TAMING POLYHEDRA IN THE TRIVIAL RANGE

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1. INTRODUCTION

One of the major problems in topology is that of determining the equivalence classes in the set of imbeddings of one topological space into another, under some suitable definition of equivalence of two imbeddings. In the special case in which the domain space is a polyhedron of dimension k and the imbedding space is a combinatorial n-manifold (without boundary), it has been shown [1], [11] that whenever $2k + 2 \leq n$, then any two "sufficiently close" piecewise linear imbeddings are equivalent by an ambient isotopy. Gluck [8], [9], Greathouse [10], and Homma [12] showed that under the same conditions, each locally tame imbedding is equivalent to a piecewise linear imbedding by a homeomorphism of the manifold onto itself. Indeed, Gluck [8], [9] went further to show that this equivalence can actually be realized by an ambient isotopy.

The purpose here is to show that for $2k + 2 \leq n$, the set of locally tame imbeddings of a k-dimensional polyhedron into a combinatorial n-manifold is actually larger than it appears to be at first. The following is our main theorem.

THEOREM 1. *Suppose* f *is an imbedding of a* k*-dimensional polyhedron* X^k *into a combinatorial* n*-manifold* M^n, *where* $2k + 2 \leq n$, *and* P *is a tame polyhedron in* M^n, *with* $\dim P \leq \frac{n}{2} - 1$, *such that* $F \mid (X^k - f^{-1}(P))$ *is locally tame. Then* f *is* ε*-tame.*

2. DEFINITIONS

A k*-dimensional polyhedron* is the underlying space of some finite k-dimensional simplicial complex; it will usually be denoted by X^k. A *combinatorial* n*-manifold* is a locally finite simplicial complex for which the link of each vertex is combinatorially equivalent to the boundary of the standard n-simplex (that is, we shall only consider manifolds without boundary), and it will usually be denoted by M^n.

An imbedding f of X^k into M^n is said to be *tame* if there exists a homeomorphism h of M^n onto itself such that hf: $X^k \to M^n$ is piecewise linear. An imbedding f of X^k into M^n is *locally tame at a point* x *of* X^k if there exists a polyhedral neighborhood Z of x in X^k such that $f \mid Z$ is tame. An imbedding will be called ε-tame if for each $\varepsilon > 0$ there exists an ε-push h of $(M^n, f(X^k))$ (see Section 3) such that hf: $X^k \to M^n$ is piecewise linear.

3. DENSE AND SOLVABLE SETS OF IMBEDDINGS

The techniques employed to prove Theorem 1 are based upon the following notions, introduced by Gluck in [8].

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If A is a closed subset of the topological manifold M, then an ε-push h of the pair (M, A) is an ε-homeomorphism of M onto itself satisfying the two conditions

1) $h \mid M - U_\varepsilon(A)$ is the identity, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A,

2) h is ε-isotopic to the identity by an isotopy h_t ($t \in I$) such that $h_t \mid M - U_\varepsilon(A)$ is the identity for each t in the unit interval $I = [0, 1]$.

Now let M be a topological manifold with complete metric d. Suppose X is a metric space and A is a subset of X such that $\bar{A}$ is compact. Hom (X, A; M) will denote an arbitrary set of imbeddings of X into M, all of which agree on X - A. Hom (X, A; M) then becomes a metric space if for f, g $\in$ Hom (X, A; M) we define

$$d(f, g) = \sup_{x \in A} d(f(x), g(x)).$$

A subset F of Hom(X, A; M) is said to be *dense* in Hom (X, A; M) if to each $\varepsilon > 0$ and each g $\in$ Hom (X, A; M) there corresponds an element f in F with $d(f, g) < \varepsilon$. A subset F of Hom (X, A; M) is said to be *solvable* if to each $\varepsilon > 0$ there corresponds a $\delta = \delta(F, \varepsilon) > 0$ such that whenever $f, f' \in F$ and $d(f, f') < \delta$, then there exists an ε-push h of (M, f(A)) with $hf = f'$.

By using Homma's techniques [12], Gluck proved the following proposition.

THEOREM 2. *The union of two dense, solvable subsets of* Hom(X, A; M) *is dense and solvable.*

4. LEMMAS

LEMMA 3. *Suppose* G *is an open subset of* X^k *and* f *is an imbedding of* X^k *into Euclidean* n-*space* E^n $(2k + 2 \leq n)$ *such that* $f \mid G$ *is locally tame. Then for each* $\varepsilon > 0$ *there is an* ε-*push* h *of* $(E^n, f(G))$ *such that* $hf \mid G$ *is locally piecewise linear and* $h \mid f(X^k - G)$ *is the identity.*

Proof. Let $X_1, X_2, \cdots$ be subpolyhedra of X^k such that

$$G = \bigcup_{i=1}^{\infty} X_i \quad \text{and} \quad X_i \subset \operatorname{int} X_{i+1}$$

(the interior of X_{i+1} is taken relative to the space X^k).

Step I. Assuming that X_0 is the empty set, define $Y_i = \operatorname{Cl}(X_{2i+1} - X_{2i})$ for $i = 0, 1, \cdots$. Then $\{Y_0, Y_1, \cdots\}$ is a mutually exclusive collection of subpolyhedra of X^k lying in G. Thus $f \mid Y_i$ is locally tame for $i = 0, 1, \cdots$.

Suppose ε is a positive number. Let $\varepsilon_0, \varepsilon_1, \cdots$ be a sequence of positive numbers such that

1) $\varepsilon_i < \varepsilon/2$ for $i = 0, 1, \cdots$,

2) $\varepsilon_i \to 0$ as $i \to \infty$,

3) $U_i \cap U_j = \emptyset$ if $i \neq j$, where U_i is the ε_i-neighborhood of $f(Y_i)$ in E^n, and

4) $U_i \cap f(X^k - G) = \emptyset$ for $i = 0, 1, \cdots$.

By Gluck's results [8], [9], there exists, for $i = 0, 1, \cdots$, an ε_i-push h_i of $(E^n, f(Y_i))$ such that

$$h_i f\colon Y_i \to E^n$$

is piecewise linear.

Define $h\colon E^n \to E^n$ by

$$h(x) = \begin{cases} x & \text{if } x \notin \bigcup_{i=0}^{\infty} U_i, \\ h_i(x) & \text{if } x \in U_i. \end{cases}$$

Then h is an $\varepsilon/2$-push of $(E^n, f(G))$ such that $hf\colon Y_i \to E^n$ is piecewise linear for $i = 0, 1, \cdots$, and $h \mid f(X^k - G)$ is the identity.

Step 2. Now let $f' = hf$, and for $i = 1, 2, \cdots$ let $Z_i = \mathrm{Cl}(X_{2i} - X_{2i-1})$. Choose a sequence $\delta_1, \delta_2, \cdots$ of positive numbers such that

1) $\delta_i < \varepsilon/2$ for $i = 1, 2, \cdots$,

2) $\delta_i \to 0$ as $i \to \infty$,

3) $V_i \cap V_j = \emptyset$ if $i \neq j$, where V_i denotes the δ_i-neighborhood of $f'(Z_i)$ in E^n, and

4) $V_i \cap f'(X^k - G) = \emptyset$ for $i = 1, 2, \cdots$.

By Gluck's Modification of Homma's Theorem [8], [9], there exists for each i a δ_i-push g_i of $(E^n, f'(Z_i))$ such that

$$g_i f'\colon Z_i \to E^n$$

is piecewise linear and $g_i \mid f'(Y_i) \cup f'(Y_{i-1})$ is the identity.

Define $g\colon E^n \to E^n$ by

$$g(x) = \begin{cases} x & \text{if } x \notin \bigcup_{i=1}^{\infty} V_i, \\ g_i(x) & \text{if } x \in V_i. \end{cases}$$

Then g is an $\varepsilon/2$-push of $(E^n, f'(G))$ such that $gf' \mid G$ is locally piecewise linear and $g \mid f'(X^k - G)$ is the identity.

Therefore gh is an ε-push of $(E^n, f(G))$ with the desired properties.

Suppose that f is an imbedding of X^k into E^n as described in the hypothesis of Lemma 3, with the additional property that $f(X^k - G)$ lies in a polyhedron P, piecewise linearly imbedded in E^n, with $\dim P \leq \frac{n}{2} - 1$. In view of Lemma 3, we may assume that $f \mid G$ is locally piecewise linear. Let $\mathrm{Hom}(X^k, G; E^n)$ denote the set of all imbeddings of X^k into E^n that agree with f on $X^k - G$. Consider the set F of all f' in $\mathrm{Hom}(X^k, G; E^n)$ for which there exists a polyhedron Z in G such that

1) $f' \mid X^k - Z = f \mid X^k - Z$ and

2) $f' \mid Z$ is piecewise linear.

LEMMA 4. *The set* F *defined above is a dense, solvable subset of* $\mathrm{Hom}(X^k, G; E^n)$.

Proof. If $\varepsilon > 0$ and $g \in \mathrm{Hom}(X^k, G; E^n)$ are given, choose a polyhedron Z in G such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X^k - Z$. By using standard extension theorems and general position arguments, we can extend $f \mid X^k - Z$ to an imbedding f' of X^k into E^n so that $f' \mid Z$ is piecewise linear and $d(f', g) < \varepsilon$. By definition, $f' \in F$; hence F is dense.

To see that F is solvable, let $\varepsilon > 0$, and suppose that $f_0, f_1 \in F$ with $d(f_0, f_1) < \varepsilon$. Notice that f_0 and f_1 agree with f except on a polyhedron Z in G, and that both f_0 and f_1 are piecewise linear on Z. Now we may use the fact that $f(X^k - G)$ ($= f_i(X^k - G)$ for $i = 0, 1$) is contained in the polyhedron P with $\dim P \leq \frac{n}{2} - 1$, together with a suitable modification of a theorem of Bing and Kister [1], to obtain an ε-push h of $(E^n, f_0(Z))$ such that $hf_0 \mid Z = f_1 \mid Z$ and $h \mid P \cup f(X^k - Z)$ is the identity. Thus F is solvable.

LEMMA 5. *Suppose* Y *is a closed subset of* X^k *and* f_0, f_1 *are imbeddings of* X^k *into* E^n $(2k + 2 \leq n)$ *satisfying the conditions*

1) $f_i \mid X^k - Y$ *is locally piecewise linear for* $i = 0, 1$,

2) $f_0 \mid Y = f_1 \mid Y$,

3) $f_0(Y)$ *lies in a polyhedron* P *piecewise linearly imbedded in* E^n, *with* $\dim P \leq \frac{n}{2} - 1$, *and*

4) $d(f_0, f_1) < \varepsilon$.

Then there exists a 2ε*-push* h *of* $(E^n, f_0(X^k - Y))$ *such that* $hf_0 = f_1$.

Proof. Let $G = X^k - Y$ and let $\mathrm{Hom}(X^k, G; E^n)$ be the set of all imbeddings of X^k into E^n that agree with f_0 on Y. For $i = 0, 1$, let F_i be the subset of $\mathrm{Hom}(X^k, G; E^n)$ associated with f_i as described previously. Then F_0 and F_1 are dense, solvable subsets of $\mathrm{Hom}(X^k, G; E^n)$, so that $F_0 \cup F_1$ is also dense and solvable.

Let $\delta = \delta(F_0 \cup F_1, \varepsilon) > 0$. Since F_0 is dense in $F_0 \cup F_1$, there exists an f_0' in F_0 such that $d(f_0', f_1) < \delta$ and $d(f_0', f_0) < \varepsilon$. From the proof of Lemma 4 it follows that there exists an ε-push h_0 of $(E^n, f_0(G))$ such that $h_0 f_0 = f_0'$. By the choice of δ, there exists an ε-push h_1 of $(E^n, f_0'(G))$ such that $h_1 f_0' = f_1$. Thus $h = h_1 h_0$ is a 2ε-push of $(E^n, f_0(G))$ with the required properties.

5. A SPECIAL CASE

THEOREM 6. *Suppose* Y *is a subpolyhedron of* X^k *and* f *is an imbedding of* X^k *into* E^n $(2k + 2 \leq n)$ *such that* $f \mid Y$ *is locally tame and* $f \mid X^k - Y$ *is locally tame. Then* f *is* ε*-tame.*

Proof. Since Gluck's results [8], [9] and Lemma 3 allow us to assume that $f \mid Y$ is piecewise linear and $f \mid X^k - Y$ is locally piecewise linear, the theorem follows immediately from Lemma 4 and Theorem 2, because the set of piecewise linear extensions of $f \mid Y$ in $\mathrm{Hom}(X^k, X^k - Y; E^n)$ is known to be dense and solvable.

Using Theorem 6, we can prove, by induction on k, the following theorem, which has been established for $k = 1$ and $k = 2$ by Cantrell [4] and Edwards [7], respectively.

THEOREM 7. *A* k-*complex* K *in* E^n $(2k + 2 \leq n)$ *is* ε-*tame if and only if each simplex of* K *is tame.*

From this and a theorem of Klee [13], we obtain an alternate proof of a theorem of Bing and Kister [1].

THEOREM 8. *If* f *is an imbedding of* X^k *into the* n-*plane* E^n *in* E^{n+k} $(n \geq k + 2)$, *then* f: $X^k \to E^{n+k}$ *is* ε-*tame.*

6. PROOF OF THEOREM 1

Case 1. $M^n = E^n$. Let $Y = f^{-1}(P)$ and let $G = X^k - Y$. We may assume that P is piecewise linearly imbedded in E^n, by Gluck's results [8], [9], and that $f \mid G$ is locally piecewise linear, by Lemma 3.

Let Φ denote the set of all piecewise homeomorphisms of E^n onto itself. For each $\phi \in \Phi$, the imbedding ϕf of X^k into E^n satisfies the conditions that $\phi f \mid G$ is locally piecewise linear and $\phi f(Y) \subset \phi(P)$, a polyhedron piecewise linearly imbedded in E^n, with $\dim \phi(P) \leq \frac{n}{2} - 1$.

For each $\phi \in \Phi$, let $\mathrm{Hom}_\phi(X^k, G; E^n)$ be the set of all imbeddings of X^k into E^n that agree with ϕf on Y, and let F_ϕ denote the subset of $\mathrm{Hom}_\phi(X^k, G; E^n)$ associated with ϕf as described in Section 4. Define

$$F = \bigcup \{F_\phi \mid \phi \in \Phi\}.$$

A. F *is dense in* $\mathrm{Hom}(X^k; E^n)$, the set of all imbeddings of X^k into E^n.

Suppose $\varepsilon > 0$ and $g \in \mathrm{Hom}(X^k; E^n)$. Let $\psi: P \to E^n$ be an extension of $gf^{-1}: f(Y) \to E^n$. Then there exists a piecewise linear imbedding $\bar{\phi}: P \to E^n$ such that $d(\bar{\phi}, \psi) < \varepsilon$, and a piecewise linear homeomorphism ϕ of E^n onto E^n such that $\phi \mid P = \bar{\phi}$. Thus, for each $x \in Y$, $d(\phi f(x), g(x)) < \varepsilon$. By Dugundji's extension theorem [6], there exists an extension $f'': X^k \to E^n$ of $\phi f \mid Y$ such that $d(f'', g) < \varepsilon$.

Choose $\delta > 0$ so that each map of X^k into E^n within δ of f'' lies within ε of g. By applying general position arguments, we can construct an imbedding $f': X^k \to E^n$ such that

1) $d(f', f'') < \delta$,
2) $f' \mid Y = f'' \mid Y$,
3) $f' \mid G$ is locally piecewise linear, and
4) $f'(G) \cap \phi(P) = \emptyset$.

Then $f' \in F$ and $d(f', g) < \varepsilon$.

B. F *is solvable.*

Given $\varepsilon > 0$, choose $\delta = \delta(F, \varepsilon) = \varepsilon/6$. Suppose that $f_0, f_1 \in F$ and $d(f_0, f_1) < \delta$. Then there exist elements $\phi_0, \phi_1 \in \Phi$ for which

$$\phi_i f \mid Y = f_i \mid Y \qquad (i = 0, 1).$$

Since $d(\phi_0(y), \phi_1(y)) < \delta$ for each $y \in f(Y)$, there exists a polyhedral neighborhood Q of f(Y) in P such that $d(\phi_0(y), \phi_1(y)) < \delta$ for each $y \in Q$ and

$$f_0(Y) \subset \phi_0(Q) \subset U_\delta(f_0(Y)),$$

where $U_\delta(f_0(Y))$ is the δ-neighborhood of $f_0(Y)$ in E^n.

From the results of Bing and Kister [1], we obtain a δ-push h_0 of $(E^n, \phi_0(Q))$ such that

$$h_0: E^n \to E^n \text{ is piecewise linear} \quad \text{and} \quad h_0\phi_0 \mid Q = \phi_1 \mid Q.$$

Since $U_\delta(\phi_0(Q)) \subset U_{2\delta}(f_0(Y))$, h_0 is a 2δ-push of $(E^n, f_0(Y))$, and hence a 2δ-push of $(E^n, f_0(X^k))$.

Let $f' = h_0 f_0$. Then $d(f', f_1) < \varepsilon/3$, $f' \mid G$ is locally piecewise linear, and $f' \mid Y = f_1 \mid Y$. By Lemma 5, there exists a $\frac{2}{3}\varepsilon$-push h of $(E^n, f'(G))$ such that $h_1 f_1' = f_1$. Thus $h = h_1 h_0$ is an ε-push of $(E^n, f_0(X^k))$ and $hf_0 = f_1$.

The special case now follows from Theorem 2, since the set of piecewise linear imbeddings of X^k into E^n is dense and solvable.

Case 2 (General Case). Suppose $x \in X^k$. Let U be a combinatorial n-ball in M^n that contains f(x), and let g be a piecewise linear homeomorphism of U onto E^n. Choose a polyhedral neighborhood Z of x in X^k and a subpolyhedron Q of P such that

1) $f(Z) \subset U$ and

2) $P \cap f(Z) \subset Q \subset U$.

Then, by Case 1, $gf \mid Z: Z \to E^n$ is tame, so that $f \mid Z: Z \to M^n$ is also tame. Hence, f is a locally tame imbedding of X^k into M^n, so that, by Gluck's results [8], [9], f is ε-tame.

THEOREM 9. *Theorems* 6 *and* 7 *remain valid if* E^n *is replaced by an arbitrary* M^n.

7. FURTHER APPLICATIONS

It follows immediately from Theorem 1 that an imbedding of X^k into M^n $(2k + 2 \leq n)$ cannot fail to be locally tame at precisely one point. Applying this fact to a polyhedral neighborhood of a point of X^k, we obtain the following theorem.

THEOREM 10. *If* S *is the set of points at which an imbedding* f *of* X^k *into* M^n $(2k + 2 \leq n)$ *fails to be locally tame, then* S *contains no isolated points and is therefore uncountable or empty.*

Dancis [3] has shown by different methods that the dimension of the polyhedron P in Theorem 1 can be increased to n - k. Actually, Theorem 1 (as well as the theorem proved by Dancis) can be applied to prove a further result.

THEOREM 11. *Suppose that each of* $A_1, A_2, \cdots$ *is a locally tame simplex of dimension at most* n - k *in* M^n. *Then every imbedding of* X^k *into* $\bigcup_{i=1}^{\infty} A_i$ *is an* ε*-tame imbedding of* X^k *into* M^n.

Proof. It is sufficient to establish the theorem in the case $M^n = E^n$, because an argument similar to that given in the proof of Case 2 of Theorem 1 can then be used to obtain the desired generalization. Also, in view of Theorem 7, we may assume that X^k is a k-simplex.

First, notice that for each i, A_i is the union of a finite number of simplexes each of which can be carried into the hyperplane E^{n-k} by a homeomorphism of E^n onto itself. Hence, we may assume that each A_i has this property.

Now suppose that f is an imbedding of the k-simplex A into $\bigcup_{i=1}^{\infty} A_i$, and let A_0 be the union of all the open subsets of A on which f is locally tame. From (Baire) category arguments, it follows that there exist an open subset U of A and a positive integer i such that $f(U) \subset A_i$. Let h be a homeomorphism of E^n onto itself such that $h(A_i) \subset E^{n-k}$. Then $hf \mid U$ is locally tame (Theorem 8), and therefore $f \mid U$ is also locally tame; that is, A_0 is not empty.

Suppose that $A_0 \neq A$. Let $B = A - A_0$. Then there exists an open subset V of B and a positive integer i such that $f(V) \subset A_i$. Let W be an open subset of A such that $W \cap B = V$. Then $f \mid W - B$ is locally tame.

Suppose that x is a point of V. Choose a k-simplex Δ, containing x in its interior (relative to A), such that $\Delta \subset W$. Assume that $A_i \subset E^{n-k}$, and let Δ' be a k-simplex in the orthogonal complement of E^{n-k}.

Choose a homeomorphism g of Δ onto Δ'. By a theorem of Klee [13], the homeomorphism

$$gf^{-1}\colon f(\Delta \cap B) \to \Delta'$$

can be extended to a homeomorphism h of E^n onto itself. Thus $hf \mid \Delta - B$ is locally tame. However, $hf(\Delta \cap B) \subset \Delta'$, and therefore, by Theorem 1, $hf \mid \Delta$ is tame. Hence, $f \mid W$ is locally tame. But $W \not\subset A_0$, a contradiction. Therefore $A_0 = A$, and the theorem follows from [8] and [9].

An imbedding f of a topological manifold M^m into a topological manifold N^n is said to be *locally flat* at the point x in M if there exists a neighborhood U of f(x) in N such that the pair $(U, U \cap f(M))$ is homeomorphic to the pair (E^n, E^m). An imbedding f of a k-cell D^k into E^n is *flat* if there exists a homeomorphism of E^n onto itself carrying $f(D^k)$ into the k-plane E^k in E^n. Similarly, f is *locally flat* at $x \in D^k$ if x lies in the interior (relative to D^k) of a k-cell D' in D^k such that $f \mid D'$ is flat. It is clear that whenever $2k + 2 \leq n$, an imbedding of D^k into E^n is flat if and only if it is tame. Thus Theorem 11 implies the following result.

LEMMA 12. *If* f *is an imbedding of the* k*-cell* D^k *into* E^n $(2k + 2 \leq n)$ *such that* $f \mid (D^k - f^{-1}(E^{\ell}))$ $(\ell \leq n - k)$ *is locally flat, then* f *is flat.*

Cantrell and Edwards [5] have shown that if $2m + 2 \leq n$, then there exists no "almost locally flat" imbedding of a topological m-manifold into a topological n-manifold. (An imbedding is "almost locally flat" if it is locally flat except at a countable number of points.) Applying Lemma 12, one can easily show that the following more general statement holds.

THEOREM 13. *Suppose that* M, N, *and* Q *are topological manifolds of dimensions* m, n, *and* q, *respectively, with* $m \leq n/2 - 1$ *and* $q \leq n - k$, *and that* Q *is a locally flat submanifold of* N. *Suppose that* f *is an imbedding of* M *into* N *such that* $f \mid (M - f^{-1}(Q))$ *is locally flat. Then* f *is locally flat.*

REFERENCES

1. R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes*, Duke Math. J. 31 (1964), 491-511.

2. M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. (2) 75 (1962), 331-341.

3. J. Dancis, *Some nice embeddings in the trivial range* (to appear).

4. J. C. Cantrell, n-*frames in Euclidean* k-*space*, Proc. Amer. Math. Soc. 15 (1964), 574-578.

5. J. C. Cantrell and C. H. Edwards, Jr., *Almost locally flat imbeddings of manifolds*, Michigan Math. J. 12 (1965), 217-223.

6. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. 1 (1951), 353-367.

7. C. H. Edwards, Jr., *Taming* 2-*complexes in high-dimensional manifolds*, Duke Math. J. (to appear).

8. H. Gluck, *Embeddings in the trivial range*, Bull. Amer. Math. Soc. 69 (1963), 824-831.

9. ———, *Embeddings in the trivial range*, Ann. of Math (2) 81 (1965), 195-210.

10. C. A. Greathouse, *Locally flat, locally tame, and tame embeddings*, Bull. Amer. Math. Soc. 69 (1963), 820-823.

11. V. K. A. M. Gugenheim, *Piecewise linear isotopy and embedding of elements and spheres, (I)* and *(II)*, Proc. London Math. Soc. (3) 3 (1953), 29-53 and 129-152.

12. T. Homma, *On the imbedding of polyhedra in manifolds*, Yokohama Math. J. 10 (1962), 5-10.

13. V. L. Klee, Jr., *Some topological properties of convex sets*, Trans. Amer. Math. Soc. 78 (1955), 30-45.

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