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Affine Dimension: Measuring the Vestiges of Curvature

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1. Introduction

The purpose of this paper is to introduce a set function, which we call affine dimension, and then apply it to the study of convex curves in $\mathbb{R}^2$. Though we believe the geometric results obtained here—as well as questions concerning their higher-dimensional generalizations—to be intrinsically interesting, the original motivation for introducing affine dimension stems from its connection (as a natural necessary condition) to certain important problems in harmonic analysis. This connection is spelled out in Proposition 2 at the end of this section (see also Propositions 3 and 4 in Section 3). For earlier applications to harmonic analysis of special cases of the affine measures introduced here, see [1; 2; 7; 8].

The definition of affine dimension is largely analogous to the definition of Hausdorff dimension. But there is an important difference that, as we shall see, renders affine dimension sensitive to curvature. For example, the Hausdorff dimensions of a circle and a line segment are equal, but their affine dimensions will differ. The following observation is intended to motivate the definition of affine dimension: For $p \geq 2$, consider the smallest rectangle containing the portion of the curve (t, t^p) corresponding to $0 \leq t \leq \varepsilon$; that rectangle has measure on the order of ε^{1+p}, which tends to 0 as p increases—that is, as the curve (t, t^p) becomes "flatter". Thus, for $E \subseteq \mathbb{R}^n$ and $\alpha, \delta > 0$, we consider sums of the form $\sum |R_j|^{\alpha/n}$, where $E \subseteq \bigcup R_j$, each R_j is a rectangle of diameter $< \delta$, and $|R_j|$ is the Lebesgue measure of R_j. By analogy to the definition of Hausdorff measures, we define $A^\alpha_\delta(E)$ to be the infimum of such sums. Next we define

$$A^\alpha(E) \doteq \lim_{\delta \to 0^+} A^\alpha_\delta(E).$$

One sees in the usual way that A^α is an outer measure on $\mathbb{R}^n$ that restricts to a measure on the σ-algebra of Borel subsets of $\mathbb{R}^n$. We will refer to this measure as α-dimensional affine measure on $\mathbb{R}^n$. The equivalent definition with parallelepipeds instead of rectangles is clearly invariant under equiaffine transformations (as defined in [5], these are the affine mappings on $\mathbb{R}^n$ that preserve Lebesgue measure). Finally, we define $\dim_a(E)$ to be the infimum of $\{\alpha > 0 : A^\alpha(E) < \infty\}$. If $\dim(E)$ stands for the Hausdorff dimension of $E \subseteq \mathbb{R}^n$, then it is clear from the

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definitions that $\dim_a(E) \leq \dim(E)$. It is easy to check that if E is contained in an $(n-1)$-dimensional hyperplane then $\dim_a(E) = 0$. On the other hand, if Γ is any sufficiently smooth curve in $\mathbb{R}^2$ that is not a line segment, we will see that $\dim_a(\Gamma) = \frac{2}{3}$.

The main results of this paper deal with a class of convex curves Γ in $\mathbb{R}^2$. The curves we consider will be graphs over finite closed intervals $[a, b]$ of continuous convex functions ϕ. Any such ϕ will be absolutely continuous with nondecreasing derivative ϕ'. When Γ is (say) $C^{(2)}$, there is a well-known relationship between the curvature of Γ and ϕ''. Under our less restrictive requirement that ϕ be continuous and convex, ϕ'' will still exist pointwise a.e. with respect to Lebesgue measure. However, the role of ϕ'' in considerations of curvature will be taken by the Lebesgue–Stieltjes measure μ induced on (a, b) by the nondecreasing function ϕ'. (Technically, as ϕ' may exist only almost everywhere on $[a, b]$, we consider $\mu([x, y))$ to be defined by the equation

$$\mu([x, y)) = \lim_{t \to y^-} \phi'(t) - \lim_{t \to x^-} \phi'(t)$$

for $a < x < y < b$.) In what follows, $[a, b]$, ϕ, Γ, and μ will be as just described. We will also use the notation $\mathrm{Lip}(\alpha)$ to stand for the space of functions f on $[a, b]$ satisfying a Lipschitz condition of order α.

THEOREM 1. *The inequality* $\dim_a(\Gamma) \leq \frac{2}{3}$ *holds.*

THEOREM 2. *Suppose that, for some* $\alpha, \beta \in (0, 1)$, *we have* $\phi' \in \mathrm{Lip}(\alpha)$ *and* $\dim(\mathrm{supp}(\mu)) = \beta$. *Then* $\dim_a(\Gamma) \leq \frac{2\beta}{2+\alpha}$.

THEOREM 3. *If* $\dim_a(\Gamma) < \frac{2}{3}$, *then* μ *is singular with respect to Lebesgue measure on* $[a, b]$.

COROLLARY. *If* ϕ'' *exists everywhere on* $[a, b]$ *and if* $\dim_a(\Gamma) < \frac{2}{3}$, *then* Γ *is a line segment.*

THEOREM 4. *If* $\phi' \in \mathrm{Lip}(\alpha)$ *for some* $\alpha \in (0, 1)$ *and if* $\dim_a(\Gamma) < \frac{2\alpha}{2+\alpha}$, *then* Γ *is a line segment.*

THEOREM 5. *There is a constant* C *such that—if* $\phi \in C^{(2)}$, λ *is the measure on* Γ *given by* $d\lambda = \phi''(x)^{1/3}dx$, *and* ν *is the restriction to* Γ *of* $\frac{2}{3}$*-dimensional affine measure—then* $C^{-1}\lambda \leq \nu \leq C \cdot \lambda$.

COMMENTS. (a) It follows from the Corollary to Theorem 3 that if Γ is a (not necessarily convex) $C^{(2)}$ curve with $\dim_a(\Gamma) < \frac{2}{3}$ then Γ is a line segment. This is analogous to the fact that if Γ is *any* curve (continuous image of $[0, 1]$) with $\dim(\Gamma) < 1$ then Γ is a point.

(b) It follows from Theorems 2 and 4 that if $\alpha \in (0, 1)$, $\dim(\mathrm{supp}(\mu)) = \alpha$, and $\phi' \in \mathrm{Lip}(\alpha)$, then $\dim_a(\Gamma) = \frac{2\alpha}{2+\alpha}$. Thus, for example, if ϕ' is the Cantor–Lebesgue function on $[0, 1]$ (see e.g. [3, p. 38]) then $\dim_a(\Gamma) = \frac{\log 4}{\log 18}$. More generally, for any $\tau \in \left[0, \frac{2}{3}\right]$ there will be Γ with $\dim_a(\Gamma) = \tau$. The curves Γ

produced in this manner will have the property that $[a, b]$ has a dense open subset such that, over each of its components, Γ is a line segment. Rather different examples of Γ with $\dim_a(\Gamma) = \tau$ will be the subject of Section 3.

(c) We will see in Section 3 (comment (i) after the proof of Proposition 4) that there are curves Γ with $\dim_a(\Gamma) = \frac{2}{3}$ and μ discrete. This bears on the question of a converse to Theorem 3.

(d) The measure λ in Theorem 5 is called *affine arclength* measure on Γ; Theorem 5 is the statement that, on curves Γ, affine arclength is uniformly equivalent to $\frac{2}{3}$-dimensional affine measure. This is analogous to the relationship between Euclidean arclength and 1-dimensional Hausdorff measure.

(e) We will address questions concerning higher-dimensional analogues of some of these results in a later paper.

The remainder of this paper is organized as follows. Section 2 contains proofs for the results just stated, and in Section 3 we construct some interesting curves and look at certain of their geometric and harmonic-analytic properties. In the balance of this section we complete the comparison between affine dimension and Hausdorff dimension and then point out the relationship between affine dimension and certain problems in harmonic analysis.

PROPOSITION 1. *Suppose $n \geq 2$ and $E \subseteq \mathbb{R}^n$. Then*

$$\max\{0, n(\dim(E) - n + 1)\} \leq \dim_a(E) \leq \dim(E).$$

Proof. It is enough to establish the lower bound for $\dim_a(E)$. Suppose that $\dim_a(E) < \alpha \leq n$. Then, for any $\delta \in \left(0, \frac{1}{2}\right)$, there is a covering of E by a countable collection of rectangles R_j such that each R_j has diameter bounded by δ and such that

$$\sum |R_j|^{\alpha/n} \leq 1.$$

Let δ_j be the smallest of the side lengths of R_j and choose nonnegative integers N_l^j such that, if $s_2, \ldots, s_n$ are the other side lengths of R_j, then

$$(N_l^j - 1)\delta_j < s_l \leq N_l^j \delta_j \quad \text{for } l = 2, \ldots, n.$$

Let

$$M_j = \prod_{l=2}^{n} N_l^j$$

and note that

$$M_j \leq \delta_j^{-(n-1)} \tag{1}$$

since $N_l^j \delta_j \leq 1$ (since the diameter of R_j is bounded by $\frac{1}{2}$). Now R_j is contained in the union of M_j cubes of side length δ_j. If we set $\beta = \alpha/n + n - 1$, then it follows from (1) that

$$M_j \delta_j^{\beta} \leq (M_j \delta_j^n)^{\alpha/n}.$$

Thus $M_j \delta_j^{\beta} \leq C|R_j|^{\alpha/n}$ for some constant C depending only on n. Then E is contained in a union of cubes C_k such that each C_k has diameter bounded by δ and such that

$$\sum \operatorname{diam}(C_k)^\beta \le C \sum |R_j|^{\alpha/n} \le C.$$

Since this is true for every $\delta \in \left(0, \frac{1}{2}\right)$ and for every α with $\dim_a(E) < \alpha \le n$, it follows from $\beta = \alpha/n + n - 1$ that $\dim(E) \le \dim_a(E)/n + n - 1$. $\square$

It is neither very difficult nor very interesting to check that the bounds in Proposition 1 cannot be improved.

Next we consider a relationship between affine dimension and two problems in harmonic analysis. Let λ be a nonnegative Borel measure on $\mathbb{R}^n$. The two problems are:

(C) determine the indices p and q such that the convolution estimate

$$\|\lambda * f\|_q \le C(\lambda, p, q)\|f\|_p \tag{2}$$

holds for $f \in L^p(\mathbb{R}^n)$;

(R) determine the indices p and q such that the Fourier restriction estimate

$$\|\hat{f}\|_{L^q(\lambda)} \le C(\lambda, p, q)\|f\|_{L^p} \tag{3}$$

holds for $f \in L^p(\mathbb{R}^n)$.

We have the following result.

PROPOSITION 2. *Suppose the nonnegative and nontrivial Borel measure λ is supported on E. If* (2) *holds, then* $\dim_a(E) \ge n(1/p - 1/q)$. *If* (3) *holds, then* $\dim_a(E) \ge nq(1 - 1/p)$.

Proof. Assume that (2) holds. Let R be a rectangle in $\mathbb{R}^n$ and set $R' = R - R$. Then

$$|R|\langle\lambda, \chi_R\rangle \le \langle\lambda, \chi_R * \chi_{R'}\rangle = \langle\lambda * \chi_{R'}, \chi_R\rangle \le C(\lambda, p, q)|R|^{1/p}|R'|^{1-1/q}.$$

Thus, if $\alpha = n(1/p - 1/q)$ then there exists a C such that $\lambda(R) \le C|R|^{\alpha/n}$. So if $E \subseteq \bigcup R_j$, it follows that $\|\lambda\| \le C \sum |R_j|^{\alpha/n}$. Hence $\dim_a(E) \ge n(1/p - 1/q)$ as desired. The second assertion will follow similarly from the inequality $\lambda(R) \le C|R|^{q - q/p}$. And this is a consequence of (3) and the existence of f such that

$$\chi_R \le \hat{f} \quad \text{and} \quad \|f\|_p \le C|R|^{1-1/p}.$$

(If R is centered at the origin and has axes parallel to the coordinate axes, then such an f can be obtained by taking $\hat{f} = \prod_i \exp(-a_i x_i^2)$ for suitable a_i; one deals with the general case by translation and rotation.) $\square$

2. Proofs

Proof of Theorem 1

By replacing (a, b) with any (a', b') such that $a < a' < b' < b$, we may assume that ϕ is Lipschitz on $[a, b]$. Suppose $a \le c \le d \le b$. Consider the region R' in $\mathbb{R}^2$ given by

$$\left\{(x, y) : c \le x \le d,\ \phi(x) \le y \le \left(\phi(c) + (x-c)\frac{\phi(d)-\phi(c)}{d-c}\right)\right\}$$

and the smallest rectangle R containing R' and whose top edge is the segment forming the top part of the boundary of R'. (Such an R will exist if, for example, ϕ is monotone on $[c, d]$.) We will say that "R is the rectangle over $[c, d]$". Then

$$\begin{aligned}
|R| \le 2|R'| &= 2\int_c^d \left(\phi(c) + (x-c)\frac{\phi(d)-\phi(c)}{d-c} - \phi(x)\right) dx \\
&\le 2\int_c^d \int_c^x \left|\frac{\phi(d)-\phi(c)}{d-c} - \phi'(s)\right| ds\, dx \\
&= \frac{2}{d-c}\int_c^d \int_c^x \left|\int_c^d (\phi'(t) - \phi'(s))\, dt\right| ds\, dx \\
&\le 2(d-c)^2 \mu([c, d)).
\end{aligned}$$

Now fix $\delta > 0$ and let $\{x_j\}$ be a partition of $[a, b]$ such that $x_j - x_{j-1} < \delta$ for all j and such that ϕ is monotone on each $[x_{j-1}, x_j]$. Let R_j be the rectangle over $[x_{j-1}, x_j]$. Then

$$\sum |R_j|^{1/3} \le \sum (x_j - x_{j-1})^{2/3}\mu([x_{j-1}, x_j))^{1/3} \le (b-a)^{2/3}\mu([a, b))^{1/3}$$

by Hölder's inequality. Since ϕ is Lipschitz, the diameter of R_j is $O(\delta)$. Thus $\dim_a(\Gamma) \le \frac{2}{3}$. □

Proof of Theorem 2

Fix $\beta' > \beta$. Since $\dim(\mathrm{supp}(\mu)) \le \beta$, for any $\delta > 0$ one can find a finite covering of $\mathrm{supp}(\mu)$ by disjoint intervals $[a_j, b_j]$ with each $b_j - a_j < \delta$ and with

$$\sum (b_j - a_j)^{\beta'} \le 1, \tag{4}$$

say. If R_j is the rectangle over $[a_j, b_j]$ then, as in the proof of Theorem 1,

$$|R_j| \le (b_j - a_j)^2 \mu([a_j, b_j)) \le C(b_j - a_j)^{(2+\alpha)},$$

where the last inequality follows from the assumption $\phi' \in \mathrm{Lip}(\alpha)$. Thus

$$\sum |R_j|^{\frac{\beta'}{2+\alpha}} \le C \tag{5}$$

by (4). Note also that, since ϕ is Lipschitz, each R_j has diameter $O(\delta)$. Now $\Gamma \sim \bigcup R_j$ is a finite union of line segments and can therefore be covered by a finite union of rectangles S_l each having diameter $< \delta$ and such that

$$\sum |S_l|^{\frac{\beta'}{2+\alpha}} \le 1.$$

With (5) this shows that $\dim_a(\Gamma) \le \frac{2\beta'}{2+\alpha}$. Since $\beta' > \beta$ was arbitrary, $\dim_a(\Gamma) \le \frac{2\beta}{2+\alpha}$ as desired. □

Proof of Theorem 3

We will require an elementary covering lemma whose proof we omit.

LEMMA. *Given a finite collection of closed intervals, there is a subcollection with the same union and with the property that no point belongs to more than two intervals of the subcollection.*

Find $\delta > 0$ such that $\dim_a(\Gamma) < \frac{2}{3+\delta}$. Writing m for Lebesgue measure on $\mathbb{R}$, assume that

$$(s\chi_E)dm \leq d\mu \tag{6}$$

for some $s > 0$ and some Borel $E \subseteq \mathbb{R}$. It will be enough to show that $m(E) = 0$. Fix $\varepsilon > 0$ and find a finite collection $\{I_j\}$ of pairwise disjoint closed intervals such that

$$m\Big(E\Delta\Big(\bigcup I_j\Big)\Big) < \varepsilon.$$

Let β be the infimum of $\{\mathrm{dist}(I_j, I_k) : j \neq k\}$. Cover Γ by a finite collection $\{R_j\}$ of closed rectangles each having diameter $< \min(\varepsilon, \beta)$ and such that

$$\sum |R_j|^{\frac{1}{3+\delta}} \leq 1.$$

This is possible since $\dim_a(\Gamma) < \frac{2}{3+\delta}$. Let $\{[a_l, b_l]\}$ be the collection of intervals obtained by projecting the intersections $\Gamma \cap R_j$ onto the x-axis. Each R_j will give rise to at most three such intervals. Suppose $[a_l, b_l]$ and R_j are associated in this way. Since the map

$$(x, y) \mapsto (x - y, \phi(x) - \phi(y))$$

takes $[a_l, b_l] \times [a_l, b_l]$ into $R_j - R_j$ and has Jacobian equal to $|\phi'(x) - \phi'(y)|$, it follows that

$$\int_{a_l}^{b_l}\int_{a_l}^{b_l} |\phi'(x) - \phi'(y)|\, dx\, dy \leq |R_j - R_j| = 4|R_j|.$$

Because

$$\int_{a_l}^{b_l}\int_{y}^{b_l} |\phi'(x) - \phi'(y)|\, dx\, dy = \int_{[a_l, b_l)} (b_l - u)(u - a_l)\, d\mu(u),$$

it follows that

$$\sum \Big(\int_{[a_l, b_l)} (b_l - u)(u - a_l)\, d\mu(u)\Big)^{\frac{1}{3+\delta}} \leq C \sum |R_j|^{\frac{1}{3+\delta}} \leq C$$

$\big($where $C = 2^{\frac{1}{3+\delta}}\big)$. Now each $[a_l, b_l]$ intersects at most one of the intervals I_k, since the diameter of each R_j is $< \beta$. By applying the covering lemma and then shrinking or discarding each remaining $[a_l, b_l]$ as necessary, we replace $\{[a_l, b_l]\}$ by a collection $\{[c_k, d_k]\}$ satisfying

$$\sum \chi_{[c_k,d_k]} \le 2,$$

$$\bigcup I_k = \bigcup [c_k, d_k],$$

and

$$\sum \left(\int_{[c_k,d_k)} (d_k - u)(u - c_k)\, d\mu(u) \right)^{\frac{1}{3+\delta}} \le C. \tag{7}$$

Let $[\tilde{c}_k, \tilde{d}_k]$ have the same center as $[c_k, d_k]$ but $\frac{9}{10}$ the length. We will need the fact that

$$\sum m([\tilde{c}_k, \tilde{d}_k] \cap E) \ge \frac{4}{5} m(E) - 2\varepsilon. \tag{8}$$

To see this, note that

$$\sum (d_k - c_k) \le 2 \sum m(I_j),$$

so

$$\sum m([c_k, d_k] \sim [\tilde{c}_k, \tilde{d}_k]) \le \frac{1}{5} \sum m(I_j).$$

Then

$$\begin{aligned} m\Big(E \cap \bigcup [\tilde{c}_k, \tilde{d}_k]\Big) &\ge m\Big(\bigcup I_k \cap \bigcup [\tilde{c}_k, \tilde{d}_k]\Big) - \varepsilon \\ &\ge m\Big(\bigcup I_k \cap \bigcup [c_k, d_k]\Big) - \sum m([c_k, d_k] \sim [\tilde{c}_k, \tilde{d}_k]) - \varepsilon \\ &\ge m\Big(\bigcup I_k \cap \bigcup [c_k, d_k]\Big) - \frac{1}{5} \sum m(I_k) - \varepsilon \\ &= \frac{4}{5} m\Big(\bigcup I_k\Big) - \varepsilon \ge \frac{4}{5}(m(E) - \varepsilon) - \varepsilon. \end{aligned}$$

Now let $\tau = \frac{5+2\delta}{6+2\delta}$ so that $2(3\tau - 2) = \frac{6+4\delta}{6+2\delta}$ and $2(1-\tau) = \frac{1}{3+\delta}$. Then, by (8) and (6),

$$\begin{aligned} \frac{4}{5} m(E) - 2\varepsilon \le \sum m([\tilde{c}_k, \tilde{d}_k] \cap E) &\le \sum (d_k - c_k)^{\tau} \left(\frac{\mu([\tilde{c}_k, \tilde{d}_k] \cap E)}{s} \right)^{1-\tau} \\ &= \sum (d_k - c_k)^{\tau - 2(1-\tau)} \left(\frac{\mu([\tilde{c}_k, \tilde{d}_k] \cap E)}{s} (d_k - c_k)^2 \right)^{1-\tau} \\ &\le \frac{C}{s^{1-\tau}} \sum (d_k - c_k)^{3\tau - 2} \left(\int_{[c_k,d_k)} (d_k - u)(u - c_k)\, d\mu(u) \right)^{1-\tau}. \end{aligned}$$

By the Schwarz inequality and the definition of τ, this last sum is dominated by

$$\frac{C}{s^{1-\tau}} \left(\sum (d_k - c_k)^{\frac{6+4\delta}{6+2\delta}} \right)^{\frac{1}{2}} \left(\sum \left(\int_{[c_k,d_k)} (d_k - u)(u - c_k)\, d\mu(u) \right)^{\frac{1}{3+\delta}} \right)^{\frac{1}{2}}. \tag{9}$$

Since we have $d_k - c_k < \varepsilon$ for each k (the diameter of each R_j is $< \varepsilon$),

$$\sum (d_k - c_k)^{\frac{6+4\delta}{6+2\delta}} \le \varepsilon^{\frac{2\delta}{6+2\delta}} \sum (d_k - c_k) \le 2\varepsilon^{\frac{2\delta}{6+2\delta}} (b - a).$$

With (7) and (9) this leads to

$$\frac{4}{5}m(E) - 2\varepsilon \leq \frac{C}{s^{1-\tau}}\varepsilon^{\frac{\delta}{6+2\delta}}(b-a)^{\frac{1}{2}}.$$

Since ε is arbitrary, it follows that $m(E) = 0$ as desired. □

Proof of Corollary

If ϕ' is differentiable at each point of $[a, b]$ then it follows that $d\mu = \phi''(x)dx$. □

Proof of Theorem 4

Suppose $0 < \beta < \alpha/2$. Since $\phi' \in \mathrm{Lip}(\alpha)$ it is easy to see that there exists a C (depending on ϕ and β) such that, if $a \leq c < d \leq b$, then

$$\int_c^d \frac{d\mu(u)}{((d-u)(u-c))^\beta} \leq C(d-c)^{\alpha-2\beta}. \tag{10}$$

Now fix ν with $\dim_a(\Gamma) < \nu < \frac{2\alpha}{2+\alpha}$ and fix $\delta > 0$. Cover Γ by rectangles R_j each having diameter $< \delta$ and such that

$$\sum |R_j|^{\nu/2} \leq 1.$$

Let the intervals $\{[a_l, b_l]\}$ be the projections of the intersections $\Gamma \cap R_j$ onto the x-axis. Then each of the $[a_l, b_l]$ has length $< \delta$ and, as in the proof of Theorem 3,

$$\sum \left(\int_{a_l}^{b_l} (b_l - u)(u - a_l)\, d\mu(u) \right)^{\nu/2} \leq C. \tag{11}$$

Hölder's inequality implies that

$$\mu([a_l, b_l]) \leq \left(\int_{a_l}^{b_l} \frac{d\mu(u)}{((b_l-u)(u-a_l))^{\gamma r'}} \right)^{1/r'} \left(\int_{a_l}^{b_l} ((b_l-u)(u-a_l))^{\gamma r}\, d\mu(u) \right)^{1/r}$$

if r and r' are conjugate exponents. If $\left(\frac{1}{r}, \frac{1}{r'}\right) = \left(\frac{\nu}{2}, 1 - \frac{\nu}{2}\right)$ and $\gamma = \frac{\nu}{2}$ then $\gamma r' = \frac{\nu}{2-\nu} < \frac{\alpha}{2}$ while $\gamma r = 1$ and $\alpha - 2\gamma r' \doteq \tau > 0$. Thus (10), (11), and $b_l - a_l < \delta$ lead to

$$\mu([a, b]) \leq \sum \mu([a_l b_l]) \leq C\delta^{\tau/r'}.$$

Since this holds for any $\delta > 0$, it follows that $\mu = 0$. □

Proof of Theorem 5

Suppose $a \leq c < d \leq b$. If the rectangle R contains the arc $\{(x, \phi(x)) : c \leq x \leq d\}$ of Γ, it follows as in the proof of Theorem 3 that

$$\int_c^d \phi''(x)(d-x)(x-c)\, dx \leq 2|R|.$$

Thus

$$\int_c^d \phi''(x)^{1/3}\, dx = \int_c^d \phi''(x)^{1/3}((d-x)(x-c))^{1/3}((d-x)(x-c))^{-1/3}\, dx$$
$$\leq (2|R|)^{1/3} \left(\int_c^d ((d-x)(x-c))^{-1/2}\, dx \right)^{2/3} = 2^{1/3}\pi^{2/3}|R|^{1/3},$$

where the first inequality is a consequence of Hölder's inequality. With a regularity argument, this shows that $\lambda \leq 2^{5/3}\nu$.

For the reverse inequality, fix $\varepsilon, \delta > 0$ and $[e, f] \subseteq [a, b]$. Write $\gamma([e, f])$ for the portion of Γ over $[e, f]$. It will be enough to assume that ϕ is monotone on $[a, b]$ and to show that

$$A_\delta^{2/3}(\gamma([e, f])) \leq 2^{4/3} \int_e^f \phi''(x)^{1/3}\, dx + 2^{4/3}\varepsilon^{1/3}(b - a). \tag{12}$$

Choose $\delta' \in (0, \delta)$ so small that if $[c, d] \subseteq [a, b]$ and $d - c < \delta'$ then the rectangle R over $[c, d]$ (see the proof of Theorem 1) has diameter $< \delta$. Recall from that proof that we then have

$$|R| \leq (d - c)^2 \mu([c, d]).$$

Now cover $[e, f] \cap \{\phi'' < \varepsilon\}$ by a finite collection of intervals $[a_j, b_j]$ such that $b_j - a_j < \delta'$, $[a_j, b_j] \subseteq \{\phi'' < 2\varepsilon\}$, and $\sum(b_j - a_j) \leq 2(b - a)$. Then, if R_j is the interval over $[a_j, b_j]$, it follows that

$$\sum |R_j|^{1/3} \leq 2^{4/3}\varepsilon^{1/3}(b - a). \tag{13}$$

Next, cover $[e, f] \cap \{\phi'' \geq \varepsilon\}$ by a finite collection of intervals $[c_k, d_k] \subseteq [e, f]$ such that, for each k, we have $d_k - c_k < \delta'$ and

$$\max\{\phi''(x) : x \in [c_k, d_k]\} \leq 2 \min\{\phi''(x) : x \in [c_k, d_k]\}.$$

By the covering lemma from the proof of Theorem 3 we can also assume that $\sum \chi_{[c_k, d_k]} \leq 2$. If T_k is the rectangle over $[c_k, d_k]$, it follows that

$$\begin{aligned} |T_k| &\leq (d_k - c_k)^3 \max\{\phi''(x) : x \in [c_k, d_k]\} \\ &\leq 2(d_k - c_k)^3 \min\{\phi''(x) : x \in [c_k, d_k]\} \end{aligned}$$

and so

$$|T_k|^{1/3} \leq 2^{1/3} \int_{c_k}^{d_k} \phi''(x)^{1/3}\, dx.$$

Then

$$\begin{aligned} \sum |T_k|^{1/3} &\leq 2^{1/3} \sum \int_{c_k}^{d_k} \phi''(x)^{1/3}\, dx \\ &= 2^{1/3} \int_e^f \phi''(x)^{1/3} \sum \chi_{[c_k, d_k]}\, dx \leq 2^{4/3} \int_e^f \phi''(x)^{1/3}\, dx. \end{aligned}$$

With (13), this gives (12). □

3. Some Examples

In this section we will construct a family of curves $\Gamma = \Gamma_\eta$ indexed by a parameter $\eta \in (0, 1)$. It will turn out that $\dim_a(\Gamma_\eta) = \frac{2}{3+\eta}$ and that, from the point of view of the problems (C) and (R) mentioned in Section 1, these curves are as nice as possible given their affine dimensions. We will also observe that the curves Γ_η possess a certain homogeneity property.

To begin, for a nonnegative integer n let D_n be the set $\left\{\frac{j}{2^n}\right\}_{j=0}^{2^n-1}$. Fix $\eta \in (0,\infty)$ and define the measure μ $(=\mu_\eta)$ by

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^{n(1+\eta)}} \sum_{x\in D_n} \delta_x .$$

For $0 \le x \le 1$, define $\phi(x)$ by

$$\phi(x) = \int_0^x \mu([0,x)). \tag{14}$$

Then Γ, the graph of ϕ over $[0,1]$, is a curve of the class described in Section 1.

LEMMA. *We have* $\dim_a(\Gamma) \le \frac{2}{3+\eta}$.

Proof. It is enough to show that there exists a C such that, for any $\delta > 0$, there is a covering of Γ by a finite collection $\{R_k\}$ of rectangles each having diameter $O(\delta)$ and such that $\sum |R_k|^{\frac{1}{3+\eta}} \le C$. So fix δ. Choose a positive integer n with $2^{-n} \le \delta$. For $j = 0,\dots,2^n-1$, choose a_j, b_j with $\frac{j}{2^n} < a_j < b_j < \frac{j+1}{2^n}$ in such a manner that the union of the $2\cdot 2^n$ segments of Γ over intervals of the form $\left[\frac{j}{2^n}, a_j\right]$ or $\left[b_j, \frac{j+1}{2^n}\right]$ can be covered by a finite collection of rectangles S_l with $\sum |S_l|^{\frac{1}{3+\eta}} \le 1$ and such that each S_l has diameter $< \delta$. (Since ϕ is continuous, this is easily accomplished by taking a_j close to $\frac{j}{2^n}$ and b_j close to $\frac{j+1}{2^n}$.) As observed in the proof of Theorem 1, the portion of Γ over $[a_j, b_j]$ lies inside a rectangle T_j with

$$|T_j| \le (b_j - a_j)^2 \mu([a_j, b_j)).$$

It is easy to check that $\mu([a_j,b_j)) \le C2^{-n(1+\eta)}$ so that $\sum |T_j|^{\frac{1}{3+\eta}} \le C$. The fact that ϕ is Lipschitz shows that each of the rectangles T_j has diameter $O(\delta)$. Now let $\{R_k\} = \{S_l\} \cup \{T_j\}$.

That $\dim_a(\Gamma) \ge \frac{2}{3+\eta}$ is a consequence of Proposition 2 and either of the next two propositions. □

PROPOSITION 3. *Let* λ *be the measure on* $\mathbb{R}^2$ *given by* $d\lambda = dx$ *on* $\Gamma = \{(x,\phi(x)) : 0 \le x \le 1\}$. *Then the convolution inequality* (2) *holds with* $\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{4+\eta}{6+2\eta}, \frac{2+\eta}{6+2\eta}\right)$.

The proof of Proposition 3 requires a lemma.

LEMMA. *There is a positive constant* C *such that, if* $c, d \in \mathbb{R}$, *then*

$$\left| \int_0^1 e^{i(cx+d\phi(x))}\,dx \right| \le C|d|^{-\frac{1}{2+\eta}}.$$

Proof. Split the integral into integrals over

$$\left\{|c + d\phi'| \le |d|^{\frac{1}{2+\eta}}\right\} \quad \text{and} \quad \left\{|c + d\phi'| \ge |d|^{\frac{1}{2+\eta}}\right\}.$$

Van der Corput's lemma [9, p. 197] bounds the second of these. For the first, note that

$$\left\{|c + d\phi'| \le |d|^{\frac{1}{2+\eta}}\right\} \subseteq [\alpha, \beta],$$

where

$$\phi'(\beta) - \phi'(\alpha) \le 2|d|^{-\frac{1+\eta}{2+\eta}}.$$

One easily checks that, for some positive C,

$$(\beta - \alpha)^{1+\eta} \le C(\phi'(\beta) - \phi'(\alpha)).$$

This leads to $\beta - \alpha \le C|d|^{-\frac{1}{2+\eta}}$ and the desired bound for the first integral. □

Proof of Proposition 3. The proof is a standard application of complex interpolation; see [1, Sec. 2] for a similar argument presented in more detail. Define an analytic family $\{\Phi_z\}$ of distributions by

$$\langle f, \Phi_z\rangle = \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_0^1 \int_{-\infty}^{\infty} f(x, \phi(x) + s)|s|^z\, ds\, dx.$$

If $\Re z = 0$, then Φ_z is a bounded function on $\mathbb{R}^2$ and so convolution with Φ_z maps L^1 $(= L^1(\mathbb{R}^2))$ into L^∞. If $\Re z = -\frac{3+\eta}{2+\eta}$ then the formula (see [4, p. 359]) for the Fourier transform of

$$\frac{|s|^z}{\Gamma\left(\frac{z+1}{2}\right)}$$

combined with the lemma just proved shows that convolution with Φ_z maps L^2 into L^2. Stein's interpolation theorem thus implies that convolution with Φ_{-1} maps L^p into L^q when $\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{4+\eta}{6+2\eta}, \frac{2+\eta}{6+2\eta}\right)$. Since Φ_{-1} is a nonzero multiple of λ, the proof of Proposition 3 is complete. □

The following corollary is analogous to Theorem 5.

Corollary. *There is a constant C (depending on η) such that, if λ is as before and if ν is the restriction to Γ of $\frac{2}{3+\eta}$-dimensional affine measure, then $C^{-1}\lambda \le \nu \le C\lambda$.*

Proof. Proposition 3 and the proof of Proposition 2 yield the inequality $\lambda(R) \le C|R|^{\frac{1}{3+\eta}}$ for rectangles R. This implies that $\lambda \le C \cdot \nu$. For the reverse inequality, fix $[e, f] \subseteq [a, b]$. Writing $\gamma([e, f])$ for the portion of Γ over $[e, f]$, it is enough to establish that

$$A_\delta^{\frac{2}{3+\eta}}(\gamma([e, f])) \le C(f - e)$$

for any $\delta > 0$. This can be done by slightly modifying the proof of the lemma preceding Proposition 3: observe that the "1" in that proof can be replaced by any $\varepsilon > 0$ and use only as many T_j as required to ensure that $\gamma([e, f]) \subseteq \bigcup R_k$. □

Proposition 4. *With λ as in Proposition 3, the restriction inequality* (3) *holds whenever* $\frac{1}{q} = (3 + \eta)\left(1 - \frac{1}{p}\right)$ *and* $1 \le p < \frac{4+2\eta}{3+2\eta}$.

Proof. The proof is analogous to the proof in [8]. The only significant modification is that (6) in [8] must be replaced by the inequality

$$\int_0^1\Big(\int_0^1 \chi_T(t-s,\phi(t)-\phi(s))\,ds\Big)^{2+\eta}dt \le C|T|, \tag{15}$$

where $|T|$ is the Lebesgue measure of an arbitrary Borel $T\subseteq\mathbb{R}^2$. Following an argument from [6], we regard (15) as an

$$(L^{2+\eta,1}(\mathbb{R}^2),\,L^{2+\eta}(d\lambda))$$

estimate for the operator $f\mapsto \lambda * f$. As such, (15) is equivalent to the adjoint

$$\big(L^{\frac{2+\eta}{1+\eta}}(d\lambda),\,L^{\frac{2+\eta}{1+\eta},\infty}(\mathbb{R}^2)\big)$$

estimate

$$\int_{\{y|\phi'(s)-\phi'(t)|\le g(t)\}}|\phi'(s)-\phi'(t)|\,ds\,dt \le \int_0^1\Big(\frac{g(t)}{y}\Big)^{\frac{2+\eta}{1+\eta}}dt$$

for nonnegative g and y. By integration in t, that will follow from

$$\int_{\{|\phi'(s)-\phi'(t)|\le B\}}|\phi'(s)-\phi'(t)|\,ds \le B^{\frac{2+\eta}{1+\eta}}$$

for $B>0$. And this is a consequence of the inequality, noted in the proof of Proposition 3, $(s-t)^{1+\eta}\le C(\phi'(s)-\phi'(t))$. □

The only properties of ϕ that were used in the proofs of Propositions 3 and 4 were the convexity of ϕ and the inequality $(s-t)^{1+\eta}\le C(\phi'(s)-\phi'(t))$ if $t<s$. Here are two consequences of this observation.

(i) If μ in (14) is replaced by

$$\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}\sum_{x\in D_n}\delta_x$$

then, for $0\le t<s\le 1$, the inequality

$$(s-t)^{1+\eta}\le C_\eta(\phi'(s)-\phi'(t))$$

holds for each $\eta\in(0,1)$. It follows that $\dim_a(\Gamma)=\frac{2}{3}$ even though μ is discrete.

(ii) If we extend μ by 1-periodicity to all of $[0,\infty)$, extend ϕ to $[0,\infty)$ by (14), let $\Gamma=\{(x,\phi(x)) : 0\le x<\infty\}$, and set $d\lambda=dx$ on Γ, then Propositions 3 and 4 are still valid. For the remainder of the paper we consider these extensions of μ, ϕ, and Γ.

In conclusion we point out a certain homogeneity property enjoyed by the curves Γ. As motivation, observe that for $t,\varepsilon\in\mathbb{R}$ we have the equation

$$\begin{pmatrix} t \\ t^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} \cdot \begin{pmatrix} \varepsilon \\ \varepsilon^2 \end{pmatrix} = \begin{pmatrix} t+\varepsilon \\ (t+\varepsilon)^2 \end{pmatrix},$$

which we interpret as the statement that the curve (t, t^2) is *equiaffinely homogeneous.* The curves Γ of this section are locally equiaffinely homogeneous at dyadic t in the following sense: if $t = j/2^N$ for nonnegative integers j and N, then there is $c(t) \in \mathbb{R}$ such that, if $0 \le \varepsilon < 2^{-N}$,

$$\begin{pmatrix} t \\ \phi(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ c(t) & 1 \end{pmatrix} \cdot \begin{pmatrix} \varepsilon \\ \phi(\varepsilon) \end{pmatrix} = \begin{pmatrix} t+\varepsilon \\ \phi(t+\varepsilon) \end{pmatrix}.$$

This is only the statement that

$$\phi(t+\varepsilon) - \phi(t) = \varepsilon c(t) + \phi(\varepsilon) - \phi(0)$$

or, equivalently,

$$\int_t^{t+\varepsilon} \phi'(s)\,ds = \varepsilon c(t) + \int_0^{\varepsilon} \phi'(s)\,ds.$$

But

$$\begin{aligned}
\int_t^{t+\varepsilon} \phi'(s)\,ds &= \int_t^{t+\varepsilon} (\phi'(s) - \phi'(t))\,ds + \varepsilon\phi'(t) \\
&= \varepsilon\phi'(t) + \int_t^{t+\varepsilon} \mu([t,s))\,ds \\
&= \varepsilon(\phi'(t) + \mu(\{t\}) - \mu(\{0\})) + \int_0^{\varepsilon} \mu([0,s))\,ds \\
&= \varepsilon(\phi'(t) + \mu(\{t\}) - \mu(\{0\})) + \int_0^{\varepsilon} \phi'(s)\,ds
\end{aligned}$$

since $\mu((t, t+u)) = \mu((0,u))$ if $t = j/2^N$ and $0 < u < 2^{-N}$.

For $C^{(2)}$-curves, such local equiaffine homogeneity is a very restrictive condition: If $\psi \in C^{(2)}$ and

$$\psi(t+\varepsilon) - \psi(t) = \varepsilon c(t) + \psi(\varepsilon) - \psi(0)$$

holds for a dense collection of t and (for each such t) for $0 < \varepsilon < \varepsilon(t)$, then $\psi''(t+\varepsilon) = \psi''(\varepsilon)$ leads to $\psi''(t) = \psi''(0)$ for all t and hence to $\psi(t) = at^2 + bt + c$ for constants a, b, c.

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