AUTOMORPHISMS OF COMPACT ABELIAN GROUPS AS MODELS FOR MEASURE-PRESERVING INVERTIBLE TRANSFORMATIONS

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The aim of the present note is to prove the following result.

THEOREM. *Let* (S, Σ, m) *be a finite measure space, and* T *a measure-preserving invertible transformation on* S. *There exist a compact abelian group* G, *an automorphism* $\mathscr{T}$ *of* G, *and a Borel measure* $\mu \geq 0$ (*on* G), *preserved by* $\mathscr{T}$, *such that* T *is conjugate to* $\mathscr{T}$ (in the sense of [3, pp. 42-45]).

Moreover, if (S, Σ, m) *is separable, then* G *can be chosen to be metrizable.*

Proof. Denote by U the unitary operator induced by T in $L(S, \Sigma, m) = L^2$. Let Γ_T be the group of (the classes of) functions f of L^2 such that $|f| = 1$, and let Γ be a subgroup of Γ_T satisfying the two conditions

(i) Γ spans L^2,

(ii) Γ is invariant under U and U^{-1}.

Evidently, if (S, Σ, m) is separable, Γ may be chosen to be countable. We shall take G to be the dual group of the abelian discrete group Γ, so that G is a compact abelian group which, in case (S, Σ, m) is separable, is also metrizable. Since, by (ii), U is an automorphism of Γ, there exists an automorphism $\mathscr{T}$ of G such that

$$\langle \mathscr{T}x, \gamma \rangle = \langle x, U\gamma \rangle \qquad (x \in G, \gamma \in \Gamma). \tag{1}$$

It remains to define the measure μ on G and the conjugation operation between $\mathscr{T}$ and T. To this end, we define the function

$$\phi(\gamma) = \int_S \gamma \, dm \qquad (\gamma \in \Gamma) \tag{2}$$

on Γ. Obviously, for every system $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of complex numbers and for $\{\gamma_1, \gamma_2, \cdots, \gamma_n\} \subset \Gamma$, we have the relations

$$\sum_{k,j=1}^{n} \alpha_k \overline{\alpha}_j \phi(\gamma_k \gamma_j^{-1}) = \sum_{k,j=1}^{n} \alpha_k \overline{\alpha}_j \int_S \gamma_k \overline{\gamma}_j \, dm = \int_S \left| \sum_{j=1}^{n} \alpha_j \gamma_j \right|^2 dm, \tag{3}$$

so that ϕ is positive definite. It is evidently continuous on Γ, since Γ is discrete. By Bochner's theorem, there exists a positive (finite) Borel measure μ on G such that

$$\phi(\gamma) = \int_G \langle x, \gamma \rangle \, d\mu(x) \qquad (\gamma \in \Gamma). \tag{4}$$

But by (1), (2), and (3),

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$$\phi(\gamma) = \phi(U\gamma) = \int_G \langle x, U\gamma \rangle \, d\mu(x) = \int_G \langle \mathcal{T} x, \gamma \rangle \, d\mu(x) = \int_G \langle x, \gamma \rangle \, d\nu(x),$$

where ν is the measure defined by $\nu(A) = \mu(\mathcal{T}^{-1} A)$, A being a Borel set contained in G. The uniqueness of the measure in Bochner's theorem implies that

$$\mu(A) = \mu(\mathcal{T}^{-1} A)$$

for every Borel set A in G, so that μ is invariant under $\mathcal{T}$.

To define the conjugation operation, we put

$$V\left(\sum_{k=1}^{n} \alpha_k \gamma_k \right) = \sum_{k=1}^{n} \alpha_k \langle \cdot, \gamma_k \rangle.$$

Then, by virtue of (3) and (4),

$$\int_S \left| \sum_{k=1}^{n} \alpha_k \gamma_k \right|^2 dm = \sum_{k,j=1}^{n} \alpha_k \bar{\alpha}_j \phi(\gamma_k \gamma_j^{-1}) = \int_G \left| \sum_{k=1}^{n} \alpha_k \langle x, \gamma_k \rangle \right|^2 d\mu(x)$$

$$= \int_G \left| V\left(\sum_{k=1}^{n} \alpha_k \gamma_k \right) \right|^2 d\mu ;$$

hence V is an isometry which, on account of (i), can be extended uniquely to an isometry of L^2 in $L^2(G, \mu)$. Using the fact that the characters on G span C(G), we deduce that V is a map onto $L^2(G, \mu)$, that is, V is a unitary operator of L^2 on $L^2(G, \mu)$. Since $V^{-1} \langle \cdot, \gamma \rangle = \gamma$ and

$$\langle \mathcal{T} \cdot, \gamma \rangle = \langle \cdot, U\gamma \rangle = VU\gamma = VUV^{-1} \langle \cdot, \gamma \rangle \qquad (\gamma \in \Gamma),$$

it follows from linearity that $U = V \mathcal{U} V^{-1}$, where $\mathcal{U}$ is the unitary operator induced by $\mathcal{T}$ in $L^2(G, \mu)$.

It remains to show that

$$\text{(5)} \qquad VL^{\infty}(S, \Sigma, m) = L^{\infty}(G, \mu)$$

and that V is multiplicative on $L^{\infty}(S, \Sigma, m)$.

Denote by $\mathcal{A}_S$, respectively, $\mathcal{A}_G$, the space of polynomials

$$\sum_{k=1}^{n} \alpha_k \gamma_k, \qquad \text{respectively,} \quad \sum_{k=1}^{n} \alpha_k \langle \cdot, \gamma_k \rangle.$$

Evidently $\mathcal{A}_S$ (respectively, $\mathcal{A}_G$) is an algebra containing together with each polynomial p also its complex conjugate $\bar{p}$. The operator V is multiplicative on $\mathcal{A}_S$, since

$$V(\gamma_1 \gamma_2) = \langle \cdot, \gamma_1 \gamma_2 \rangle = \langle \cdot, \gamma_1 \rangle \langle \cdot, \gamma_2 \rangle = V\gamma_1 \cdot V\gamma_2 ;$$

moreover, V is real, that is, $V\bar{p} = \overline{Vp}$ $(p \in \mathscr{A}_S)$. Thus, for each $p \in \mathscr{A}_S$,

$$\int_G |Vp|^{2n}\,d\mu = \int_G (Vp)^n \overline{(Vp)^n}\,d\mu = \int_G Vp^n \cdot V\bar{p}^n\,d\mu$$
$$= \int_G V|p|^{2n}\,d\mu = \int_S |p|^{2n}\,dm \le m(S)\,\|p\|_\infty^{2n},$$

where

$$\|p\|_\infty = m\text{ - ess. max}\,|p|.$$

Consequently

$$\left(\int_G |Vp|^{2n}\,d\mu\right)^{1/2n} \le m(S)^{1/2n}\|p\|_\infty;$$

letting n tend to ∞, we deduce that

$$\|Vp\|_\infty = \mu\text{ - ess. max}\,|Vp| \le \|p\|_\infty.$$

The same argument applied to V^{-1} shows that $\|V^{-1}q\|_\infty \le \|q\|_\infty$ for every $q \in \mathscr{A}_G$, so that finally, for $p \in \mathscr{A}_S$,

$$\|Vp\|_\infty = \|p\|_\infty. \tag{6}$$

It is evident from the definition that also

$$V\mathscr{A}_S = \mathscr{A}_G. \tag{7}$$

From (6) and (7) it follows that if $\tilde{\mathscr{A}}_S$ denotes the closure of $\mathscr{A}_S$ in L^∞, then

$$V\tilde{\mathscr{A}}_S = C(G), \tag{8}$$

V remains multiplicative on $\tilde{\mathscr{A}}_S$, and (6) is valid for every $f \in \tilde{\mathscr{A}}_S$. From the real algebraic isometry V and from (8) we deduce that if ϕ is a real continuous function defined on the complex plane, then $\phi \circ f = \phi(f) \in \tilde{\mathscr{A}}_S$ whenever $f \in \tilde{\mathscr{A}}_S$. Let now $f \in L^\infty(S, \Sigma, m) = L^\infty$. There exists a sequence $\{p_n\} \subset \mathscr{A}_S$ such that $p_n \to f$ in L^2 and also m-everywhere in S. Take $\infty > b > a > \|f\|_\infty$, and choose a continuous function ϕ defined on the complex plane, with $0 \le \phi \le a$ and

$$\phi(\lambda) = \lambda \quad \text{if } |\lambda| \le a, \qquad \phi(\lambda) = 0 \quad \text{if } |\lambda| \ge b.$$

Then the functions $f_n = \phi \circ p_n$ also converge (in L^2) to f, and

$$f_n \in \tilde{\mathscr{A}}_S, \qquad \|f_n\|_\infty \le a.$$

Since $Vf_n \to Vf$ in $L^2(G, \mu)$, we may suppose (taking a subsequence, if necessary) that $Vf_n \to Vf$ μ-almost everywhere, so that by an application of (6) we obtain the inequality $\|Vf\|_\infty \le a$, from which it follows easily that

$$\|Vf\|_\infty \le \|f\|_\infty \quad (f \in L^\infty). \tag{9}$$

Let now $p \in \mathscr{A}_S$, $g \in L^2$, and choose $\{p_n\} \subset \mathscr{A}_S$ so that $p_n \to g$ (in L^2). By (6) (or by (9)) we have (in $L^2(G, \mu)$) the equalities

$$V(pg) = \lim V(pp_n) = \lim Vp \cdot Vp_n = Vp \cdot \lim Vp_n = Vp \cdot Vg,$$

that is,

$$V(pg) = Vp \cdot Vg \quad (p \in \mathscr{A}_S,\ g \in L^2). \tag{10}$$

Letting $g = f \in L^\infty$ in (10), and using (9) and the fact that $\mathscr{A}_S$ is dense in L^2, we deduce that

$$V(gf) = Vg \cdot Vf \tag{11}$$

for every $g \in L^2$. In this manner and in view of (9) and (11), we conclude that V maps L^∞ into $L^\infty(G, \mu)$ and is multiplicative on L^∞. Using (8) and replacing V with V^{-1}, we see further that V actually maps L^∞ onto $L^\infty(G, \mu)$.

This completes the proof.

Remarks. 1. One can always take $\Gamma = \Gamma_T$. In this case, the automorphism $\mathscr{T}'$ is a conjugate invariant; this means that if T_1 is conjugate with T_2, and G_1, $\mathscr{T}_1'$, μ_1 and G_2, $\mathscr{T}_2'$, μ_2 are constructed as in the theorem, with $\Gamma = \Gamma_{T_1}$ (respectively, $\Gamma = \Gamma_{T_2}$), then there exists a homeomorphic isomorphism h of G_1 on G_2 such that $h\mathscr{T}_1' = \mathscr{T}_2' h$.

2. Subgroups (of generalized proper functions) of Γ_T were already considered long ago, by P. R. Halmos and J. von Neumann [4]. A detailed study of these groups was carried out by L. M. Abramov [1].

3. The last part of the proof, concerning (5) and the multiplicativity of V, is similar to one given in [2].

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