the same reasons as for the set from (4.2), and we have $\tilde{F} \subset (\alpha \times U_1') \cup (\tilde{F} \cap \pi^{-1}(V \setminus \alpha))$. The first part $\alpha \times U_1'$ has dimension n. Furthermore, $\pi : \tilde{F} \to V$ is locally proper over $V \setminus \alpha$, since in this situation $\hat{F}(Q_z \cap \Omega)$ has dimension $n - 1$ and hence there are only finitely many possible points z' with $\hat{F}(Q_z \cap \Omega) \subset Q'_{z'}$. Therefore, the dimension of the second part $\tilde{F} \cap \pi^{-1}(V \setminus \alpha) \le n$. Together, then, we have $\dim \tilde{F} \le n$. $\square$

We now assume additionally that ${}_s w^0 \in U$. Then, by Lemma 4.9, the sets F and $\tilde{F}$ coincide near the points of the form $({}_s w^0, w'^0)$ with $w'^0 \in \hat{F}({}_s w^0)$. We delete from $\tilde{F}$ those components that do not contain at least one of these points and denote the new analytic set again by $\tilde{F}$. Then $\dim \tilde{F} \equiv n$. By the uniqueness theorem we now have the following lemma.

Lemma 5.2. *If (w^0, z^0) is a pair of reflection, then $\tilde{F} = \tilde{F}(w^0, \Omega, V)$ contains F near every point $(z^0, z'^0) \in F$.*

Proof. Let $(z, z') \in F$ with $z \in \Omega(z^0)$ and take $w \in Q_z \cap \Omega(w^0)$. Then $z \in Q_w \cap \Omega(z^0)$ and $\hat{F}(Q_w \cap \Omega(z^0)) \subset Q'_{\hat{F}(w)}$. By Lemma 4.9 we have $\hat{F}(Q_z \cap \Omega(w^0)) \subset Q'_{\hat{F}(z)} = Q'_{z'}$. Hence $(z, z') \in \tilde{F}$. $\square$

As an immediate consequence we obtain our next result.

Corollary 5.3. *In the situation of Lemma 5.2, after deleting low-dimensional components from $\tilde{F}$ one has* $\dim \tilde{F} = n$ *and $\tilde{F} \supset F \cap (V \times U_1')$. More precisely, $F \cap (V \times U_1')$ is the union of suitable irreducible components of $\tilde{F} \cap (V \times U_1')$.*

Let $w^0, \Omega, V, \tilde{F}$ be as before and suppose that (w^0, z^0) is a pair of reflection with $z^0 \in \Sigma = M \cap U$. Let us denote by $S(w^0, z^0)$ the irreducible component of $\tilde{F} \cap [(Q_{w^0} \cap U_1) \times U_1']$ containing the germ of the graph of f at $(z^0, f(z^0))$. Obviously, $S(w^0, z^0)$ does not depend on the choice of Ω or V and is an analytic set of dimension $n - 1$ in $(Q_{w^0} \cap U_1) \times U_1'$.

LEMMA 5.4. *Let* (w^0, z^0) *be a pair of reflection with* $z^0 \in \Sigma$. *Then*:

(i) $S(w^0, z^0) \subset [(U_1 \cap Q_{w^0}) \times (U_1' \times Q'_{\hat{F}(w^0)})] \cap F$;

(ii) $S(w^0, z^0)$ *is an analytic set in* $(U_1 \cap Q_{w^0}) \times (U_1' \cap Q'_{\hat{F}(w^0)})$ *and* $\pi(S(w^0, z^0)) \cap (M \setminus \Sigma) = \emptyset$;

(iii) *the projection* $\pi : S(w^0, z^0) \to \pi(S(w^0, z^0)) \subset U \cap Q_{w^0}$ *is proper.*

REMARK 5.5. We do not claim that the projection $\pi : S(w^0, z^0) \to U_1 \cap Q_{w^0}$ is proper.

Proof of Lemma 5.4. Part (i) follows immediately from the uniqueness theorem, Lemma 5.2, and $\hat{F}(Q_{w^0} \cap \Omega(z^0)) \subset Q'_{\hat{F}(w^0)}$. Part (ii) follows from (i), Corollary 5.3, and Lemma 4.10(iv). Part (iii) follows from the properness of $\pi : F \to U$. □

The set $S(w^0, z^0)$ may be considered as the maximal analytic continuation of the germ of the graph of f at $(z^0, f(z^0))$ along $Q_{w^0} \cap U_1$. From Lemma 4.9 and the definitions of $\tilde{F}$ and $S(w^0, z^0)$, the next lemma follows immediately.

LEMMA 5.6. *For any* $z \in \pi(S(w^0, z^0))$, *the point* (w^0, z) *is a pair of reflection.*

We remind the reader of the following notation.

DEFINITION 5.7. Let A_ν be a sequence of (closed) subsets of a domain $D \subset \mathbb{C}^n$ (or $\mathbb{R}^n$). We define

$$\mathrm{cl}(A_\nu) := \{z \in D : \exists z_\nu \in A_\nu : z \text{ is a point of accumulation of } (z_\nu)\}.$$

Next we want to show holomorphic extendability of our map $f : M \to M'$ in certain points in M by studying cluster sets of sequences of certain sequences $S(w^\nu, z^\nu)$.

PROPOSITION 5.8. *Let* $(w^\nu, z^\nu) \in U \times \Sigma$ *be a sequence of pairs of reflection and choose* $w'^\nu \in \hat{F}(w^\nu)$. *Assume that* $(w^\nu, z^\nu) \to (0, 0)$ *and* $w'^\nu \to w'^0 \in U_1'$. *Suppose, furthermore, that the cluster set* $S := \mathrm{cl}(S(w^\nu, z^\nu))$ *contains a point* $(\zeta^0, \zeta'^0) \in U_1 \times U_1'$ *with* $\zeta_0 \in U$. *Then* $0 \in \Sigma$.

Proof. Let $(\zeta^\nu, \zeta'^\nu) \in S(w^\nu, z^\nu)$ be chosen such that $\zeta^{\nu_\mu} \to \zeta^0$ and $\zeta'^{\nu_\mu} \to \zeta'^0$ for a certain subsequence (ν_μ). By Lemmas 4.9 and 5.6, $(\zeta^{\nu_\mu}, w^{\nu_\mu})$ is a pair of reflection for any μ. Let $\Omega \subset U$ and $V \subset U_1$ be connected open neighborhoods of ζ^0 and $Q_{\zeta^0} \cap U_1$ (respectively) such that, for all $w \in V$, the intersection $Q_w \cap \Omega$ is connected and nonempty. Then $\tilde{F}(\zeta^0, \Omega, V)$ is an analytic set in $V \times U_1'$. After shrinking U_1 we have $\zeta^{\nu_\mu} \in \Omega$ and $Q_{\zeta^{\nu(\mu)}} \cap U_1 \subset V$ for $\mu \gg 1$, $\nu(\mu) = \nu_\mu$, and thus by (5.1) $\tilde{F}(\zeta^0, \Omega, V) = \tilde{F}(\zeta^{\nu_\mu}, \Omega, V)$. By Lemma 5.2, the set $\tilde{F}(\zeta^{\nu_\mu}, \Omega, V)$ contains the graph of f near $(z^{\nu_\mu}, f(z^{\nu_\mu}))$ and hence $\tilde{F}(\zeta^0, \Omega, V)$ contains $(0, 0') = \lim_\mu(z^{\nu_\mu}, f(z^{\nu_\mu}))$. This means that the graph of f extends as an analytic set to a neighborhood of $(0, 0')$. From the result of [30] it then follows that $0 \in \Sigma$. □

Unfortunately, the situation of Proposition 5.8 cannot always be established, since $\pi(S(w^\nu, z^\nu))$ cannot always be shown to be analytic because we do not know that π

is proper on these sets. Therefore, we now begin with the construction of some new sequences of analytic sets that will, at the end, allow us to overcome this difficulty.

LEMMA 5.9. *There exist sequences $(w^\nu, z^\nu) \in U \times \Sigma$ and $w'^\nu \in \hat{F}(w^\nu)$ and analytic sets $\sigma_\nu \subset U_1$ such that*:

(1) (w^ν, z^ν) *is a pair of reflection for any* ν;
(2) $(w^\nu, z^\nu) \to (0, 0)$;
(3) $w'^\nu \to w'^0 \in U_1'$;
(4) *there is an integer* $p \geq 1$ *such that the* σ_ν *are analytic sets of pure dimension* p;
(5) $z^\nu \in \sigma_\nu \subset \pi(S(w^\nu, z^\nu))$ *for all* ν.

Proof. Choose an arbitrary sequence $z^\nu \in \Sigma$, $z^\nu \to 0$. If there is a radius $r > 0$ such that, for any $\nu \gg 1$, the set $\pi(S(z^\nu, z^\nu))$ contains $Q_{z^\nu} \cap B(z^\nu, r)$, then (after shrinking U_1) properties (1)–(5) of the lemma are satisfied for $w^\nu = z^\nu$, $w'^\nu = f(z^\nu)$, and $\sigma_\nu = Q_{z^\nu} \cap U_1$. Thus we may now assume that there is no such radius r. This means that, for any (small enough) $r' > 0$, there exists a sequence $(w^\nu, w'^\nu) \in S(z^\nu, z^\nu)$ such that $w^\nu \to 0$ and $w'^\nu \to w'^0$ with $|w'^0| = r'$. The conditions $w^\nu = z^\nu$ and $w'^\nu = f(z^\nu)$ of course no longer hold; moreover, $w^\nu \in U^+$. By Lemma 4.10(ii) we have $w'^0 \in Q'_{0'} \cap U_1'$. Since $r' > 0$ is arbitrary and M' is of finite type, we may assume that $Q'_{w'^0} \neq Q'_{0'}$. By Lemmas 4.9 and 5.6, (w^ν, z^ν) is a pair of reflection for every ν. We put $S_\nu := S(w^\nu, z^\nu)$.

It remains to show that $\pi(S_\nu)$ contains an analytic set $\sigma_\nu \subset U_1$, $z^\nu \in \sigma_\nu$, of some fixed pure dimension $p \geq 1$. Since $w'^0 \in Q'_{0'}$ we have $Q_{w'^0} \ni 0'$. Since $Q'_{w'^0} \neq Q'_{0'}$, there exists a normal coordinate system in the image space such that $0'$ is an isolated point of $Q'_{w'^0} \cap \{z' : z_2' = \cdots = z_n' = 0\}$. Hence there exists an $\varepsilon > 0$ such that, after shrinking U_1', the intersection $q'^0 := Q'_{w'^0} \cap \{z' \in U_1' : z_2' = \cdots = z_{n-1}' = 0, |z_n'| < \varepsilon\}$ has no limit points on $\partial U_1'$. Notice that $q'^0 \ni 0'$ and is an analytic set of dimension 1 in $U_1' \cap \{|z_n'| < \varepsilon\}$. Thus, for $\zeta'^\nu := f(z^\nu)$ and $\nu \gg 1$, the sets

$$q'^\nu := Q'_{w'^\nu} \cap \{z' \in U_1' : z_k' = \zeta_k'^\nu \text{ for } k = 2, \ldots, n-1, |z_n'| < \varepsilon\}$$

contain ζ'^ν and are analytic sets of dimension 1 in $U_1' \cap \{|z_n'| < \varepsilon\}$ without limit points on $\partial U_1'$. Since $S_\nu \subset (U_1 \cap Q_{w^\nu}) \times (U_1' \cap Q'_{w'^\nu})$ and $S_\nu \ni (z^\nu, f(z^\nu))$, the intersections

$$s_\nu := S_\nu \cap \{(z, z') : z_k' = \zeta_k'^\nu \text{ for } k = 2, \ldots, n-1\}$$

are analytic sets of dimension ≥ 1 in $U_1 \times (U_1' \cap \{|z_n'| < \varepsilon\})$. Since the sets q'^ν have no limit points on $\partial U_1'$, the sets s_ν have no limit points on $U_1 \times (\partial U_1' \cap \{|z_n'| < \varepsilon\})$. By Lemma 4.10, we have $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'} = \{z_n' = 0\}$. Thus, for small enough $U_1 \ni 0$, the s_ν have no limit points on $U_1 \times (U_1' \cap \{|z_n'| = \varepsilon\})$. This means that, for $\nu \gg 1$, the projections $\pi : s_\nu \to U_1$ are proper and the images $\sigma_\nu := \pi(s_\nu)$ are analytic sets of dimension ≥ 1 in U_1 with $z^\nu \in \sigma_\nu$. □

The following lemma, which is now easy to show, is crucial for our further considerations.

LEMMA 5.10. *Let w^ν, z^ν, w'^ν, σ_ν be sequences with all the properties stated in Lemma 5.9. Assume that $0 \in M \setminus \Sigma$. Then* $\mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma$.

Proof. Suppose there is a point $\zeta^0 \in \mathrm{cl}(\sigma_\nu) \cap (U_1 \setminus (M \setminus \Sigma))$. By Lemma 5.9 there is a sequence $(w^\nu, z^\nu) \in U \times \Sigma$ with $(w^\nu, z^\nu) \to (0, 0)$ and a sequence $(\zeta^\nu, \zeta'^\nu) \in S_\nu = S(w^\nu, z^\nu)$ with $\zeta^\nu \in \sigma_\nu$ such that $\zeta^\nu \to \zeta^0$, $\zeta'^\nu \to \zeta'^0 \in U_1'$. Since $\zeta^\nu \in U$ we have $\zeta^0 \in \bar{U} \cap U_1$. But since $\zeta^0 \notin M \setminus \Sigma$, Lemma 4.10 implies $\zeta'^0 \in \partial U_1'$, a contradiction. $\square$

6. Final Steps for Proving Theorems 1.2 and 1.4

What must still be done in order to finish the proofs of Theorems 1.1, 1.2, and 1.4? Let us suppose that, in the situation introduced at the beginning of Section 2, there is a point $a \in M \setminus \Sigma$ left. Using Lemma 5.10, we shall derive a contradiction.

Notice that this would be done if the following conjecture were known to have a positive answer.

CONJECTURE 6.1. *Let $N \subset W \subset \mathbb{C}^n$ be a real-analytic CR manifold of finite type in the sense that there are no complex-analytic germs of positive dimension in N, and let $A_\nu \subset W$ be closed complex-analytic sets of pure fixed dimension $p \geq 1$. Then* $\mathrm{cl}(A_\nu) \nsubseteq N$.

Unfortunately, this important conjecture is in general open. It amounts to having a kind of more global uniform Lojaziewicz inequality. One of the difficulties with Conjecture 6.1 arises because $\mathrm{cl}(A_\nu)$ need not contain any even 1-dimensional complex-analytic germ, as examples of Wermer [35] and Stolzenberg [32] show. (If this were the case then Lemma 5.10—together with the fact that M is of finite type—would give a contradiction.)

However, under suitable extra hypotheses on N, the conjecture can be proved. This is, for instance, the case if the point $a \in N$ is a peak point for the restriction to N of continuous plurisubharmonic functions on W. Then the maximum principle for plurisubharmonic functions will yield the desired conclusion. We will use this to bring the proof of Theorem 1.2 to a quick end. The same strategy together with a small additional argument can also be used to finish the proof of Theorem 1.4.

For proving the general Theorem 1.1, a new strategy will have to be used. It is related to the theorem of Bishop [9], which states that any sequence of analytic sets A_ν of pure p dimension (as considered in Conjecture 6.1) contains a subsequence converging to a complex-analytic set of dimension p if, for any relatively compact subset $V \subset\subset W$, the $2p$-dimensional area of $A_\nu \cap V$ is uniformly bounded in ν. Namely, we will prove in Section 7 a new criterion for uniform area boundedness of certain sequences of analytic sets. Later in the article we will replace the sequences (σ_ν) from Lemma 5.9 by new sequences with sufficiently large dimensions for which we can then prove a variant of Conjecture 6.1. The details needed to carry out this strategy will, however, still be considerable.

We now come to the proofs of Theorems 1.2 and 1.4. First we introduce the following simple way of speaking.

DEFINITION 6.2. Let $U \subset \mathbb{C}^n$ be an open set, let $a \in U$, and let $a \in B \subset U$ be a closed subset. Suppose $A_\nu \subset U$ are closed complex-analytic subsets of pure dimension $p \geq 1$. We say that the sequence (A_ν) *clusters along* B *at* a if $a \in \mathrm{cl}(A_\nu) \subset B$.

LEMMA 6.3. *Suppose that, for a point $a \in B \subset U$, there is a continuous plurisubharmonic function $\varphi \in \mathrm{PSH}(U)$ such that $\varphi(a) > \varphi(z)$ for all $z \in B \setminus \{a\}$. Let (A_ν) be a sequence of closed complex-analytic subsets of U of pure dimension $p \geq 1$. Then (A_ν) does not cluster along B at a.*

Proof. Suppose (A_ν) were to cluster along B at a, and fix an open neighborhood $W \subset\subset U$ of a. Then $A_\nu \cap \partial W \neq \emptyset$ for all ν and hence $\mathrm{cl}(A_\nu) \cap \partial W \neq \emptyset$. Therefore also $B \cap \partial W \neq \emptyset$. One has $\sup\{\varphi(z) : z \in B \cap \partial W\} < \varphi(a)$. Hence, because of the continuity of φ, there is a sufficiently small open neighborhood V of $B \cap \partial W$ so that $c := \sup\{\varphi(z) : z \in V\} < \varphi(a)$. Next we can choose an open neighborhood V_1 of a such that $\varphi(z) > c$ on V_1. Note, however, that for ν sufficiently large, $A_\nu \cap \partial W \subset V$ and $A_\nu \cap V_1 \neq \emptyset$. But this is a contradiction to the maximum principle applied to the plurisubharmonic function $\varphi|A_\nu$. □

It is now important to observe for which B as in Lemma 6.3 the required plurisubharmonic peak functions φ are known to exist. Here is a list of the most important cases.

PROPOSITION 6.4. *Let $a \in B \subset U$ be a closed subset. Then there is a continuous plurisubharmonic peak function φ on U with $\varphi(a) > \varphi(z)$ for all $z \in B \setminus \{a\}$ in each of the following cases*:

(1) *B is a $\mathcal{C}^2$-smooth strictly pseudoconvex hypersurface*;
(2) *B is a smooth totally real submanifold*;
(3) *B is a $\mathcal{C}^\infty$-smooth pseudoconvex hypersurface of finite type.*

Proof. The cases of strictly pseudoconvex hypersurfaces and of totally real manifolds are well known and easy. If B is a $\mathcal{C}^\infty$-smooth pseudoconvex hypersurface then, according to Catlin [10], B is B-regular. (If B is even a $\mathcal{C}^\omega$-smooth hypersurface, this result was first shown in [14] together with [13].) □

COROLLARY 6.5. *Let $B \subset U \subset \mathbb{C}^2$ be a $\mathcal{C}^\omega$-smooth hypersurface of finite type. Then Conjecture 6.1 holds for $N = B$.*

Proof. According to Proposition 6.4(2) and (3), the conjecture holds if the point a lies in either the pseudoconvex or the pseudoconcave region of B. So it remains to consider the case where a is a point of degeneracy of the Levi form $\mathcal{L}_B$ of B. The set E of these points, however, is itself real-analytic and can (in $\mathbb{C}^2$) be stratified by totally real manifolds of dimensions 2, 1, and 0. If we now assume that $\mathrm{cl}(A_\nu) \subset B$ then, for $a \in E$, the set $\mathrm{cl}(A_\nu)$ must lie in E because otherwise we can move on $\mathrm{cl}(A_\nu)$ to a point in the pseudoconvex or pseudoconcave region. By the same argument, we can then work our way down the strata of E. □

We now want to apply these considerations on special cases of Conjecture 6.1 to the holomorphic extendability of CR maps $f\colon M \to M'$ under the hypotheses of Theorem 1.2 and Theorem 1.4. From Lemma 5.10 and Lemma 6.3 applied to $A_\nu := \sigma_\nu$ and $B := M$, we immediately obtain the following technical result.

PROPOSITION 6.6. *Let $a \in B := (M \setminus \Sigma)$ be arbitrary. Then there is no open neighborhood U of a with a continuous plurisubharmonic function φ on U such that $\varphi(a) > \varphi(z)$ for all $z \in B \setminus \{a\}$.*

Proof. Lemma 5.10 tells us that necessarily $a \in \operatorname{cl}(\sigma_\nu) \subset B$. According to Lemma 6.3, however, this is impossible. □

The proofs of Theorem 1.2 and Theorem 1.4 are now obvious.

Proof of Theorem 1.2. Since M is supposed to be $\mathcal{C}^\omega$-smooth, pseudoconvex, and of finite type, there are (by Proposition 6.4) plurisubharmonic peak functions for every $a \in M$. Hence, according to Proposition 6.6, the set Σ must be all of M.
□

Proof of Theorem 1.4. We put $M := \partial D$ and $M' := \partial D'$. Then all considerations of this article apply to $f|M\colon M \to M'$. Notice, however, that in this case the set $E := M \setminus \Sigma \subset M$ is compact. Let us assume that it is nonempty. Then we can choose a suitable origin of a global coordinate system and take as point $a \in E$ the (unique) point on E of farthest distance from this origin. The function $\varphi(z) := |z|^2$ is a plurisubharmonic peak function as needed for $a \in E$. This is a contradiction to Proposition 6.6. □

7. A Convergence Theorem for Families of Analytic Sets

As explained at the beginning of Section 6, we will need a criterion telling us when certain sequences of complex-analytic sets contain convergent subsequences with complex-analytic sets as limit sets. We published our result in this direction in [20]. For the convenience of the reader and the completeness of this article, we repeat the proof here.

In general, the cluster set of a sequence of analytic subsets $A_\nu \subset U$ of dimension $p > 0$ ($U \subset \mathbb{C}^n$ open) need not contain a germ A of an analytic subset of positive dimension—even if all A_ν pass through a fixed point $z_0 \in U$. This can already happen in codimension 1 and if all A_ν are of the form $A_\nu = \{z \in U : g_\nu(z) = 0\}$ for suitable holomorphic functions $g_\nu \neq 0$ with $g_\nu \to 0$ on U. For examples see [35] and [32].

Probably the most important positive result in this direction is the theorem of Bishop (see [9] and also [11, Thm. 15.5]), which can be formulated in the following way.

THEOREM 7.1. *Let $A_\nu \subset U$ be a sequence of pure p-dimensional analytic subsets of a complex manifold X converging to some set $A \subset X$ and such that, for any compact subset $K \subset X$, there exists a constant $M_K \geq 0$ with*

$$\mathrm{vol}_{2p}(A_\nu \cap K) \le M_K$$

for all ν. Then A is also a pure p-dimensional analytic subset of X.

Theorem 7.1 immediately implies the following.

COROLLARY 7.2. *Let $A_\nu \subset U$ be a sequence of pure p-dimensional analytic subsets of a complex manifold X with locally uniformly bounded $2p$-dimensional Hausdorff measures*:

$$\mathrm{vol}_{2p}(A_\nu \cap K) \le M_K \quad \forall\nu$$

for a suitable constant M_K associated to an arbitrary compact subset $K \subset U$. Then we can extract a subsequence from (A_ν) converging in U to a pure p-dimensional analytic subset $A \subset U$ or to $\emptyset$.

For the convenience of the reader, we formulate here explicitly what is meant by saying that the sequence (A_ν) converges to the set A.

DEFINITION 7.3. We say that a sequence of subsets $E_j \subset U$ *converges to a set* $E \subset U$ if (a) E consists exactly of all the limit points of convergent sequences (x_{ν_j}) with $x_{\nu_j} \in E_{\nu_j}$ and (b) for any compact subsets $K \subset E$ and any $\varepsilon > 0$, there exists an index $\nu(\varepsilon, K)$ such that K belongs to the ε-neighborhood of E_ν in U for all $\nu > \nu(\varepsilon, K)$.

The goal of this paper is to show that a strong analogue of Montel's theorem holds for families of analytic sets $A_w \subset U$ depending holomorphically on a parameter $w \in V \subset \mathbb{C}^m$. Precisely speaking, our result will be as follows.

THEOREM 7.4. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets, and let $g_j(z, w) \in \mathcal{O}(U \times V)$ for $j = 1, \dots, k$ with a positive integer $k \le n$. For $w \in V$, put*

$$A_w := \{z \in U : g_j(z, w) = 0 \text{ for } j = 1, \dots, k\}.$$

Let $E := \{w \in V : \dim_{\mathbb{C}} A_w > n - k\}$. Then, for any $\tilde U \subset\subset U$ and $\tilde V \subset\subset V$, there exists a constant $c = c(\tilde U, \tilde V) > 0$ such that

$$\mathrm{vol}_{2(n-k)}(A_w \cap \tilde U) < c$$

for all $w \in \tilde V \setminus E$. In particular, we can extract from any sequence (A_{w_ν}), $w_\nu \in \tilde V \setminus E$, a subsequence converging in U to an analytic subset A of pure dimension $n - k$. (Observe that the sequence (w_ν) might converge to a point in E.)

REMARK 7.5. (a) Theorem 7.4 is in some sense purely local so that it easily extends to open subsets U, V of complex manifolds.

(b) For the proof of Theorem 7.4, we may assume that U, V are polydiscs centered at 0 and that $g_j(0, 0) = 0$ for all $j = 1, \dots, k$.

In Section 7.1 we will first show a crucial lemma that is equivalent to Theorem 7.4 in the case $k = 1$. In Section 7.2, we generalize this lemma to arbitrary codimension. The proof of Theorem 7.4 will then be given in Section 7.3 using the lemma from Section 7.2.

7.1. A Lemma in Codimension 1

The case $k = 1$ of Theorem 7.4 is equivalent to the following lemma.

LEMMA 7.6. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be polydiscs centered at 0. Let $g(z, w) \in \mathcal{O}(U \times V)$ be a function such that $g(z, 0) \equiv 0$ and*

$$g(z, w) = \sum_p \alpha_p(w) z^p.$$

Suppose that there is a sequence $(w^\nu) \subset V$, $w^\nu \to 0$, and that $g(z, w^\nu) \not\equiv 0$ for all ν. Then there exist a multiindex $l = (l_1, \ldots, l_m)$, an open neighborhood $\tilde{V}$ of $0 \in \mathbb{C}^n$, a holomorphic map $h\colon \tilde{V} \to V$, and a sequence $(\tilde{w}_\nu) \subset \tilde{V}$, $\tilde{w}^\nu \to 0$, such that (after possibly passing to a subsequence)

(1) $w^\nu = h(\tilde{w}^\nu)$ *for all* ν,
(2) $\beta_p(\tilde{w}) := \alpha_p(h(\tilde{w}))/\alpha_l(h(\tilde{w})) \in \mathcal{O}(\tilde{V})$ *for all* p, *and*
(3) $\tilde{g}(z, \tilde{w}) := \sum_p \beta_p(\tilde{w}) z^p \in \mathcal{O}(U \times \tilde{V})$.

In particular, since $b_l(\tilde{w}) \equiv 1$, it follows that $\tilde{g}(z, 0) \not\equiv 0$.

Proof. After shrinking the polydisc V a little bit, we have in particular $\alpha_p \in \mathcal{O}(\bar{V})$ for all p. Because (according to a theorem of Frisch and Siu) the ring $\mathcal{O}(\bar{V})$ is Noetherian, the ideal $\mathcal{I} \subset O(\bar{V})$ generated by the functions α_p is spanned as an ideal by finitely many of them. Since $g(z, 0) \equiv 0$ we conclude from this that there is a positive integer s such that, for each multiindex p, we have a representation of the form

$$\alpha_p(w) = \sum_{1 \le |j| \le s} h_{pj}(w) \alpha_j(w) \tag{7.1}$$

on $\bar{V}$ with holomorphic functions $h_{pj} \in \mathcal{O}(\bar{V})$.

After passing to a subsequence of (w^ν), there is a multiindex l with $|l| \le s$ such that

$$|\alpha_j(w^\nu)| \le |\alpha_l(w^\nu)| \quad \forall \nu,\ \forall |j| \le s. \tag{7.2}$$

From (7.1) and (7.2) we obtain that, for all p,

$$\frac{|\alpha_p(w^\nu)|}{|\alpha_l(w^\nu)|} \le C_p \quad \forall \nu \tag{7.3}$$

for suitable constants $C_p > 0$. Since, moreover, $\alpha_l(0) = 0$, it follows from (7.3) that 0 is a point of indeterminacy of the meromorphic functions α_p/α_l for all p.

Let $j_1, \ldots, j_N$ be any numbering of the multiindices j with $|j| \le s$. We put

$$\hat{\alpha}(w) := \left(\frac{\alpha_{j_1}}{\alpha_l}, \ldots, \frac{\alpha_{j_N}}{\alpha_l} \right) : V \to (\mathbb{P}^1)^N$$

as a meromorphic map and $A := \{w \in V : \alpha_l(w) = 0\}$. Then $\hat{\alpha}$ is holomorphic on $V \setminus A$ and the graph $\Gamma_{\hat{\alpha}} \subset V \times (\mathbb{P}^1)^N$ of the meromorphic map $\hat{\alpha}$ is given by

$$\Gamma_{\hat{\alpha}} = \overline{\{(w, \xi) \in (V \setminus A) \times (\mathbb{P}^1)^N : \xi = \hat{\alpha}(w)\}},$$

which is a complex-analytic subset of $V \times (\mathbb{P}^1)^N$ of dimension n. We denote by $\pi_1\colon \Gamma_{\hat{\alpha}} \to V$ the projection to the first coordinate and by $\pi_2\colon \Gamma_{\hat{\alpha}} \to (\mathbb{P}^1)^N$ the projection to the second coordinate. Both are, of course, holomorphic maps. By (7.3) we may assume, after passing again to a subsequence of (w^ν), that

$$\lim_{\nu\to\infty} \hat{\alpha}(w^\nu) = \hat{c} = (\hat{c}_1, \dots, \hat{c}_N) \in \mathbb{C}^N.$$

In general, the graph $\Gamma_{\hat{\alpha}}$ will be singular at $\pi^{-1}(A)$. Yet according to the Hironaka theorem there is a neighborhood $\hat{V}$ of $(0, \hat{c}) \in \Gamma_{\hat{\alpha}}$ and a complex manifold X of dimension m together with a proper holomorphic map $\sigma : X \to \hat{V}$ such that $\sigma|\sigma^{-1} \circ \pi_1^{-1}(V \setminus A) : \sigma^{-1} \circ \pi_1^{-1}(V \setminus A) \to \pi_1^{-1}(V \setminus A)$ is biholomorphic. After passing again to a subsequence we find a convergent sequence $\tilde{w}^\nu \in X$, $\tilde{w}^\nu \to x$, with $\sigma(\tilde{w}^\nu) = (w^\nu, \hat{\alpha}(w^\nu))$. Hence $\sigma(x) = (0, \hat{c})$. Let $\tilde{V}$ be a small coordinate neighborhood of x on X with $x = 0$. Call the coordinates $\tilde{w}$. On $\tilde{V}$ we define the map $h(\tilde{w}) := \pi_1 \circ \sigma(\tilde{w}) \colon \tilde{V} \to V$. It is holomorphic: $h(0) = 0$ and $h(\tilde{w}^\nu) = w^\nu$. Next we define the holomorphic map

$$\hat{\beta}(\tilde{w}) := \pi_2 \circ \sigma(\tilde{w}). \tag{7.4}$$

On $(\pi_1 \circ \sigma)^{-1}(V \setminus A) \cap \tilde{V}$ we have

$$\hat{\beta}(\tilde{w}) = \hat{\alpha}(h(\tilde{w})) \tag{7.5}$$

and $\hat{\beta}(0) = \hat{c} \in \mathbb{C}^N$, so we may assume that $\hat{\beta} \colon \tilde{V} \to W(\hat{c}) \subset \mathbb{C}^N$, an open neighborhood of $\hat{c}$. Hence all components $\hat{\beta}_i(h(\tilde{w}))$, $i = 1, \dots, N$, are holomorphic functions on $\tilde{V}$. Together with (7.1) it follows that, for all multiindices p,

$$\beta_p(\tilde{w}) := \frac{\alpha_p(h(\tilde{w}))}{\alpha_l(h(\tilde{w}))}$$

may be considered as holomorphic functions on $\tilde{V}$. This proves Lemma 7.1. $\square$

7.2. A Lemma in Arbitrary Codimension

LEMMA 7.7. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be polydiscs centered at* 0, *and let $g_j(z, w) \in \mathcal{O}(U \times V)$, $j = 1, \dots, k$ for some $k \le n$, be holomorphic functions with $g_j(0, 0) = 0$. Let $(w^\nu)_{\nu=1}^\infty \subset V$ be a sequence with $w^\nu \to 0$ and put*

$$A_\nu := \{z \in U : g_j(z, w^\nu) = 0 \ \forall j = 1, \dots, k\}.$$

Assume that $A_\nu \subset U$ is an analytic set of pure dimension $n - k$ for all ν. Then there exists a subsequence of (w^ν), which we again denote by (w^ν), such that (A_ν) converges to an analytic subset $A \subset U$ of pure dimension $n - k$. Moreover, there exist a holomorphic coordinate system in $\mathbb{C}^n$ with the same origin, a polydisc $U_1 \subset\subset U$ with center 0, *a neighborhood $\tilde{V} \subset \mathbb{C}^m$ of the origin, a holomorphic map $h \colon \tilde{V} \to V$, a sequence $(\tilde{w}^\nu) \subset \tilde{V}$, $\tilde{w}^\nu \to$* 0, *and holomorphic functions $\tilde{g}_j(z, \tilde{w}) \in \mathcal{O}(U_1 \times \tilde{V})$ such that*:

(1) $h(\tilde{w}^\nu) = w^\nu$;

(2) *for each $j = 1, \dots, k$, the function $\tilde{g}_j$ is a polynomial in z_j with coefficients holomorphic in $z_{j+1}, \dots, z_n, \tilde{w}$ and leading coefficient* 1;

(3) *$A_\nu \cap U_1 \subset \{z \in U_1 : \tilde{g}_j(z, \tilde{w}^\nu) = 0 \ \forall j = 1, \dots, k\} =: \tilde{A}_\nu$ for all ν and $\tilde{A}_\nu$ is of pure dimension $n - k$*;

(4) $A_0 := \{z \in U_1 : \tilde{g}_j(z, 0) = 0 \ \forall j = 1, \ldots, k\}$ *is an analytic set in* U_1 *of pure dimension* $n - k$; *and*

(5) $A_0 \supset \lim(A_\nu \cap U_1)$*—in particular, this limit exists.*

REMARK 7.8. In general we do not have $A_\nu = \tilde{A}_\nu$ in Lemma 7.7(3), because we pass at a certain step of the following proof to a resultant of two pseudopolynomials that might add some additional roots.

Proof of Lemma 7.7. The proof is given by induction over k. Namely, for $k = 1$ the claim follows from Lemma 2.1. By the induction hypothesis we then may assume that, for $j = 1, \ldots, k-1$, the g_j are already Weierstraß polynomials in z_j with coefficients holomorphic in $z_{j+1}, \ldots, z_n, w$. We put

$$A'_\nu := \{z \in U : g_j(z, w^\nu) = 0 \ \forall j = 1, \ldots, k-1\} \quad \forall \nu,$$
$$A'_0 := \{z \in U : g_j(z, 0) = 0 \ \forall j = 1, \ldots, k-1\},$$

and we have

$$A_\nu \subset A'_\nu \cap \{z \in U : g_k(z, w^\nu) = 0\}. \tag{7.6}$$

The A'_ν are analytic sets in U of pure dimension $n - k + 1$ and $A'_0 = \lim A'_\nu$. By Lemma 2.1 we may also assume that $g_k(z, 0) \not\equiv 0$. Since the analytic sets

$$A'_\nu \cap \{z \in U : g_k(z, w^\nu) = 0\}$$

are of pure dimension $n - k$, the functions $g_k(\cdot, w^\nu)|A'_\nu$ are not identically 0 for any ν.

Replacing g_k by $g_k + g_1$ if necessary, we may assume that $g_k(z_1, 0, \ldots, 0) \not\equiv 0$. After that we replace g_k by the resultant of g_1 and the Weierstraß polynomial of g_k (with respect to z_1). The new g_k is obviously holomorphic in $U \times V$ (with possibly smaller $U \ni 0$) and does not depend on z_1. At the same time, (7.6) still holds. After repeating this procedure consecutively for $z_2, \ldots, z_{k-1}$, we obtain a function holomorphic in $U \times V$, again denoted by g_k, that does not depend on $z_1, \ldots, z_{k-1}$ and for which (7.6) holds.

Finally, we apply Lemma 2.1 to this g_k and obtain a holomorphic map $h : \tilde{V} \to V$, a sequence $\tilde{w}^\nu \in \tilde{V}$, and a function $\tilde{g}_k \in \mathcal{O}(U \times \tilde{V})$ such that $h(\tilde{w}^\nu) = w^\nu$ and

$$A_\nu \subset A'_\nu \cap \{z \in U : \tilde{g}_k(z, \tilde{w}_\nu) = 0\}.$$

Since $\tilde{g}_k(z_k, \ldots, z_m, 0) \not\equiv 0$, the analytic set

$$A_0 := A'_0 \cap \{z \in U : \tilde{g}_k(z, 0) = 0\}$$

is of pure dimension $n - k$. After an appropriate change in the variables $z_k, \ldots, z_n$, we can replace $\tilde{g}_k$ by its Weierstraß polynomial in z_k. Thus $\lim A_\nu$ exists and is the union of some components of A_0.

In order to finish the proof of the lemma, it only remains to show that $\lim A_\nu$ is an analytic set in the whole domain $U \subset \mathbb{C}^n$. However, the preceding arguments can be applied to an arbitrary point in $\lim A_\nu$. Hence $\lim A_\nu$ is an analytic set of pure dimension $n - k$ in a neighborhood of each of its points. At the same time, $\lim A_\nu$ is closed in U. □

7.3. Proof of Theorem 7.4

Note that Theorem 7.4 is purely local. We may assume that $0 \in U$, $0 \in V$, and $V \setminus E \neq \emptyset$. Furthermore, we need only consider the case $0 \in E$. It then suffices to show that, for any sequence $(w^\nu) \subset V \setminus E$ with $w^\nu \to 0$, there is a constant $c > 0$ such that

$$\mathrm{vol}_{2(n-k)}(A_{w^\nu} \cap \tilde{U}) < c.$$

Observe that it is enough to consider the situation where $g_j(0,0) = 0$. We put

$$A_\nu := \{z : g_j(z, w^\nu) = 0\}$$

and, after applying Lemma 7.7, we obtain new coordinates $\tilde{w}$ (instead of w) as well as holomorphic functions $\tilde{g}_j(z, \tilde{w}) \in \mathcal{O}(U_1 \times \tilde{V})$ such that

$$A_\nu \subset \{z : \tilde{g}_j(z, \tilde{w}^\nu) = 0 \ \forall j = 1, \ldots, k\}.$$

As in Lemma 7.7, we put

$$A_0 := \{z : \tilde{g}_j(z, 0) = 0 \ \forall j = 1, \ldots, k\};$$

we know that A_0 is a purely $(n-k)$-dimensional analytic set with the property that the projection $\pi' \colon \mathbb{C}^n \to \mathbb{C}^{n-k}$ with $\pi'(z_1, \ldots, z_n) := (z_{k+1}, \ldots, z_n)$ is locally proper on A_0 near 0. As before, the limit $\lim A_\nu$ is the union of some irreducible components of A_0 and thus is an analytic set of pure dimension $n-k$. After an arbitrarily small change of coordinates, we even have that, for any $(n-k)$-tuple $I = (i_1, \ldots, i_{n-k})$ with $1 \le i_1 < \cdots < i_{n-k} \le n$, the projection $\pi_I \colon (z_1, \ldots, z_n) \to (z_{1_1}, \ldots, z_{i_{(n-k)}})$ is locally proper on A_0 near 0. Because of the analyticity of our family A_w, Rouché's theorem then implies that there exist $M > 0$ and $N > 0$ such that, for all multiindices I as described previously and for all $\nu \ge N$, the projections $\pi_I \colon A_\nu \to \mathbb{C}_I^{n-k}$ have local multiplicities $\le M$ near 0. Hence, the Wirtinger theorem implies

$$\mathrm{vol}(A_\nu \cap \tilde{U}) \le c \quad \forall \nu \ge N$$

for a suitable constant $c > 0$. This finishes the proof of the theorem. □

8. Conjecture 6.1 for Large Dimensions

In Lemma 5.9 we constructed for any $a \in M \setminus \Sigma$ a neighborhood $V \ni a$ and a sequence $\sigma_\nu \subset V$ of analytic sets such that

(i) $\dim \sigma_\nu = p \ge 1$,
(ii) $a \in \mathrm{cl}(\sigma_\nu)$, and
(iii) $\mathrm{cl}(\sigma_\nu) \subset E := M \setminus \Sigma \subset M$.

In order to finish the proof of Theorem 1.1 it therefore suffices to show that, for any sequence $\sigma_\nu \subset V$ of analytic sets satisfying (i) and (ii), condition (iii) can not be satisfied. (Then it follows that $M = \Sigma$.) This statement can easily be shown for the case $p = n-1$. However, from Lemma 5.9 we know only that $p \ge 1$.

Our goal is thus to establish for which dimensions $p \ge 1$ we really can exclude (iii); then we can refine the construction of Lemma 5.9 in order to obtain analytic sets σ_ν with sufficiently large dimension p.

The following result of Tumanov [33] will give us lower estimates for those p that still allow us to exclude condition (iii).

THEOREM 8.1 (Tumanov). *Let $N \subset U \subset \mathbb{C}^n$ be a closed real-analytic submanifold of finite type of an open set U. Then N can be stratified as $N = \bigcup_{j=1}^m N_j$ such that each stratum N_j is a real-analytic CR manifold and locally is contained in a Levi nondegenerate real real-analytic hypersurface.*

We need to apply this to a situation somewhat more general than our given hypersurface M from Theorem 1.1. Namely, we suppose that $N \subset V \subset \mathbb{C}^n$ is a closed real-analytic submanifold, pick an $a \in N$, and assume that there is a sequence of closed analytic sets $\sigma_\nu \subset V$ such that

(i) $\dim \sigma_\nu \equiv p \geq 1$,
(ii) $a \in \mathrm{cl}(\sigma_\nu)$, and
(iii) $\mathrm{cl}(\sigma_\nu) \subset N$.

PROPOSITION 8.2. *Let N and a be as before and suppose that $\sigma_\nu \subset V \subset \mathbb{C}^n$ is a sequence of analytic sets satisfying the conditions* (i) *and* (ii) *just stated with $p \geq n/2$. Then* $\mathrm{cl}(\sigma_\nu) \nsubseteq N$.

Proof. Consider Tumanov's stratification of N and rewrite it in the form $N = \bigcup_{j=1}^{2n-1} N_j$, where N_j is the union of all strata of dimension j from Tumanov's original stratification. Each N_j is locally contained in a real-analytic hypersurface $\tilde{M}_j$ with nondegenerate Levi form. Assume that $\mathrm{cl}(\sigma_\nu) \subset N$ and let j_0 be the largest index such that $\mathrm{cl}(\sigma_\nu) \cap N_j \neq \emptyset$. Then $\mathrm{cl}(\sigma_\nu) \cap N_{j_0} \neq \emptyset$, but $\mathrm{cl}(\sigma_\nu) \cap N_j = \emptyset$ for all $j > j_0$. We pick a point $b \in \mathrm{cl}(\sigma_\nu) \cap N_{j_0}$. After replacing V by a small neighborhood V_1 of b, we will have

$$\mathrm{cl}(\sigma_\nu) \cap V_1 \subset N_{j_0} \cap V_1 \subset \tilde{M}_{j_0} \cap V_1. \tag{8.1}$$

Without loss of generality, we may assume hereafter that $b = 0$ and $b \in \sigma_\nu$ for all ν (this is the case after small translations of σ_ν that do not destroy property (8.1) and follows from the fact that $b = 0 \in \mathrm{cl}(\sigma_\nu)$).

Since $\tilde{M}_{j_0}$ has a nondegenerate Levi form, there exists a complex linear subspace $L \ni 0$ of dimension $d \geq (n+1)/2$ such that L is transversal to $\tilde{M}_{j_0}$ at 0 and $L \cap \tilde{M}_{j_0}$ is a strictly pseudoconvex hypersurface in L near 0. For $\tilde{\sigma}_\nu := \sigma_\nu \cap L$ we have

(1) $\tilde{\sigma}_\nu \ni 0$,
(2) $\dim(\tilde{\sigma}_\nu) \geq 1$ at 0, and
(3) $\mathrm{cl}(\tilde{\sigma}_\nu) \subset \tilde{M}_{j_0} \cap L$.

This, however, contradicts Lemma 6.3 and Proposition 6.4(1). □

The next statement generalizes the previous proposition.

PROPOSITION 8.3. *Let $V \subset \mathbb{C}^n$ be an open set, $N \subset V$ a $\mathcal{C}^\omega$ real hypersurface of finite type, $A \subset V$ an analytic set of dimension $p \geq 1$, and (σ_ν) a sequence of* *analytic sets in V of pure dimension $p_1 \geq p/2$. Suppose that* $\emptyset \neq \mathrm{cl}(\sigma_\nu) \subset A$. *Then* $\mathrm{cl}(\sigma_\nu) \nsubseteq N$.

Proof. We stratify A in the form $A = A_0 \cup A_1 \cup \cdots \cup A_p$, where each A_d is a complex manifold of dimension d. Since $\mathrm{cl}(\sigma_\nu) \cap A \neq \emptyset$ we have $\mathrm{cl}(\sigma_\nu) \cap A_d \neq \emptyset$ for some d. Without loss of generality we may assume that $\mathrm{cl}(\sigma_\nu) \cap A_p \cap N \neq \emptyset$. Let $A_p \cap N = N_0 \cup N_1 \cup \cdots \cup N_{2p-1}$ be a stratification of the real-analytic set $A_p \cap N$ such that each N_j is a real-analytic manifold of dimension j, let j_0 be the largest index such that $\mathrm{cl}(\sigma_\nu) \cap N_{j_0} \neq \emptyset$, and let $a \in \mathrm{cl}(\sigma_\nu) \cap N_{j_0}$. Furthermore, choose a holomorphic projection $\pi : V \to A_p$ after possibly shrinking $V \ni a$. Then $\pi : V \cap \sigma_\nu \to A_p$ is proper for $\nu \gg 1$ and so $\tilde{\sigma}_\nu := \pi(\sigma_\nu) \subset A_p$ are analytic sets of pure dimension p. Since $\mathrm{cl}(\sigma_\nu) \subset A_p \cap V$ (we shrink V), we also have $\mathrm{cl}(\sigma_\nu) = \mathrm{cl}(\tilde{\sigma}_\nu) \subset A_p$. Because N_{j_0} is a real-analytic manifold of finite type, Proposition 8.2 tells us that $\mathrm{cl}(\sigma_\nu) = \mathrm{cl}(\tilde{\sigma}_\nu) \nsubseteq N_{j_0}$ near a. Since j_0 was chosen to be maximal, it follows that $\mathrm{cl}(\sigma_\nu) \nsubseteq N$. $\square$

9. Intersections of Segre Varieties

In this section we study intersection properties of Segre varieties of a closed real-analytic smooth real hypersurface $M \subset W \subset \mathbb{C}^n$ of finite type. We may assume that $0 \in M$ and set up the machinery needed for the Segre varieties as in Section 2. (We will later apply what we do here to the target manifold M' of Theorem 1.1.)

We introduce the following polarization process.

Definition 9.1. Let $0 \in S \subset Q_0 \cap U_1$ be an arbitrary subset. The set

$$S^* := \{w \in Q_0 \cap U_1 : Q_w \supset S\}$$

is called the *polarization* of S.

Lemma 9.2. *The polarization S^* of S can be characterized as*

$$S^* = \bigcap_{z \in S} Q_z \subset Q_0.$$

Proof. Notice that $w \in S^*$ iff $Q_w \supset S$. This means that $z \in Q_w$ for all $z \in S$, which is equivalent to having $w \in Q_z$ for all $z \in S$. This, however, is the same as $w \in \bigcap_{z \in S} Q_z$. $\square$

We have the following consequence.

Corollary 9.3. *The polarization $S^* \subset Q_0 \cap U_1$ is a complex-analytic set.*

Next we define the Segre completion of a set $S \subset Q_0$.

Definition 9.4. Let $0 \in S \subset Q_0 \cap U_1$ be an arbitrary subset and let

$$\mathcal{I}_S := \{\tilde{S} \subset U_1 : \tilde{S} \text{ is closed complex-analytic with } S \subset \tilde{S}\}.$$

The analytic set

$$\hat{S} := \bigcap_{\tilde{S} \in \mathcal{I}_S} \tilde{S}$$

is called the *Segre completion* of S.

LEMMA 9.5. *One always has $S \subset \hat{S}$ and $\hat{S} \cap S^* \subset M$.*

Proof. The fact that $S \subset \hat{S}$ is obvious. Next, take $w \in \hat{S} \cap S^*$. Then $Q_w \supset S$ and hence, according to the definition of $\hat{S}$, we have that even $Q_w \supset \hat{S} \ni w$. This shows $w \in M$. □

Since M is supposed to be of finite type, we obtain the following.

COROLLARY 9.6. *After possibly shrinking $U_1 \ni 0$, we have $\hat{S} \cap S^* = \{0\}$ and hence*

$$\dim \hat{S} + \dim S^* \leq n - 1.$$

In particular: If S contains a germ of a complex-analytic set at 0 of dimension p, then $\dim S^* \leq n - p - 1$.

We now come to a more specific study of intersections of Segre varieties. For this we assume again that $0 \in S \subset Q_0 \cap U_1$ is a closed subset and denote $m := \dim \hat{S}$. Furthermore, for any k-tuple $(w^1, \ldots, w^k)$ of points in S we define

$$q^k := \bigcap_{j=1}^{k} Q_{w^j}, \qquad \tilde{q}^k := q^k \cap Q_0. \tag{9.1}$$

LEMMA 9.7. *Suppose the k-tuple $(w^1, \ldots, w^k)$ of points from S has been chosen in such a way that*

$$\dim_0(\hat{S} \cap q^k) = m - k.$$

Then, for every irreducible component $q^\#$ of q^k at 0,

$$\dim_0(q^\#) = n - k \quad \textit{and} \quad \dim_0(\tilde{q}^k) = n - k - 1. \tag{9.2}$$

Proof. Obviously $\dim q^\# \geq n - k$. In order to show that $\dim q^k = n - k$, we choose an irreducible component $\hat{S}^\# \ni 0$ of $\hat{S}$ of dimension m. If $\dim q^\# > n - k$ then one also has $\dim(\hat{S}^\# \cap q^\#) > m - k$. This, however, contradicts the choice of the k-tuple $(w^1, \ldots, w^k)$, which was done in a such a way that $\dim_0(\hat{S} \cap q^k) = m - k$. Thus $\dim q^\# = n - k$.

In order to show that $\dim \tilde{q}^k = n - k - 1$, we assume by contradiction that $\dim \tilde{q}^k = n - k$. This would mean that there is a component $q^\#$ of q^k at 0 such that $q^\# \subset Q_0$. Since $\hat{S} \subset Q_0$, we have

$$\dim(q^\# \cap \hat{S}) \geq \dim q^\# + \dim \hat{S} - \dim Q_0 = n - k + m - (n - 1) = m - k + 1.$$

This contradicts the fact that $\dim(q^\# \cap \hat{S}) = m - k$. □

LEMMA 9.8. *Let S be as in Definition 9.1 and denote by m the dimension of the Segre completion $\hat{S}$ of S at 0. Then, after possibly shrinking U_1, there are points $w^1, \dots, w^k \in S$ $(k \le n-1)$ such that one of the following two cases holds true*:

(1) $k = m$ *and* $\dim(\hat{S} \cap Q_{w^1} \cap \cdots \cap Q_{w^k}) = 0$; *or*
(2) $k \ge 2m - n + 1$ *and* $\dim(\hat{S} \cap Q_{w^1} \cap \cdots \cap Q_{w^k}) = m - k$.

Proof. Let $0 \le k \le m$ be the largest integer for which the k-tuple $w^1, \dots, w^k \in S$ can be chosen in such a way that

$$\dim(\hat{S} \cap q^k) = m - k. \tag{9.3}$$

If $k = m$ then there is nothing to prove, so we assume that $k < m$. From (9.2) we have that $\dim q^k \equiv n - k$ and $\dim \tilde{q}^k = n - k - 1$. The maximality of k means that for any (additional) $z \in S$ there is at least one irreducible component $(q^k \cap \hat{S})^\#$ of $q^k \cap \hat{S}$ of dimension $m - k$ such that $(q^k \cap \hat{S})^\# \subset Q_z$. If we therefore denote by $\mathcal{Q}_k$ the collection of all irreducible components of $q^k \cap \hat{S}$ of dimension $m - k$, then using Definition 9.1 immediately yields

$$S \subset \bigcup_{q^\# \in \mathcal{Q}_k} q^{\#*};$$

because (according to Corollary 9.3) the sets $q^{\#*}$ are complex-analytic, by Definition 9.4 we can even surmise that

$$\hat{S} \subset \bigcup_{q^\# \in \mathcal{Q}_k} q^{\#*}. \tag{9.4}$$

Let now $\hat{S}^\#$ be an irreducible component of $\hat{S}$ of dimension m. Then (9.4) implies that there is one component $q^\# \in \mathcal{Q}_k$ such that even

$$\hat{S}^\# \subset q^{\#*}.$$

Notice that $\dim q^\# = m - k$. Hence, using Corollary 9.6 we obtain the following estimates for dimensions:

$$m = \dim \hat{S}^\# \le \dim q^{\#*} \le n - 1 - \dim q^\# = n - 1 - (m - k).$$

This implies

$$k \ge 2m - n + 1$$

and so finishes the proof of Lemma 9.8. □

10. Proof of Theorem 1.1

We now have all tools needed for finishing the proof of Theorem 1.1. After these preparations, we may assume that we are in the situation of Lemma 5.9. We want to show that $0 \in \Sigma$. For this we choose a sequence of points $a_\nu \in \Sigma$, $a_\nu \to 0$, and put $S_\nu := S(a_\nu, a_\nu) \subset U_1 \times U_1'$ as defined after Corollary 5.3. According to Proposition 5.8, it would suffice to show that $\pi(\mathrm{cl}(S_\nu)) \cap U \neq \emptyset$.

Put $S' := \pi'(\mathrm{cl}(S_\nu) \cap (\{0\} \times U_1')) \subset Q_0'$ and $m := \dim \hat{S}'$, where $\hat{S}'$ is given by Definition 9.4. If $m = 0$ then $0'$ is an isolated point of S'. After shrinking U_1 and U_1', the cluster set $\mathrm{cl}(S_\nu) \subset U_1 \times U_1'$ has no limit points on $U_1 \times \partial U_1'$ and the projections $\pi : S_\nu \to U_1$ are proper for $\nu \gg 1$. Thus $\pi(S_\nu) = Q_{a_\nu} \cap U_1$ and $\pi(\mathrm{cl}(S_\nu)) = Q_0 \cap U_1$. If $0 \in M \setminus \Sigma$ then Lemma 5.10 applied to $\sigma_\nu := \pi(S_\nu)$ would imply that $Q_0 \cap U_1 = \mathrm{cl}(\sigma_\nu) \subset M \setminus \Sigma \subset M$. However, this is not the case, since M is of finite type. We must therefore have $\pi(\mathrm{cl}(S_\nu) \cap U) \neq \emptyset$ and $0 \in \Sigma$ in this case ($m = 0$).

From now on we may assume that $m > 0$ and apply Lemma 9.8 to S'. There are two possibilities which we will consider as separate cases.

Case 1. There are m points $w'^1, \dots, w'^m \in S'$ such that

$$\dim(\hat{S}' \cap q'^m) = 0 \tag{10.1}$$

for $q'^m := Q'_{w'^1} \cap \dots \cap Q'_{w'^m}$. In this case, for every $\nu \in \mathbb{N}$ we can find, associated to the given m-tuple $(w'^1, \dots, w'^m)$, m-tuples $(w^{1\nu}, \dots, w^{m\nu})$ and $(w'^{1\nu}, \dots, w'^{m\nu})$ such that $(w^{\mu\nu}, w'^{\mu\nu}) \in S_\nu \subset Q_{a_\nu} \times Q'_{f(a_\nu)}$ and $w^{\mu\nu} \in U$, with

$$w^{\mu\nu} \xrightarrow[\nu\to\infty]{} 0 \quad \text{and} \quad w'^{\mu\nu} \xrightarrow[\nu\to\infty]{} w'^{\mu} \quad \text{for } \mu = 1, \dots, m.$$

Furthermore, we claim that we may also assume that

$$q^{m\nu} := Q_{w^{1\nu}} \cap \dots \cap Q_{w^{m\nu}} \text{ has dimension } n - m. \tag{10.2}$$

Indeed, suppose that, for some $k \in \{1, \dots, m-1\}$, we already have $\dim(Q_{w^{1\nu}} \cap \dots \cap Q_{w^{k\nu}}) = n - k$. If

$$\dim(Q_{w^{1\nu}} \cap \dots \cap Q_{w^{k\nu}} \cap Q_{w^{k+1,\nu}}) = n - k$$

for the next point $w^{k+1,\nu}$, then for some irreducible component $q_w^{\#\nu}$ of $q^{k\nu} := Q_{w^{1\nu}} \cap \dots \cap Q_{w^{k\nu}}$ we have

$$w^{k+1,\nu} \in (q_w^{\#\nu})^*. \tag{10.3}$$

Let, furthermore, $q^{\#\nu}$ be any irreducible component of $q^{k\nu}$. Since $\dim q^{\#\nu} = n - k \geq n - (m-1) \geq 1$, by Corollary 9.6 we have $\dim(q^{\#\nu})^* \leq n - 2$. Hence

$$\dim\Big(\bigcup (q^{\#\nu})^*\Big) \leq n - 2,$$

where the union is taken over all irreducible components of $q^{k\nu}$. Notice, however, that $w^{k+1,\nu} \in \pi(S_\nu)$ and that $\pi(S_\nu) \subset Q_{a^\nu} \cap U$ is relatively open. We can therefore move the original point $w^{k+1,\nu}$ slightly to obtain a point $w^{k+1,\nu} \in \pi(S_\nu) \setminus \bigcup (q^{\#\nu})$. Because of (10.3), we then have

$$\dim(Q_{w^{1\nu}} \cap \dots \cap Q_{w^{k\nu}} \cap Q_{w^{k+1,\nu}}) = n - k - 1. \tag{10.4}$$

The property $w'^{k+1,\nu} \xrightarrow[\nu\to\infty]{} w'^{k+1}$ will not be destroyed by this if we move by only a sufficiently small amount. After $m - 1$ steps we will have $\dim(q^{m\nu}) = n - m$. This proves (10.2).

We now consider

$$S_\nu^m := \bigcap_{\mu=1}^{m} S(w^{\mu\nu}, a^\nu) \subset (q^{m\nu} \times q'^{m\nu}) \cap (U_1 \times U_1').$$

Since $\dim \hat{S}' = m$ and $\dim(\hat{S}' \cap q'^m) = 0$, by Lemma 9.7 we have

$$\dim q'^m \equiv n - m. \tag{10.5}$$

Subcase 1a: $m = n - 1$. In this situation we obviously have $\hat{S}' = Q'_{0'}$ and, as a result, $\dim(q'^{n-1}) = 0$ such that Lemma 9.8 implies $\dim(q'^{n-1} \cap Q'_{0'}) = 0$. Since $\mathrm{cl}(q'^{n-1,\nu}) \subset q'^{n-1}$ and $\mathrm{cl}_{\hat{F}}(0) \subset Q'_{0'}$ (see Lemma 4.10(ii)), it follows that $0'$ is an isolated point of

$$S'^{n-1} := \pi'(\mathrm{cl}(S^{n-1}_\nu) \cap (\{0\} \times U'_1)) \subset q'^{n-1} \cap Q'_{0'} = \{0'\}.$$

By Lemma 4.10(ii), this means that (after shrinking U_1) the projections

$$\pi : S^{n-1}_\nu \to U_1$$

are proper for $\nu \gg 1$ and thus

$$\pi(S^{n-1}_\nu) = q^{n-1,\nu} \cap U_1.$$

At this point, the convergence criterion for families of analytic sets (from Theorem 7.4) becomes important. It tells us that, after passing to a subsequence, the sequence $q^{m\nu} \cap U_1$—which does, indeed, (anti-)analytically depend on the m-tuple of points defining it—converges to an analytic set $A \subset U_1$ of pure dimension 1 with $0 \in A$. Since M is of finite type, the set A contains a point $\zeta^0 \notin M$ and

$$\zeta^0 \in \pi(\mathrm{cl}(S^{n-1}_\nu)) \subset \pi(\mathrm{cl}(S(w'^\nu, a^\nu))).$$

Together with Lemma 5.10, this yields $0 \in \Sigma$.

Subcase 1b: $m < n - 1$. We remind the reader of our notation $S_\nu = S(a^\nu, a^\nu)$ and put $\tilde{S}^m_\nu := S^m_\nu \cap S_\nu$. We have $\dim \tilde{S}^m_\nu \geq n - m - 1 > 0$. Since we still are in Case 1, the point $0'$ is an isolated point in $\tilde{S}'^m := \pi'(\mathrm{cl}(\tilde{S}^m_\nu) \cap (\{0\} \times U'_1)) \subset q'^m \cap \hat{S}' = \{0'\}$. Hence, for appropriate choices of U_1 and U'_1, the projections $\pi : \tilde{S}^m_\nu \to U_1$ are proper. We also have $\pi(\tilde{S}^m_\nu) = q^{m\nu} \cap Q_{a^\nu} \cap U_1$ and, after possibly passing to a subsequence, the sets $q^{m\nu} \cap Q_{a^\nu} \cap U_1$ converge to an analytic set A of positive dimension. The same arguments as in Subcase 1a now show that $0 \in \Sigma$. This finishes Case 1.

Case 2. There exist $k < m$ points $w'^1, \dots, w'^k \in S'$ such that $\dim(\hat{S}' \cap q'^k) = m - k$ and $k \geq 2m - n + 1$ ($q'^k := Q'_{w'^1} \cap \cdots \cap Q'_{w'^k}$).

As done in Case 1, we can find two sequences of k-tuples $(w^{1\nu}, \dots, w^{k\nu})$ and $(w'^{1\nu}, \dots, w'^{k\nu})$, $\nu = 1, 2, 3, \dots$, such that one has, for all ν and $\mu = 1, \dots, m$:

(i) $w^{\mu\nu} \in Q_{a_\nu} \cap U$, $w^{\mu\nu} \to 0$ as $\nu \to \infty$;
(ii) $w'^{\mu\nu} \in Q'_{f(a_\nu)}$, $w'^{\mu\nu} \to w'^\mu$ as $\nu \to \infty$;
(iii) $(w^{\mu\nu}, w'^{\mu\nu}) \in S_\nu \subset Q_{a_\nu} \times Q'_{f(a_\nu)}$;
(iv) the sets $q^{k\nu} := Q_{w^{1\nu}} \cap \cdots \cap Q_{w^{k\nu}}$ and $\tilde{q}^{k\nu} := Q_{a_\nu} \cap q^{k\nu}$ have dimensions $n - k$ and $n - k - 1$, respectively (Lemma 9.8).

Analogously, we put

$$q'^{k\nu} := \left(\bigcap_{\mu=1}^{k} Q'_{w'^{\mu\nu}} \right) \cap U'_1 \quad \text{and} \quad \tilde{q}'^{k\nu} := q'^{k\nu} \cap Q'_{f(a_\nu)}.$$

By Theorem 7.4, we may assume that the sequence $(q^{k\nu})$ converges to an analytic set $A \subset U_1$ and that the sequence $(\tilde{q}^{k\nu})$ converges to an analytic set $\tilde{A} \subset U_1$, where

$$\dim A = n - k \quad \text{and} \quad \dim \tilde{A} = n - k - 1. \tag{10.6}$$

We introduce

$$S_\nu^k := \bigcap_{\mu=1}^{k} S(w^{\mu\nu}, a^\nu) \subset (q^{k\nu} \times q'^{k\nu}) \cap (U_1 \times U_1').$$

Since $k < m$, the inequality $k \geq 2m - n + 1$ implies that $m < n - 1$ and $k < n - 2$. Thus $\dim \tilde{S}_\nu^k = n - k - 1 > 1$. Since $\dim(\hat{S}' \cap q'^k) = m - k$, there exists a coordinate system in $\mathbb{C}^n$ such that

$$\hat{S}' \cap q'^k \cap \{z' \in U_1' : z_1' = \cdots = z_{m-k}' = 0\} = \{0'\}.$$

Now we consider the sets

$$T_\nu := \{(z, z') \in \tilde{S}_\nu^k : z_1' = a_1'^\nu, \ldots, z_{m-k}' = a_{m-k}'^\nu\},$$

where $a_\nu' = (a_{\nu 1}', \ldots, a_{\nu n}') := f(a_\nu)$. Since $\dim \tilde{S}_\nu^k = n - k - 1$ and $(a_\nu, a_\nu') \in \tilde{S}_\nu^k$, the sets T_ν are analytic sets in $U_1 \times U_1'$ of dimension $n - k - 1 - (m - k) = n - m - 1 > 0$. We also have

$$\tilde{s}'^m := \pi'(\mathrm{cl}(T_\nu) \cap (\{0\} \times U_1')) \subset \hat{S}' \cap q'^k \cap \{z' : z_1' = \cdots = z_{m-k}' = 0\} = \{0\}.$$

Consequently, for appropriate U_1 and U_1', the projections $\pi : T_\nu \to U_1$ are proper.

We now redefine the sets σ_ν from Section 5 by putting

$$\sigma_\nu := \pi(T_\nu).$$

We have:

(1) the sets σ_ν are analytic in U_1;
(2) $\sigma_\nu \subset \tilde{q}^{k\nu}$;
(3) $\dim \sigma_\nu = n - m - 1$ whereas $\dim \tilde{q}^{k\nu} = n - k - 1$;
(4) $\mathrm{cl}(\sigma_\nu) \subset \tilde{A}$.

Property (4), together with (10.6) and Lemma 9.8, yields the following estimates of dimensions:

$$2(n - m - 1) = 2n - 2m - 2 \geq 2n - 2 - k - n + 1 = n - k - 1,$$

implying that

$$\dim \sigma_\nu \geq \tfrac{1}{2} \dim \tilde{A}$$

Therefore, applying Proposition 8.3 with $N := M$ immediately gives

$$\mathrm{cl}\, \sigma_\nu \not\subset M. \tag{10.7}$$

We are now in the following situation: the sequences $((w^{1\nu}, a_\nu))$ and (σ_ν) satisfy all the properties stated in Lemma 5.9. Hence Lemma 5.10 applies, and this together with (10.7) implies (by contradiction) that $0 \in \Sigma$. This finishes the proof of Theorem 1.1. □

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