ON MANIFOLD-LIKE POLYHEDRA

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1. INTRODUCTION

If Π is a collection of spaces, a metric space X is said to be Π-like if for each $\varepsilon > 0$ there exists a map

$$f: X \to Y \in \Pi$$

that is onto and all of whose point inverses have diameter less than ε. Ganea [3] has given an example of an S^3-like space that is not a manifold, and Deleanu [2] has proved that every manifold-like polyhedron of dimension less than 4 is a manifold. This note gives an example, for each $n \geq 4$, of an n-dimensional, S^n-like polyhedron that is a generalized manifold but not a manifold.

2. CONSTRUCTION

By a theorem of Curtis [1] there exists, for each $n \geq 4$, a combinatorial n-manifold M with the properties

(1) M is contractible,

(2) $\pi_1(\partial M) \neq 0$ but M is a homology sphere,

(3) $M \times I = I^{n+1}$.

Let N be the suspension of ∂M. Then N can be written as $C(\partial M \times i) \cup \partial M \times I$ $(i = 0, 1)$, where C(X) means the cone over X. Moreover, if $\varepsilon > 0$, we may take $C(\partial M \times i)$ to have diameter less than ε. Since M is contractible, there exists a map $h: C(M) \to M$ that is the identity on the base of the cone. Let

$$g = h \mid C(\partial M): C(\partial M) \to M.$$

We shall show that g is onto. Indeed, assume that this is not so, and let $x \in M - \mathrm{im}(g)$. Since g is the identity on ∂M, x is in the interior of M. Let U be the interior of a combinatorial n-cell containing x, and let it be small enough so that $\overline{U}$ misses im(g). Then M - U is an orientable combinatorial manifold with boundary $\partial M \cup \partial\overline{U}$, and therefore the fundamental (n - 1)-cycles on ∂M and $\partial\overline{U}$ are homologous in M - U. From the homology exact sequence of the pair (M, M - U) it is clear that the fundamental (n - 1)-cycle on $\partial\overline{U}$ generates $H_{n-1}(M - U) = Z$. Then the inclusion $i: \partial M \to M - U$ induces an homology isomorphism in dimension n - 1. Since g is the identity on ∂M, the diagram

$$\begin{array}{ccc} H_{n-1}(\partial M) & \xrightarrow{\quad i_* \quad} & H_{n-1}(M - U) \\ & \searrow \qquad \nearrow g_* & \\ & H_{n-1}(C(\partial M)) & \end{array}$$

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is commutative; but this is impossible, since the isomorphism $i_*: Z \to Z$ cannot be factored through 0. Thus g is onto.

THEOREM. N *is an* S^n*-like polyhedron that is a generalized manifold but not a manifold.*

Proof. From properties (1) and (2) it is clear that N is a generalized homology manifold but not a homology manifold, thus not a manifold. By property (3),

$$S^n = \partial(I^{n+1}) = \partial(M \times I) = \partial M \times I \cup M \times \{0, 1\}.$$

Given $\varepsilon > 0$, define $f: N \to S^n$ as follows:

$$f \mid C(\partial M \times i) = g: C(\partial M \times i) \to M \qquad (i = 0, 1),$$

$$f \mid \partial M \times I = \text{id}: \partial M \times I \to \partial M \times I.$$

Then f is continuous by the definition of h, and onto because g is onto; the only nontrivial point inverses are in $C(\partial M \times i)$ $(i = 0, 1)$, and these have diameter less than ε. Therefore N is S^n-like.

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