ON ORDER-PRESERVING EXTENSIONS TO REGRESSIVE ISOLS

Fred J. Sansone

1. INTRODUCTION

The extension of functions to isols was treated by Nerode in [4]. If we restrict our attention to regressive isols, the notion of an infinite series of isols, defined in [2], becomes a useful tool. In [1] and [5], such series were employed to study extensions. In particular, Barback proved that for a recursive function f, $f_\Lambda: \Lambda_R \to \Lambda_R$ if and only if f is eventually increasing. Our main concern here is the determination of the functions that are recursive and eventually increasing and have the additional property that their extensions ultimately preserve the partial ordering $\leq$ in Λ_R. We call the extension f_Λ of a recursive, eventually increasing function f *ultimately order-preserving in* Λ_R if there exists a natural number k such that f_Λ preserves the order $\leq$ in the class of regressive isols that are greater than or equal to k. Our principal result states that among the recursive, eventually increasing functions, those whose extensions are ultimately order-preserving in Λ_R are exactly the functions whose first difference is eventually increasing. Our notation and terminology is that of [5].

2. A THEOREM ON INFINITE SERIES

By a number-theoretic function, we mean any function defined on the nonnegative integers and having integral values. A number-theoretic function is said to be *recursive* if its positive and negative parts are both recursive. We recall the definitions of two additional concepts, defined in [5]: the mapping ϕ_f and the star-sum. If T is an infinite regressive isol and f is a one-to-one function, then $\phi_f(T) = \text{Req}\ \rho t_{f(n)}$, where t_n is any regressive function ranging over any set in T. The star-sum is defined as follows. If f is a recursive, number-theoretic function and T is an infinite, regressive isol, then

$$\sum_T{}^{*} f_n = \sum_T f_n^+ - \sum_T f_n^-,$$

where f_n^+, f_n^- are respectively the positive and negative parts of f.

THEOREM 1. *Let* a_n *be a recursive function. Then for all regressive isols* T *and* U

$$\left[U \leq T \Rightarrow \sum_U a_n \leq \sum_T a_n \right] \Leftrightarrow a_n \text{ is eventually increasing.}$$

Proof. Proceeding from right to left, we first assume that a_n is recursive and eventually increasing. If U is finite, the left-hand side clearly holds. Suppose U is infinite. We first dispense with the case where a_n is increasing. If $U \leq T$, then $U = \phi_f(T)$ for some strictly increasing but not necessarily recursive function f. Thus for this f, $\phi_f(T) \leq T$, and it follows that

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(1)
$$\sum_{\phi_f(T)} a_{f(n)} \leq \sum_T a_n .$$

Since a_n and f are both increasing, $a_n \leq a_{f(n)}$ for all n. Thus

(2)
$$\sum_{\phi_f(T)} a_n \leq \sum_{\phi_f(T)} a_{f(n)} .$$

The argument below suffices to show that (2) holds. If m is a number of the form $j(t_{f(k)}, p)$, where j is the well-known recursive pairing function from ε^2 onto ε, and where $p < a_{f(k)}$, then we can find $t_{f(k)}$ and hence f(k), since $t_{f(k)}$ is regressive. Having obtained f(k), one can also obtain f(0) through f(k - 1), since T is regressive and $\phi_f(T) \leq T$. Thus the number k can be found, and hence also a_k. One need only compare p with a_k to determine whether $m \in j(t_{f(k)}, \nu(a_k))$. Combining (1) and (2), we have the inequality

$$\sum_U a_n \leq \sum_T a_n .$$

If a_n is eventually increasing, but not increasing, there exists a positive number k such that a_{n+k} is increasing. The relation $U \leq T$ implies that $U - k \leq T - k$, and by the inequality above, $\sum_{U-k} a_{n+k} \leq \sum_{T-k} a_{n+k}$. Therefore, $\sum_U a_n \leq \sum_T a_n$.

In order to prove the converse, assume that a_n is recursive and not eventually increasing. We show that for some regressive isol T,

(3)
$$\sum_{T-1} a_n \not\leq \sum_T a_n .$$

From the definition of an infinite series of isols, one readily obtains the equivalence

(4)
$$\sum_{T-1} a_n \leq \sum_T a_n \Leftrightarrow \sum_{T-1} a_n \leq a_0 + \sum_{T-1} a_{n+1} .$$

Since a_n is recursive, it is clear that $\Delta a = a_{n+1} - a_n$ is a recursive, number-theoretic function. By [5, Theorem 3, Corollary 4],

$$\sum_{T-1} a_{n+1} - \sum_{T-1} a_n = \sum_{T-1}^{*} \Delta a_n .$$

Hence the left-hand side of (4) holds if and only if

$$a_0 + \sum_{T-1}^{*} \Delta a_n \in \Lambda_R ,$$

or by [5, Theorem 2], if and only if $a_\Lambda(T - 1) \in \Lambda_R$. Since a_n is not eventually increasing, it follows from [1, Theorem 4] that some regressive isol T satisfies (3).

3. THE MAIN RESULT

THEOREM 2. *Let* f *be increasing and recursive. Then* f_Λ *is order-preserving in* Λ_R *if and only if* Δf *is eventually increasing.*

Proof. Since f is increasing and recursive, Δf is a recursive function. Replacing a by Δf in Theorem 1 and applying [5, Theorem 3, Corollary 3], we obtain the desired result.

COROLLARY. *Let* f *be recursive and eventually increasing. Then* f_Λ *is ultimately order-preserving in* Λ_R *if and only if* Δf *is eventually increasing.*

Proof. If f is recursive and eventually increasing, there exists a natural number k such that f_{n+k} is recursive and increasing. With the notation $g_n = f_{n+k}$, it follows that g_Λ is order-preserving in Λ_R if and only if Δg is eventually increasing. Let T, U $\in \Lambda_R$ with $k \leq U \leq T$. Then $U - k \leq T - k$, where both are members of Λ_R. Hence

$$g_\Lambda(U - k) \leq g_\Lambda(T - k) \Leftrightarrow \Delta g \text{ is eventually increasing}.$$

However, Δg is eventually increasing if and only if Δf is eventually increasing. Finally, $g_\Lambda(U - k) = f_\Lambda(U)$ and $g_\Lambda(T - k) = f_\Lambda(T)$.

4. REMARKS

In [3], Dekker considers a relation $\leq^*$ in Λ, which he proves to be a partial ordering. We readily see that the extension of every recursive, eventually increasing function is $\leq^*$ order-preserving in Λ_R. (It follows from [3, Proposition 12] and [5, Theorem 3, Corollary 3].) As a consequence, we obtain the following simple proof of the existence of regressive isols X and Y such that $X \leq^* Y$ and $X \not\leq Y$. Let f be a function that is recursive and eventually increasing, but for which Δf is not eventually increasing. Then there exist regressive isols T and U such that $U \leq T$ and yet $f_\Lambda(U) \not\leq f_\Lambda(T)$. However, $U \leq T \Rightarrow U \leq^* T$, and hence $f_\Lambda(U) \leq^* f_\Lambda(T)$.

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Arizona State University
Tempe, Arizona