ASYMPTOTIC VALUES OF MEROMORPHIC FUNCTIONS

K. F. Barth

1. INTRODUCTION

Let $\mathfrak{D}$ denote the unit disc $\{|z| < 1\}$, and let $\mathfrak{C}$ denote the unit circle $\{|z| = 1\}$. The purpose of this paper is to derive some results on asymptotic values of functions meromorphic in $\mathfrak{D}$. G. R. MacLane [12, p. 7] considered the classes $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{L}$ of functions that are nonconstant and holomorphic in $\mathfrak{D}$. $\mathscr{A}$ is the class of functions having asymptotic values at a dense set on $\mathfrak{C}$. $\mathscr{B}$ is the class of functions for which there exists a set of Jordan arcs Γ in $\mathfrak{D}$, with end points dense on $\mathfrak{C}$, such that on each Γ either $f \to \infty$ or f is bounded. The class $\mathscr{L}$ is defined as follows: $f \in \mathscr{L}$ if and only if each level set $\{z: |f(z)| = \lambda\}$ "ends at points" of $\mathfrak{C}$ (the precise definition will be found early in Section 3). MacLane proved that $\mathscr{A} = \mathscr{B} = \mathscr{L}$. We shall consider the corresponding classes $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ of meromorphic functions.

The classes $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ are defined in Section 3. We prove that

$$\mathscr{A}_m \subset \mathscr{B}_m \quad \text{and} \quad \mathscr{L}_m \subset \mathscr{B}_m,$$

and we give examples showing that

$$\mathscr{B}_m \not\subset \mathscr{A}_m, \quad \mathscr{B}_m \not\subset \mathscr{L}_m, \quad \mathscr{A}_m \not\subset \mathscr{L}_m, \quad \mathscr{L}_m \not\subset \mathscr{A}_m.$$

Section 4 is concerned with the existence of asymptotic values on sets of positive measure. We prove (Theorem 5) that if $f \in \mathscr{A}_m$ and there exists a complex number a (possibly ∞) such that $N(r, a, f) = O(1)$, then on each subarc γ of $\mathfrak{C}$ on which f does not have the asymptotic value a, f has asymptotic values on a set of positive measure. Here $N(r, a, f)$ denotes the Nevanlinna counting function of f. Theorem 5 generalizes a theorem of MacLane [12, Theorem 11]. This result, together with Theorem 8, extends a theorem of Bagemihl [1, Theorem 1], which is a generalization of [4, Theorem 3].

In Section 5 we establish sufficient conditions for f to belong to $\mathscr{A}_m$. The fundamental condition (see Theorem 7) is as follows. If there exist a complex number a (possibly ∞) and a set Θ, dense on $[0, 2\pi]$, such that

$$\int_0^1 (1 - r)\log^+\left|\frac{1}{f(re^{i\theta}) - a}\right| dr < \infty \quad \text{and} \quad N(r, a, f) = O(1) \qquad (\theta \in \Theta,\ a \neq \infty),$$

then $f \in \mathscr{A}_m$. (If $a = \infty$, change $1/(f - a)$ to f.) A more restrictive condition is

$$\int_0^1 (1 - r)T(r)dr < \infty \qquad \text{and} \qquad N(r, a, f) = O(1),$$

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where T(r) is the Nevanlinna characteristic of f. These conditions generalize conditions (I) and (III) of MacLane [12, Section 7]. We give an example showing that the condition N(r, a) = O(1) cannot be relaxed to $\delta(a) = 1$, where $\delta(a)$ is the Nevanlinna defect of a. MacLane [11] has constructed a meromorphic function f in $\mathfrak{D}$, without any asymptotic value whatsoever, such that T(r, f) is of arbitrarily slow growth. Another important sufficient condition is that if f is nonconstant, meromorphic, and normal in the sense of Lehto and Virtanen, and if there exists a complex number a (possibly ∞) such that N(r, a, f) = O(1), then $f \in \mathscr{A}_m$ (see Theorem 8). This generalizes a theorem that was proved independently by Bagemihl and Seidel [4, Corollary 1] and by MacLane [12, Theorem 17]. It also extends [1, Corollary 1].

In Section 6 the classes $\mathscr{A}'_m$, $\mathscr{B}'_m$, and $\mathscr{L}'_m$ are defined. $f \in \mathscr{A}'_m$, $\mathscr{B}'_m$, or $\mathscr{L}'_m$ if and only if $f \in \mathscr{A}_m$, $\mathscr{B}_m$, or $\mathscr{L}_m$, respectively, and $N(r, \infty, f) = O(1)$. We prove that $\mathscr{A}'_m = \mathscr{B}'_m \supset \mathscr{L}'_m$ and that Koebe's Lemma holds for functions in $\mathscr{A}'_m$. The extension of Koebe's Lemma generalizes a result of MacLane [12, Theorem 9], and it overlaps with a theorem of Bagemihl and Seidel [4, Theorem 1].

Section 7 is devoted to results about asymptotic tracts of functions in $\mathscr{A}_m$. One of the most interesting results is that if $f \in \mathscr{A}_m$ and there exist complex numbers a, b (one of which may be ∞) such that $a \neq b$, N(r, a) = O(1), and N(r, b) = O(1), then f has *no* arc tracts.

2. PRELIMINARIES

In the following, the symbols

$$N(r, a), \quad m(r, a), \quad T(r), \quad \delta(a)$$

will have their usual meanings (see [14, p. 166]). It is convenient to make the following definition. Let $\{\gamma_n\}$ be a sequence of continuous curves, compact in $\mathfrak{D}$, and let γ be an arc $\{z: |z| = 1, \alpha \leq \arg z \leq \beta\}$.

Definition. $\gamma_n \to \gamma$ if for each $\varepsilon > 0$ there exists an n_0 such that, whenever $n > n_0$,

$$\gamma_n \subset \{1 - \varepsilon < |z| < 1\}, \quad \left|\inf_{\gamma_n} \arg z - \alpha\right| < \varepsilon, \quad \text{and} \quad \left|\sup_{\gamma_n} \arg z - \beta\right| < \varepsilon.$$

The terms *asymptotic value, asymptotic tract, end of a tract, arc tract,* and *point tract* will also have their usual meaning (see [12, Section 2] for definitions). A tract $\{\mathfrak{T}(\varepsilon), a\}$ $(\varepsilon > 0)$ will be called *global* if the end of the tract is $\mathfrak{C}$ and for each arc γ on $\mathfrak{C}$ there exists a sequence of arcs $\gamma_n \subset \mathfrak{T}(1/n)$ such that $\gamma_n \to \gamma$.

We shall say that f *has the asymptotic value* a *at* ζ $(|\zeta| = 1)$ if there exists a curve ending at ζ on which f has the asymptotic value a.

3. THE CLASSES $\mathscr{A}_m$, $\mathscr{B}_m$, $\mathscr{L}_m$

Let f be meromorphic and nonconstant in $\mathfrak{D}$. Let a be any complex number (possibly ∞), and consider any ζ such that $|\zeta| = 1$. We say that $\zeta \in A_a$ provided f has the asymptotic value a at ζ. In order to avoid confusion, A_a will sometimes be denoted by $A_a(f)$. If S is any subset of the sphere, we write

$$A(f, S) = \bigcup_{a \in S} A_a, \qquad A(f, S) = \square \text{ if } S = \square. \tag{3.1}$$

In particular, if b is any complex number, we write

$$A_b^* = \bigcup_{a \neq b} A_a, \qquad A = A_\infty^* \cup A_\infty. \tag{3.2}$$

Definition. Let f be meromorphic and nonconstant in $\mathfrak{D}$. Then $f \in \mathscr{A}_m$ provided A(f) is dense on $\mathfrak{C}$.

We now define the set B^*. A point ζ such that $|\zeta| = 1$ is said to belong to B^* provided there exists a continuous arc $\Gamma \subset \mathfrak{D}$, ending at ζ, such that f is bounded on Γ.

We write

$$B(f) = B = A_\infty \cup B^*. \tag{3.3}$$

Definition. $f \in \mathscr{B}_m$ provided f is meromorphic and nonconstant in $\mathfrak{D}$ and B is dense on $\mathfrak{C}$.

It is clear that $A_\infty^* \subset B^*$ and $A \subset B$; hence

$$\mathscr{A}_m \subset \mathscr{B}_m. \tag{3.4}$$

For any f defined on $\mathfrak{D}$ and any $\lambda > 0$, we shall denote the level set $\{z: |f| = \lambda\}$ (short notation for $\{z: |f(z)| = \lambda\}$) by $L(\lambda)$. A component of $L(\lambda)$ is called a *level curve*, and we denote it by $C(\lambda)$.

Let S be any subset of $\mathfrak{D}$. For each r $(0 < r < 1)$, let the components of $S \cap \{r < |z| < 1\}$ be $S_i(r)$, where i ranges over some index set I. Let $\delta_i(r) = \operatorname{diam} S_i(r)$, and set

$$\delta(r) \equiv \sup_{i \in I} \delta_i(r),$$

with $\delta(r) \equiv 0$ if I is void. Clearly, $\delta(r) \downarrow$ as $r \uparrow$. We shall say that S *ends at points* of $\mathfrak{C}$ provided $\delta(r) \downarrow 0$ as $r \uparrow 1$.

Definition. f belongs to the class $\mathscr{L}_m$ (the class $\mathscr{L}_m^*$) provided it is meromorphic and nonconstant in $\mathfrak{D}$, and every level set $L(\lambda)$ (every level curve $C(\lambda)$) ends at points of $\mathfrak{C}$.

It is clear that

$$\mathscr{L}_m \subset \mathscr{L}_m^*. \tag{3.5}$$

Our definitions of $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ are the same as MacLane's definitions of $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{L}$, except that we have replaced the word "holomorphic" with "meromorphic."

THEOREM 1. *Let* $f \in \mathscr{A}_m$, *and let* $\{\gamma_n\}$ *be a sequence of disjoint simple arcs in* $\mathfrak{D}$ *that tend to the arc* γ *of* $\mathfrak{C}$, *with the property that there exists a complex number* a *such that*

$$\sup_{\gamma_n} |f(z) - a| = \mu_n \to 0 \ (n \to \infty) \quad (\text{if } a \neq \infty), \tag{3.6}$$

$$\inf_{\gamma_n} |f(z)| = \mu_n \to \infty \ (n \to \infty) \quad (\text{if } a = \infty). \tag{3.7}$$

Then f *has an arc tract* $\{\mathfrak{T}(\varepsilon), a\}$ *with end* K *such that* $\gamma \subset K$ *and such that, for each point* ζ *of* K, *some curve* Γ *belonging to* $\{\mathfrak{T}(\varepsilon), a\}$ *ends at* ζ. *At any interior point* ζ *of* K, *the only asymptotic values come from this tract* $\{\mathfrak{T}(\varepsilon), a\}$. *If* f *is in* $\mathscr{L}_m$ *(but not necessarily in* $\mathscr{A}_m$*), the above conclusions are true for* $a = \infty$.

Remark. For holomorphic functions in $\mathscr{A}$, $\mathscr{B}$, or $\mathscr{L}$, this theorem was proved by G. R. MacLane [12, Theorem 3]. It is an open question whether the conclusions of Theorem 1 are true for $f \in \mathscr{L}_m$ and $a \neq \infty$.

Proof. For f in $\mathscr{L}_m$ and $a = \infty$, we omit the proof, because it is the same as that of [12, Theorem 3].

Suppose $f \in \mathscr{A}_m$ and $a = \infty$. Let $\gamma = \{e^{i\theta}: \alpha \leq \theta \leq \beta\}$, and let $S(\alpha, \beta)$ denote the sector $\{z: \alpha \leq \arg z \leq \beta, |z| < 1\}$. Then $L(\lambda) \cap S(\alpha, \beta)$ ends at points of $\mathfrak{C}$ for all $\lambda > 0$. If not, there would exist a $\lambda_1 > 0$, a subarc $\triangle$ of γ, and a sequence $\{\triangle_n\}$ of continuous arcs, compact in $\mathfrak{D}$, such that $\triangle_n \subset L(\lambda_1)$ for all n and $\triangle_n \to \triangle$ as $n \to \infty$. Let ζ be any interior point of $\triangle$. Each curve ending at ζ must cross all but a finite number of the $\triangle_n$ and γ_n, and thus f cannot have an asymptotic value at ζ. This contradicts the hypothesis that $f \in \mathscr{A}_m$. Hence $L(\lambda) \cap S(\alpha, \beta)$ ends at points of $\mathfrak{C}$; again, exactly the same proof as for [12, Theorem 3] works.

Finally, suppose a is finite. By applying Theorem 1 with $a = \infty$ to the function $1/(f - a)$, we obtain the desired result.

THEOREM 2. *Let* $f \in \mathscr{L}_m$. *Suppose* $\gamma = \{e^{i\theta}: \alpha \leq \theta \leq \beta, \alpha \neq \beta\}$ *is a subarc of* $\mathfrak{C}$ *such that no level curve of* f *ends at any point of* γ. *Then exactly one of the following two statements is valid.*

(3.8) *For each interior point* $e^{i\phi}$ $(\alpha < \phi < \beta)$ *of* γ *there exists a continuous curve* $\Gamma(e^{i\phi}) \subset \mathfrak{D}$ *ending at* $e^{i\phi}$ *and such that* f *is bounded on* $\bigcup_{\alpha < \phi < \beta} \Gamma(e^{i\phi})$. *Moreover,* f *does not have the asymptotic value* ∞ *at any interior point of* γ.

(3.9) *There exists an arc tract for* ∞ *of* f *with end* K *such that* $\gamma \subset K$.

Proof. It is easy to show that (3.8) and (3.9) cannot occur for the same γ; we shall prove that either (3.8) or (3.9) must be valid. Let $S(\alpha, \beta)$ be the sector

$$S(\alpha, \beta) = \{z: |z| < 1 \text{ and } \alpha < \arg z < \beta\}. \tag{3.10}$$

Pick $\{\lambda_n\}_{n=1}^{\infty}$ so that $0 < \lambda_n \uparrow \infty$ and $L(\lambda_n)$ has no multiple points. Then, since $f \in \mathscr{L}_m$, each $C(\lambda_n)$ is either a closed Jordan curve or a crosscut of $\mathfrak{D}$. We may suppose that the origin is not a pole of f. If it is, pick a point a near 0 and repeat the following argument, using a in place of 0. Let N be such that

$$0 \in \{z: |f| < \lambda_N\}.$$

For any $n \geq N$, let $\triangle(\lambda_n)$ denote the component of $\{z: |f| < \lambda_n\}$ that contains 0. Since $f \in \mathscr{L}_m$ and no level curve of f ends at any point of γ, at least one of the following statements must be valid for any $n \geq N$:

(3.11) $\left\{ \begin{array}{l} \text{There exists a } \tau_n \subset \partial\Delta(\lambda_n) \text{ such that } \tau_n \text{ is a crosscut of the sector} \\ S(\alpha, \beta) \text{ that joins a point of } \arg z = \alpha \text{ to a point of } \arg z = \beta. \end{array} \right.$

$$\partial\Delta(\lambda_n) \supset \gamma. \tag{3.12}$$

If (3.11) is valid for all $n \geq N$, it is clear that $\tau_n \to \gamma$, and thus, by Theorem 1, f has an arc tract for ∞ with end $K \supset \gamma$. Hence, (3.9) holds.

Now suppose (3.12) is true for some $n = M$. Let $\zeta = e^{i\phi}$ $(\alpha < \phi < \beta)$ be any interior point of γ; by (3.12), $\zeta \in \partial\Delta(\lambda_M)$. Since $f \in \mathscr{L}_m$ and no level curves of f end at points of γ, there exists a $\delta(\zeta) > 0$ such that each component of $\partial\Delta(\lambda_M)$ having a nonempty intersection with the set

$$U(\delta, \zeta) = \{z: |z - \zeta| < \delta, \quad |z| < 1\} \tag{3.13}$$

is a closed Jordan curve contained in $S(\alpha, \beta)$. This, together with the hypothesis that the diameter of the set $L(\lambda_M) \cap \{z: 1 - \varepsilon < |z| < 1\}$ tends to zero as $\varepsilon \downarrow 0$ $(f \in \mathscr{L}_m)$, implies that 0 and ζ may be connected by a continuous curve $\Gamma(e^{i\phi}) \subset \Delta(\lambda_M) \cup \zeta$.

To prove the final statement in (3.8), note that the existence of the asymptotic value ∞ at ζ implies that $L(\lambda)$ ends at ζ for all $\lambda > \lambda_M$. This is contradictory. Thus (3.8) is valid, and the proof is complete.

It is clear from the proof that Theorem 2 may be generalized as follows: the condition $f \in \mathscr{L}_m$ may be replaced by the requirement that for each $\zeta \in \gamma^0$ (^{0} denotes interior) there exists a $\delta(\zeta) > 0$ such that the set $\{|f| = \lambda\} \cap U(\delta, \zeta)$ ends at points of $\mathfrak{C}$ for all $\lambda > 0$.

Now we shall prove the promised results.

THEOREM 3. *$\mathscr{A}_m \subset \mathscr{B}_m$, $\mathscr{L}_m \subset \mathscr{B}_m$, and no other inclusion relations between $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ are valid.*

Proof. We have already shown that $\mathscr{A}_m \subset \mathscr{B}_m$ (see (3.4)); we shall now prove that $\mathscr{L}_m \subset \mathscr{B}_m$. Suppose $f \in \mathscr{L}_m$, and consider any subarc $\gamma = \{e^{i\theta}: \alpha \leq \theta \leq \beta\}$ of $\mathfrak{C}$. We shall show that there exists either a continuous curve ending at some point of γ on which f is bounded, or else a continuous curve ending at some point of γ on which f has the asymptotic value ∞. If a level curve of f ends at a point of γ, we are done. If not, Theorem 2 applies, and we see that either for each interior point $e^{i\theta}$ $(\alpha < \theta < \beta)$ of γ there exists a continuous curve $\Gamma(\theta) \subset \mathfrak{D}$ that ends at $e^{i\theta}$ and on which f is bounded, or there is an arc tract of f for ∞ with end K such that $\gamma \subset K$. If the first case occurs, we are through; in the second case, applying Theorem 1, we see that f has the asymptotic value ∞ at each point of γ. Hence $f \in \mathscr{B}_m$. Examples 1 and 2 (see below) imply that no other inclusion relations between $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ are valid.

EXAMPLE 1. We shall construct a function f, *meromorphic and nonconstant in* $\mathfrak{D}$, *such that* $f \in \mathscr{B}_m$ *and* $f \in \mathscr{L}_m$ *but* $f \notin \mathscr{A}_m$. *Thus* $\mathscr{B}_m \not\subset \mathscr{A}_m$ *and* $\mathscr{L}_m \not\subset \mathscr{A}_m$. This example is due to Lehto and Virtanen (see [9, p. 58]).

Let h be a "modular function" omitting the values 0, 1, and 5; that is, let h map the unit disc one-to-one and conformally onto the universal covering surface of the complex sphere with the points 0, 1, and 5 removed. We know that h is normal in the sense of Lehto and Virtanen [9, p. 53]. Also, h has a radial limit $h(e^{i\theta})$ at each $e^{i\theta} \in M$, and $h(e^{i\theta})$ does not exist for any $e^{i\theta}$ in $\mathfrak{C} - M$, where M is a countable dense subset of $\mathfrak{C}$. By [10, Theorem 6] there exists a function g, holomorphic and bounded in $\mathfrak{D}$, such that $g(e^{i\theta})$ exists for each $e^{i\theta} \in \mathfrak{C} - M$ and for no $e^{i\theta} \in M$. The function $f = g + h$ is normal, since g is bounded and h is normal [9, p. 53], and f cannot have any radial limits. Since f is normal, it can have no asymptotic values [9, Theorem 2]. It is now clear that $f \in \mathscr{B}_m$ but $f \notin \mathscr{A}_m$.

Next we want to show that $f \in \mathscr{L}_m$. We know that $A_0(h)$, $A_1(h)$, and $A_5(h)$ are all dense in $\mathfrak{C}$, and we may suppose that $|g| \leq 1$. Now suppose that $f \notin \mathscr{L}_m$; then, for some $\lambda > 0$, there exists a sequence $\{\gamma_n\}$ of arcs such that $\gamma_n \to \gamma$, a subarc of $\mathfrak{C}$, and $|f| = \lambda$ on γ_n. Pick $e^{i\theta_1} \in \gamma^0$ so that $h(e^{i\theta_1}) = 0$, and $e^{i\theta_2} \in \gamma^0$ so that $h(e^{i\theta_2}) = 5$. It is easy to see that this is incompatible with the condition $|f| = \lambda$ on γ_n. Thus $f \in \mathscr{L}_m$ but $f \notin \mathscr{A}_m$. Note that Example 1 also shows that *$\mathscr{A}_m$ is not a linear space.*

EXAMPLE 2. Using a theorem of Mergelyan, we shall construct a *function* f, *meromorphic in* $\mathfrak{D}$, *such that* $f \in \mathscr{A}_m$, $f \in \mathscr{B}_m$, *but* $\{z: |f| = 1\}$ *contains a sequence of arcs that approaches* $\mathfrak{C}$, *that is*, $f \notin \mathscr{L}_m$. The construction is similar to that used by Bagemihl and Seidel in [2].

Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive numbers $(r_n \uparrow 1)$; for $n \geq 1$, let

$$C_n = \{|z| = r_n\}, \tag{3.14}$$

$$D_n = \{|z| < r_n\}, \tag{3.15}$$

$$E_n = \{z: r_n \leq |z| \leq r_{n+1};\ \arg z = 2k\pi/2^n\} \quad (k = 0, 1, \cdots, 2^n - 1); \tag{3.16}$$

and for $n > 1$, let

$$F_n = D_{n-1}^- \cup E_{n-1} \cup C_n. \tag{3.17}$$

Now we shall inductively define two sequences of functions, $\{f_n(z)\}_{n=1}^{\infty}$ and $\{R_n(z)\}_{n=1}^{\infty}$.

First, let $f_1(z)$ and $R_1(z)$ be defined on D_1^- so that $f_1 \equiv R_1(z) \equiv 1/2$.

Next construct $f_2(z)$ so that it is continuous on F_2 and

$$f_2(z) = f_1(z) \quad \text{on } D_1^-, \tag{3.18a}$$

$$f_2(z) = 5/4 \quad \text{on } C_2, \tag{3.18b}$$

$$f_2(z) \text{ is linear on each component of } E_1. \tag{3.18c}$$

It is clear that F_2 is closed and that it divides the plane into a finite number of regions. Also, $f_2(z)$ is continuous on F_2 and analytic in the interior of F_2. Thus, by a remark of Mergelyan [13, p. 24], there exists a rational function $R_2(z)$ such that

$$\max_{z \in F_2} |f_2(z) - R_2(z)| < 2^{-4}. \tag{3.19}$$

Let $\{a(2, k)\}_{k=1}^{N_2}$ denote the poles of $R_2(z)$ that are contained in D_2^-, and let $P(2, k, z)$ denote the principal part of $R_2(z)$ at $a(2, k)$. Now construct $f_3(z)$ so that it is continuous (in the spherical metric) on F_3 and

(3.20a) $$f_3(z) = R_2(z) \quad \text{on } D_2^-,$$

(3.20b) $$f_3(z) = 1 - 2^{-3} \quad \text{on } C_3,$$

(3.20c) $f_3(z)$ is linear on each component of E_2.

Then the function

(3.21) $$g_3(z) = f_3(z) - \sum_{k=1}^{N_2} P(2, k, z)$$

is continuous on F_3 and analytic at interior points of F_3. As before, there exists a rational function $S_3(z)$ such that

$$\max_{z \in F_3} |g_3(z) - S_3(z)| < 2^{-5}.$$

Hence

$$\max_{z \in F_3} \left|f_3(z) - \left[S_3(z) + \sum_{k=1}^{N_2} P(2, k, z)\right]\right| < 2^{-5}.$$

Letting

$$R_3(z) = S_3(z) + \sum_{k=1}^{N_2} P(2, k, z),$$

we obtain the estimate

(3.22) $$\max_{z \in F_3} |f_3(z) - R_3(z)| < 2^{-5}.$$

We denote the poles of $R_3(z)$ by $\{a(3, k)\}_{k=1}^{N_3}$ and the principal part of $R_3(z)$ at $a(3, k)$ by $P(3, k, z)$.

In general, suppose that $f_n(z)$ is continuous (spherically) on F_n, and that

(3.23a) $$f_n(z) = R_{n-1}(z) \quad \text{on } D_{n-1}^-,$$

(3.23b) $$f_n(z) = 1 + (-1)^n 2^{-n} \quad \text{on } C_n,$$

(3.23c) $f_n(z)$ is linear on each component of E_{n-1}.

We can find an $R_n(z)$ such that

(3.24) $$\max_{z \in F_n} |f_n(z) - R_n(z)| < 2^{-n-2}.$$

A straightforward calculation shows that $\{R_n(z)\}$ converges to a meromorphic function $R(z)$ in $\mathfrak{D}$.

In order to show that $R(z) \notin \mathscr{L}_m$ it suffices to show that for each n some component of $\{z: |R| = 1\}$ separates C_n and C_{n+1}. If we prove that

(3.25) $$|R(z) - (1 + (-1)^n 2^{-n})| < 2^{-n-1} \qquad (z \in C_n),$$

it is clear that a component of $\{z: |R| = 1\}$ must separate C_n and C_{n+1}. The proof of (3.25) and the proof that f has the asymptotic value 1 on each radius of the form

(3.26) $$\{z: 0 \leq |z| < 1, \arg z = k2^{-n}\} \qquad (n = 1, 2, \cdots \text{ and } k = 0, 1, 2, \cdots, 2^n - 1)$$

consist of straightforward calculations (see [5, Example 2] for details). Since (3.26) is dense, we see that $f \in \mathscr{A}_m$. Thus $\mathscr{A}_m \not\subset \mathscr{L}_m$. Also, note that $F \in \mathscr{B}_m$, which implies that $\mathscr{B}_m \not\subset \mathscr{L}_m$.

MacLane [12, p. 18] has shown that if f is holomorphic and $f \in \mathscr{L}$, then $f + a \in \mathscr{L}$ for each finite complex number a. We shall show that this is *not* true for meromorphic functions. Consider the function f constructed in Example 2. We know that $f \notin \mathscr{L}_m$ and that $A_1(f)$ is dense in $\mathfrak{C}$. Thus $A_0(f - 1)$ is dense in $\mathfrak{C}$, which implies that $f - 1 \in \mathscr{L}_m$. Hence $f \notin \mathscr{L}_m$ but $f - 1 \in \mathscr{L}_m$.

4. ASYMPTOTIC VALUES ON SETS OF POSITIVE MEASURE

In the proof of Theorem 5, we shall need the measurability of the set A(f, S) defined in Section 3.

THEOREM 4. *Let* $f \in \mathscr{A}_m$, *and let* S *be a Borel set on the sphere. Then* A(f, S) *is measurable.*

(Here, measurable means Lebesgue measurable as a set in $[0, 2\pi]$.)

Proof. Theorem 4 was proved by MacLane [12, Theorem 10] for $f \in \mathscr{A}$ (see Section 3). Because the proof of Theorem 4 is an easy modification of the proof of [12, Theorem 10], we omit it (see [5, Theorem 4] for details).

We can now prove a generalization of [12, Theorem 11].

THEOREM 5. *Let* $f \in \mathscr{A}_m$. *Suppose* a *is a complex number (possibly* ∞*) such that* $N(r, a, f) = O(1)$, *and let* γ *be any subarc of* $\mathfrak{C}$ *such that* $A_a \cap \gamma = \Box$. *Then* $\text{meas}(A_a^* \cap \gamma) > 0$.

Remark. The inequality $\text{meas}(A_a^* \cap \gamma) < \text{meas}(\gamma)$ is possible (see [12, p. 75]).

Proof. By Theorem 4, A_a^* and hence $A_a^* \cap \gamma$ is measurable. We may suppose that $a = \infty$, since if $N(r, a) = O(1)$ for some finite a, we may obtain the conclusion by applying Theorem 5 with $a = \infty$ to the function $1/(f - a)$. Suppose also that f has a pole of order λ at $z = 0$ (where $\lambda = 0$ if f is holomorphic at $z = 0$). Let the poles of f be denoted by $b_k = |b_k| e^{i\beta_k}$, where a pole of order μ appears μ times among the b_k. It is known that if $N(r, \infty)$ is bounded for $0 \leq r < 1$, then the product

(4.1) $$B(z) = z^\lambda \prod_{k=1}^{\infty} \frac{|b_k| - ze^{-i\beta_k}}{1 - \bar{b}_k z}$$

converges subuniformly in $\mathfrak{D}$ to a holomorphic function, and $|B(z)| \leq 1$ (see [14, p. 188]). The function

(4.2) $$F(z) = f(z)B(z)$$

is holomorphic in $\mathfrak{D}$.

There are now two possibilities. Either

(4.3a) $\quad$ F is bounded in some neighborhood of some point ζ_0 on γ,

or

(4.3b) $$\limsup_{z \to \zeta} |F(z)| = \infty \quad (\text{all } \zeta \in \gamma).$$

If (4.3a) occurs, let $U = \{z: |z - \zeta_0| < \delta\} \cap \mathfrak{D}$, where δ is chosen so that F is bounded in U. Also, let $\gamma_1 = U^- \cap \mathfrak{C}$. Choose an elementary function g(Z) so that g(Z) maps $\{|Z| < 1\}$ one-to-one and conformally onto U. Now γ_1 corresponds to a subarc γ_2 of $\{|Z| = 1\}$. Write

(4.4) $$\Phi(Z) = F(g(Z)) = F(z) \quad (z \in U)$$

and

(4.5) $$\Psi(Z) = B(g(Z)) = B(z) \quad (z \in U).$$

Then Φ and Ψ are bounded in $\{|Z| < 1\}$, and thus

$$\Phi(Z)/\Psi(Z) = F(g(Z))/B(g(Z))$$

is a function of bounded characteristic in $\{|Z| < 1\}$. Hence Φ/Ψ has finite radial limits on a set $E^* \subset \gamma_2$ such that

(4.6) $$m(E^*) = m(\gamma_2) > 0.$$

Since U is a Jordan domain, each radial limit $\Phi(e^{i\theta})/\Psi(e^{i\theta})$ $(e^{i\theta} \in E^*)$ corresponds to a point asymptotic value of F in U. Also, E^* corresponds to a set $E \subset \gamma_1$ such that $m_e(E) > 0$ ($m_e(E)$ is the exterior measure of E), since g is an elementary function. Because $E \subset A_\infty^* \cap \gamma$, we see that

(4.7) $$m(A_\infty^* \cap \gamma) \geq m_e(E) > 0.$$

Now suppose that (4.3b) occurs. Pick two distinct points ζ_1 and ζ_2 of γ and two curves $\Delta(\zeta_1)$ and $\Delta(\zeta_2)$ such that $\Delta(\zeta_1)$ and $\Delta(\zeta_2)$ end at ζ_1 and ζ_2, respectively, and such that f tends to a finite limit on $\Delta(\zeta_1)$ and $\Delta(\zeta_2)$ as $|z| \to 1$. Since $|B(z)| \leq 1$, $|F|$ is bounded on $\Delta(\zeta_1)$ and $\Delta(\zeta_2)$ for $|z|$ sufficiently near 1. Hence $|F| \leq M$ on some crosscut γ_1 of $\mathfrak{D}$ that joins ζ_1 and ζ_2. The crosscut γ_1 and the arc $\gamma' = (\zeta_1, \zeta_2) \subset \gamma$ bound a domain H. Let $\chi(s)$ map $\{|s| \leq 1\}$ one-to-one onto H^- so that $\chi(s)$ is conformal in $\{|s| < 1\}$ and continuous in $\{|s| \leq 1\}$. Consider the function $F_0(s)$, holomorphic in $\{|s| < 1\}$, given by

(4.8) $$F_0(s) = F(\chi(s));$$

we shall show that $F_0 \in \mathscr{B}$. Let $\chi^{-1}[\gamma_1]$ and $\chi^{-1}[\gamma']$ denote the image by $\chi^{-1}(z)$ of γ_1 and γ', respectively. Obviously, F_0 is bounded on each arc that approaches a point of $\chi^{-1}[\gamma_1]$. Since $f \in \mathscr{A}_m$ and $A_\infty(f) \cap \gamma = \Box$, there exists a dense subset γ'' of γ' each of whose points is the end of an arc on which f has a finite limit. Since $|B(z)| \leq 1$, F is bounded on each of these arcs, for $M_0 < |z| < 1$, where M_0 depends on the particular arc. Hence each point of $\chi^{-1}[\gamma'']$ is the end of an arc on which F_0 is bounded. Thus $F_0 \in \mathscr{B}$, and by [12, Theorem 1], $F_0 \in \mathscr{A}$. Hence at each point of some dense subset of γ', F has an asymptotic value. Recall that, for each $\lambda > 0$, $L(\lambda)$ denotes the set $\{z: |F| = \lambda\}$. In view of the above, it is clear that

$$(4.9) \qquad H \cap L(\lambda) \text{ ends at points of } \gamma', \text{ for each } \lambda > 0.$$

By (4.3b), F is unbounded in H. In the following argument, it is not necessary that n be an integer, and we may assume that n is such that the level set $L(n)$ has no multiple points. For $n > M$, H will contain at least one component of $\{z: |F| > n\}$; we shall denote these components in H by $T_{n,1}, T_{n,2}, \cdots$. If F were unbounded in every set $T_{n,k}$, for $n > M$ and for all k involved, then $T_{n,1}$ would contain at least one $T_{n+1,i}$, which we shall denote by $T_{n+1,1}$. Also $T_{n+1,1}$ would contain $T_{n+2,1}, \cdots$. The domains $T_{n,1} \supset T_{n+1,1} \supset \cdots$ determine an asymptotic tract of F with asymptotic value ∞; the end K of this tract is a subset of γ. Because of (4.9) (F is "in $\mathscr{L}_m$ near γ"), it is clear from the proof of Theorem 1 that F has the asymptotic value ∞ at each point of K. Since $|F| = |f \cdot B| \leq |f|$, f has the asymptotic value ∞ at each point of K, which contradicts the hypothesis that $A_\infty(f) \cap \gamma = \Box$. Hence F is bounded on some $T_{n,k}$; we shall denote this domain by T_0. The boundary of T_0 consists of various Jordan arcs and crosscuts Γ_0, on which $|F| = n$, and a set $E_1 \subset \gamma$. Also,

$$(4.10) \qquad n < |F(z)| < N \qquad (z \in T_0).$$

The set E_1 must be nonempty. For otherwise, we would have the inequality $\limsup |F(z)| \leq n$ at every boundary point of T_0. This would imply that $|F| \leq n$ in T_0, which contradicts (4.10). Each Jordan curve in Γ_0 creates a hole in T_0. We add all such holes to T_0 to obtain a simply connected domain $T \subset H$, bounded by E_1 and by crosscuts Γ on which $|F| = n$; also,

$$(4.11) \qquad |F(z)| < N \qquad (z \in T).$$

Now, if Γ contains infinitely many crosscuts, their diameters must approach zero, by (4.9). It follows easily that the boundary of T is a Jordan curve. The set E_1 contains no arcs, because of (4.3b) and (4.11), but we shall prove that some subset of E_1 has positive measure.

We may assume (using a linear transformation on $\mathfrak{D}$, if necessary) that $z = 0 \in T$. Let $z = g_1(Z)$ map $\{|Z| < 1\}$ one-to-one and conformally onto T, with $g(0) = 0$, and write

$$(4.12) \qquad \Phi_1(Z) = F(g_1(Z)) = F(z) \qquad (z \in T),$$

$$(4.13) \qquad \Psi_1(Z) = B(g_1(Z)) = B(z) \qquad (z \in T).$$

Then $\Phi_1(Z)$ and $\Psi_1(Z)$ are bounded in $\{|Z| < 1\}$, and by Fatou's Theorem, they have radial limits $\Phi_1(e^{i\theta})$ and $\Psi_1(e^{i\theta})$ almost everywhere. Note also that $\Psi_1(e^{i\theta}) \neq 0$ for almost all θ. Also, $\Phi_1(Z)$ may be expressed by the Poisson integral with boundary function $\Phi_1(e^{i\theta})$. Since $|\Phi_1| > n$ in part of $\{|Z| < 1\}$ (corresponding to T_0), it follows that $|\Phi_1(e^{i\theta})| > n$ on a set E_1^* of positive measure.

Thus the function

$$\Phi_1(Z)/\Psi_1(Z) = F(g_1(Z))/B(g_1(Z))$$

has finite radial limits on a set $E_2^* \subset E_1^*$ such that

$$m(E_2^*) = m(E_1^*) > 0.$$

Since T is a Jordan domain, each radial limit corresponds to a point asymptotic value of f in T. The set E_2^* maps onto a set $E_2 \subset E_1$, since $|F| = n$ on Γ.

An argument used by MacLane [12, p. 27] shows that $m_e(E_2) \geq m(E_2^*) > 0$. However, since $E_2 \subset A_\infty^*$, this implies that $m(A_\infty^* \cap \gamma) > 0$, which completes the proof of Theorem 5.

COROLLARY 1. *Let* f *and* γ *satisfy the hypotheses of Theorem* 5, *and let* V *be the set of asymptotic values that occur on* γ. *Then* V *contains a closed set* V_1 *of positive harmonic measure.*

Proof. This follows immediately if we apply Priwalow's theorem [15, p. 210] either to $\Phi(Z)/\Psi(Z)$ and its angular limits on the set E^* or to $\Phi_1(Z)/\Psi_1(Z)$ and its angular limits on the set E_2^* (depending on whether (4.3a) or (4.3b) occurs).

COROLLARY 2. *Let* $f \in \mathscr{B}_m$. *Suppose* $N(r, \infty, f) = O(1)$, *and let* γ *be any subarc of* $\mathfrak{C}$ *such that* $A_\infty \cap \gamma = \square$. *Then* $m_e(A_\infty^* \cap \gamma) > 0$.

The proof is essentially the same as that of Theorem 5. The corollary will be needed in the proof of Theorem 9.

5. SOME SUFFICIENT CONDITIONS FOR f TO BELONG TO $\mathscr{A}_m$

The most important sufficient condition we shall establish in this section is that if $N(r, a, f) = O(1)$ for some complex number a (possibly ∞) and the growth of T(r) is suitably restricted (see Theorem 7 for a precise statement), then $f \in \mathscr{A}_m$. First we shall prove the following theorem.

THEOREM 6. *Let* g *and* h *be holomorphic in* $\mathfrak{D}$, *and let* g/h *be nonconstant. Suppose* $g \in \mathscr{A}$ *and* h *is bounded, and let* $f = g/h$. *Then* $f \in \mathscr{A}_m$ *and* $1/f \in \mathscr{A}_m$.

Proof. Consider any subarc γ of $\mathfrak{C}$. We shall show that there exist a point $\zeta \in \gamma$ and a curve ending at ζ, on which f tends to a limit as $|z| \to 1$. First suppose that $A_\infty(g) \cap \gamma \neq \square$. Then there exist a point $\zeta \in \gamma$ and a curve $\triangle$ ending at ζ, on which $g \to \infty$ as $|z| \to 1$. It follows readily that $f \to \infty$ as $|z| \to 1$ on $\triangle$, and thus f has the asymptotic value ∞ at ζ.

Next suppose that $A_\infty(g) \cap \gamma = \square$. If g is bounded in some neighborhood of some point of γ, the conclusion is a trivial consequence of the Fatou-Nevanlinna Theorem. Thus we may suppose

$$\lim_{z \to \zeta} \sup |g(z)| = \infty \qquad (\text{all } \zeta \in \gamma). \tag{5.1}$$

Under these hypotheses, MacLane has shown [12, p. 26] that there exists a $D \subset \mathfrak{D}$ with the following properties: D is a simply connected Jordan domain, bounded by crosscuts Γ of $\mathfrak{D}$ on which $|g| = \lambda$ for some $\lambda > 0$, and by a nonempty subset F of γ; also,

$$|g(z)| < N \qquad (z \in D). \tag{5.2}$$

Moreover,

$$\lambda < |g(z)| < N \qquad (z \in D_0),$$

where D_0 is a nonempty subdomain of D. The argument used in the latter part of the proof of Theorem 5 (begin with the paragraph that contains (4.12)) shows that f has asymptotic values at some points of γ. Hence $f \in \mathscr{A}_m$, and it is now obvious that $1/f \in \mathscr{A}_m$.

The following three sufficient conditions generalize conditions (I), (II), and (III) of MacLane [12, pp. 35-37] to meromorphic functions. We shall say that f, meromorphic in $\mathfrak{D}$, satisfies condition (I) if there exist a complex number a (possibly ∞) and a set Θ, dense on $[0, 2\pi]$, such that

$$N(r, a) = O(1) \quad \text{and} \quad \int_0^1 (1 - r)\log^+\left|\frac{1}{f(re^{i\theta}) - a}\right| dr < \infty \quad (\theta \in \Theta)$$

if $a \neq \infty$. If $a = \infty$, the integral condition is

$$\int_0^1 (1 - r)\log^+ |f(re^{i\theta})|\, dr < \infty \quad (\theta \in \Theta).$$

Here no uniformity is implied; we merely require that each individual integral converge.

We shall eventually prove (Theorem 7) that if f satisfies (I), then $f \in \mathscr{A}_m$. However, we first examine two other sufficient conditions. The form of (I) suggests that we may be able to find a sufficient condition involving the *Schmiegungsfunktion* of Nevanlinna. In order to do this, let

$$(5.3) \qquad \sigma(a, \theta) = \int_0^1 (1 - r)\log^+\left|\frac{1}{f(re^{i\theta}) - a}\right| dr \qquad (a \neq \infty).$$

Then

$$(5.4) \qquad \frac{1}{2\pi}\int_0^{2\pi} \sigma(a, \theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (1 - r)\log^+\left|\frac{1}{f(re^{i\theta}) - a}\right| dr\, d\theta$$

$$= \int_0^1 (1 - r)\, m(r, a)\, dr \qquad (a \neq \infty)$$

for any meromorphic function f. (If $a = \infty$, make the obvious modifications in (5.3) and (5.4).)

We shall say that f, meromorphic in $\mathfrak{D}$, satisfies condition (II) if there exists a complex number a (possibly ∞) such that

$$N(r, a) = O(1) \quad \text{and} \quad \int_0^1 (1 - r)\, m(r, a)\, dr < \infty .$$

Since (II) implies that $\sigma(a, \theta)$ is finite for almost all θ (see (5.4)), we see that

$$(5.5) \qquad (II) \Rightarrow (I).$$

Finally, we shall say that f, meromorphic in $\mathfrak{D}$, satisfies condition (III) if

$$N(r, a) = O(1) \quad \text{and} \quad \int_0^1 (1 - r)T(r)\,dr < \infty$$

for some complex number a (possibly ∞). If $a = \infty$, it is clear that (III) $\Rightarrow$ (II). Using Nevanlinna's First Main Theorem [14, p. 168], we deduce that if a is finite, then $\int_0^1 (1 - r)m(r, a)\,dr < \infty$, and thus

$$\text{(III)} \Rightarrow \text{(II)}. \tag{5.6}$$

THEOREM 7. *Let* f *be meromorphic and nonconstant in* $\mathfrak{D}$. *Suppose that* f *satisfies one of the conditions* (I), (II), *and* (III). *Then* $f \in \mathscr{A}_m$.

Proof. Because of (5.5) and (5.6), it suffices to prove that (I) implies $f \in \mathscr{A}_m$. First suppose that $a = \infty$. As in the proof of Theorem 5, let B(z) be the Blaschke product with zeros at the poles of f. Then the function

$$g(z) = B(z)f(z) \tag{5.7}$$

is holomorphic in $\mathfrak{D}$, and

$$\int_0^1 (1 - r)\log^+|g(re^{i\theta})|\,dr = \int_0^1 (1 - r)\log^+|B(re^{i\theta})f(re^{i\theta})|\,dr$$

$$\leq \int_0^1 (1 - r)\log^+|B(re^{i\theta})|\,dr + \int_0^1 (1 - r)\log^+|f(re^{i\theta})|\,dr \qquad (\theta \in \Theta).$$

Thus

$$\int_0^1 (1 - r)\log^+|g(re^{i\theta})|\,dr \leq \int_0^1 (1 - r)\log^+|f(re^{i\theta})|\,dr \qquad (\theta \in \Theta), \tag{5.8}$$

since $|B| \leq 1$. Using (5.8) and (I), we see that

$$\int_0^1 (1 - r)\log^+|g(re^{i\theta})|\,dr < \infty \qquad (\theta \in \Theta). \tag{5.9}$$

By [12, Theorem 14], $g \in \mathscr{A}$, and therefore $f = g/B$, where $g \in \mathscr{A}$ and $|B| \leq 1$. Thus, by Theorem 6, we see that $f \in \mathscr{A}_m$.

If $a \neq \infty$, the argument above implies that $1/(f - a) \in \mathscr{A}_m$, and thus $f \in \mathscr{A}_m$. Hence the proof of Theorem 7 is complete.

In conditions (I), (II), and (III), the global restrictions on f may be replaced by certain local restrictions. Let $|\zeta| = 1$ and $\delta > 0$. Define

$$U(\delta, \zeta) = \{z\colon |z - \zeta| < \delta \text{ and } |z| < 1\},$$

$$U^*(\delta, \zeta) = \{z: |z - \zeta| < \delta \text{ and } |z| \leq 1\}.$$

Suppose that a covering $\{U^*(\delta_i, \zeta_i)\}_{i \in I}$ of $\{|\zeta| = 1\}$ is given, and let

$$F_i(Z) = f(G_i(Z)) = f(z) \quad (z \in U(\delta_i, \zeta_i)),$$

where $G_i(Z)$ maps $\{|Z| < 1\}$ one-to-one and conformally onto $U(\delta_i, \zeta_i)$. If, for each $i \in I$, $F_i(Z)$ satisfies one of the conditions (I), (II), and (III), it is easily proved that $f(z) \in \mathscr{A}_m$. Note in particular that the value a can be different for each i.

Examples 6, 7, 8, and 9 of [12] show that the implications (5.5) and (5.6) cannot be reversed.

The following example demonstrates that the hypothesis $N(r, a) = O(1)$ in (I), (II), and (III) is both essential and best possible.

EXAMPLE 3. We shall show that the hypothesis $N(r, a) = O(1)$ *cannot be relaxed to* $\delta(a) = 1$, *where* $\delta(a)$ *is the Nevanlinna defect of* a [14, p. 269]. Specifically, we shall construct a function f, *meromorphic in* $\mathfrak{D}$, *such that*

$$T(r, f) \leq \log(1 - r)^{-1} \quad (0 \leq r < 1), \qquad \delta(\infty) = 1, \qquad f \notin \mathscr{A}_m.$$

By a theorem of MacLane [11], there exists a function g, meromorphic in $\mathfrak{D}$, such that

(5.10) $$T(r, g) \leq \log\log(1 - r)^{-1} \quad (0 \leq r < 1)$$

and such that g *has no asymptotic values*. Let

$$h(z) = (1 - z)^{-1} \quad (|z| < 1),$$

and let

(5.11) $$f(z) = g(z) + h(z) \quad (|z| < 1).$$

Since g has no asymptotic values, the only point of $\mathfrak{C}$ at which f can have an asymptotic value is $z = 1$. Thus $f \notin \mathscr{A}_m$. Also,

$$T(r, h) \sim \log(1 - r)^{-1} \quad (r \to 1).$$

Thus

(5.12) $$T(r, f) \sim \log(1 - r)^{-1} \quad (r \to 1),$$

and

(5.13) $$N(r, \infty, f) = N(r, \infty, g) \leq T(r, g) \leq \log\log(1 - r)^{-1},$$

by (5.10). After an elementary calculation we see that $\delta(\infty, f) = 1$, which completes Example 1.

Bagemihl and Seidel [4, Corollary 1] proved that if f is holomorphic and normal (see [9, p. 53]) in $\mathfrak{D}$, then the set of points at which f has an angular limit is dense on $\mathfrak{C}$; that is, if f is holomorphic, nonconstant, and normal in $\mathfrak{D}$, then $f \in \mathscr{A}$. This result was proved independently by MacLane [12, p. 43]. Bagemihl [1, Corollary 1] has also proved that if f is meromorphic and normal and omits at least one value in $\mathfrak{D}$, then the set of points at which f has an angular limit is dense on $\mathfrak{C}$. We can do somewhat better than that.

THEOREM 8. *Let* f *be nonconstant, meromorphic, and normal in* $\mathfrak{D}$. *Also, let there exist a complex number* a *(possibly* ∞*) such that* $N(r, a) = O(1)$. *Then*

1^0 $f \in \mathscr{A}_m$;

2^0 *if* $|\zeta| = 1$, *then* f *has at most one asymptotic value at* ζ; *moreover, if* f *has the asymptotic value* b *at* ζ, *then* f *has the angular limit* b *at* ζ.

Remark. The hypothesis $N(r, a) = O(1)$ is essential, for Lehto and Virtanen [9, p. 58] have constructed a normal meromorphic function without any asymptotic values.

Proof. Let $T_s(r)$ denote the spherical characteristic of Schmizu and Ahlfors [14, p. 177]. Then

$$T_s(r) = \int_0^r \frac{A(t)}{t}\,dt \qquad (0 \le r \le 1),$$

where

$$A(t) = \frac{1}{\pi} \int_{r<t}\int \frac{|f'(re^{i\theta})|^2\, r\,dr\,d\theta}{(1 + |f(re^{i\theta})|^2)^2} \qquad (0 \le t < 1).$$

Using [9, Theorem 3], we obtain the inequality

$$A(t) \le \frac{1}{\pi} \int_{r<t}\int \frac{C^2\, r\,dr\,d\theta}{(1 - r^2)^2} \qquad (0 \le r < t < 1),$$

where C is a constant. After an elementary computation, we obtain the further inequality

$$T_s(r) \le \frac{C^2}{2} \log \frac{1}{1 - r^2} \qquad (0 \le r < 1). \tag{5.14}$$

Also,

$$T(r) = T_s(r) + O(1), \tag{5.15}$$

and by (5.14) and (5.15),

$$T(r) \le \frac{C^2}{2} \log \frac{1}{1 - r^2} + O(1), \tag{5.16}$$

where $T(r)$ is the original Nevanlinna characteristic of f. It follows immediately from (5.16) that f satisfies (III), so that $f \in \mathscr{A}_m$. The conclusion 2^0 is an obvious consequence of [9, Theorem 2].

6. THE CLASSES $\mathscr{A}'_m$, $\mathscr{B}'_m$, AND $\mathscr{L}'_m$

The object of this section is to find conditions on the functions in $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ under which the conclusion of Theorem 3 may be strengthened, in the sense that $f \in \mathscr{B}_m$ implies $f \in \mathscr{A}_m$, and so forth. Also, we shall prove a generalization of Koebe's Lemma (see [8]).

We shall say that $f \in \mathscr{A}'_m$, $\mathscr{B}'_m$, or $\mathscr{L}'_m$ *if* $f \in \mathscr{A}_m$, $\mathscr{B}_m$, *or* $\mathscr{L}_m$, *respectively, and* $N(r, \infty, f) = O(1)$. The conclusion of Theorem 3 can be improved if f is in one of the classes $\mathscr{A}'_m$, $\mathscr{B}'_m$, or $\mathscr{L}'_m$.

THEOREM 9. $\mathscr{A}'_m = \mathscr{B}'_m \supset \mathscr{L}'_m$.

Remark. It is an open question whether or not $\mathscr{B}'_m \subset \mathscr{L}'_m$. An affirmative answer would be interesting because it would generalize [12, Theorem 1] ($\mathscr{A} = \mathscr{B} = \mathscr{L}$) from holomorphic functions to meromorphic functions with $N(r, \infty, f) = O(1)$.

Proof. It has already been shown that $\mathscr{A}_m \subset \mathscr{B}_m$ and $\mathscr{L}_m \subset \mathscr{B}_m$; see (3.4) and Theorem 3. Thus, in particular, $\mathscr{A}'_m \subset \mathscr{B}'_m$ and $\mathscr{L}'_m \subset \mathscr{B}'_m$, and we need only show that $\mathscr{B}'_m \subset \mathscr{A}'_m$. Suppose $f \in \mathscr{B}'_m$, and let γ be any subarc of $\mathfrak{C}$. We shall show that f has an asymptotic value at some point of γ. This is obvious if $A_\infty(f) \cap \gamma \neq \Box$; therefore we may suppose that $A_\infty \cap \gamma = \Box$. Then, by Corollary 2 of Theorem 5, $m_e(A^*_\infty \cap \gamma) > 0$. Thus f has asymptotic values at many points of γ. Hence $f \in \mathscr{A}'_m$ and $\mathscr{B}'_m \subset \mathscr{A}'_m$.

Gross [7] generalized Koebe's Lemma from bounded holomorphic functions to meromorphic functions that omit three values. Bagemihl and Seidel [4, Theorem 1] then proved the lemma for normal meromorphic functions, and MacLane [12, Theorem 9] later generalized it to functions in $\mathscr{A}$. The results of MacLane and of Bagemihl and Seidel overlap, but neither contains the other. We can improve MacLane's result.

THEOREM 10. *Let* $f \in \mathscr{A}'_m$, *and let* $\{\gamma_n\}$ *be a sequence of simple arcs in* $\mathfrak{D}$ *that tend to an arc* $\gamma \subset \mathfrak{C}$. *Let* a *be any finite complex number, and let*

$$\mu_n = \max_{z \in \gamma_n} |f - a|. \tag{6.1}$$

Then

$$\mu = \liminf_{n \to \infty} \mu_n > 0. \tag{6.2}$$

Remarks. Example 2 shows that some hypothesis other than just $f \in \mathscr{A}_m$ is necessary for the conclusion of Theorem 10 to be true. However, it will be clear from the proof that the hypotheses of Theorem 10 may be weakened. The condition $f \in \mathscr{A}'_m$ may be replaced by the requirement that $f \in \mathscr{A}_m$ and there exist a point $\zeta \in \gamma$ and a neighborhood $U(\delta, \zeta) = \{z: |z - \zeta| < \delta \text{ and } |z| < 1\}$ such that $N(R, \infty, f(G(Z))) = O(1)$, where $G(Z)$ maps $\{|Z| < 1\}$ one-to-one and conformally onto $U(\delta, \zeta)$.

Proof. Suppose $\mu = 0$. As in the proof of Theorem 5, let $B(z)$ be the Blaschke Product with zeros at the poles of f. Then

$$g(z) = f(z)B(z) \tag{6.3}$$

is holomorphic in $\mathfrak{D}$. The assumption $\mu = 0$ implies that $A_a(f) \cap \gamma^0$ is dense in γ^0. Let

$$\zeta_1, \zeta_2 \in A_a(f) \cap \gamma^0, \qquad \zeta_1 = e^{i\theta_1}, \zeta_2 = e^{i\theta_2} \text{ with } \theta_1 < \theta_2.$$

Since $|B(z)| \leq 1$, we see that

$$|g(z)| \leq |f(z)|. \tag{6.4}$$

Thus we may construct two curves Γ_1 and Γ_2 such that $\Gamma_1, \Gamma_2 \subset \mathfrak{D}$, Γ_i begins at $z = 0$ and ends at ζ_i $(i = 1, 2)$, $\Gamma_1 \cap \Gamma_2 = 0$, $|g|$ is bounded on $\Gamma_1 \cup \Gamma_2$, and f has the asymptotic value a on Γ_i $(i = 1, 2)$. Let $\gamma_0 = \{e^{i\theta}: \theta_1 \leq \theta \leq \theta_2\}$, and let H be the domain bounded by $\Gamma_1 \cup \Gamma_2 \cup \gamma_0$. Now Γ_1 and Γ_2 intersect all but a finite number of the γ_n; thus we can find a sequence $\{\tau_k\}_{k=1}^{\infty}$ of crosscuts of H such that τ_k joins a point of Γ_1 to a point of Γ_2, τ_{k+1} separates τ_k from γ_0 in H, for each k there exists an n such that $\tau_k \subset \gamma_n$, and $\tau_k \rightarrow \gamma_0$. Let D_k be the subdomain of H bounded by τ_k, τ_{k+1}, and subarcs of Γ_1 and Γ_2. Using (6.4), applying the maximum principle to D_k, and letting $k \rightarrow \infty$, we see that

$$\limsup_{\substack{z \rightarrow \gamma_0 \\ z \in H}} |g(z)| \leq |a|.$$

Thus

(6.5) g is bounded in H.

Now map H one-to-one and conformally onto $\{|Z| < 1\}$ by $z = G(Z)$. Let

$$\Phi(Z) = g(G(Z)) = g(z) \qquad (z \in H),$$

$$\Psi(Z) = B(G(Z)) = B(z) \qquad (z \in H).$$

By (6.5), $\Phi(Z)$ and $\Psi(Z)$ are bounded in $\{|Z| < 1\}$. Hence the function

$$F(Z) = \frac{\Phi(Z)}{\Psi(Z)} = f(z) \qquad (z \in H)$$

is the quotient of two bounded holomorphic functions in $\{|Z| < 1\}$ and is thus of bounded characteristic. γ_0 corresponds to a subarc γ_0' of $\{|Z| = 1\}$. Thus, by (6.1) and the assumption that $\mu = 0$ (see (6.2)), F has the angular limit a at almost every point of γ_0'. By [14, p. 209], $F \equiv a$, which implies that $f \equiv a$. This contradicts the hypothesis that $f \in \mathscr{A}_m'$; therefore μ must be positive.

7. RESULTS ON ASYMPTOTIC TRACTS

Theorems 1 and 2 give information about asymptotic tracts of functions in $\mathscr{A}_m$. Our next theorems give conditions for the existence of arc tracts for functions in $\mathscr{A}_m$. MacLane [12, p. 61] has pointed out that if $f \in \mathscr{A}$ is unbounded, then the growth of M(r) has nothing to do with the existence of arc tracts. However, we shall see (Theorem 12) that conditions on the growth of N(r, a, f) are relevant to the existence of arc tracts. We first prove a generalization of [12, Theorem 4].

THEOREM 11. *Let* $f \in \mathscr{A}_m$, *and let* a *be any complex number (possibly* ∞) *such that* $N(r, a, f) = O(1)$. *If* $\{\mathfrak{T}(\varepsilon), b\}$ $(b \neq a)$ *is a tract of* f, *then* $\{\mathfrak{T}(\varepsilon), b\}$ *is a point tract.*

Proof. Suppose that $\{\mathfrak{T}(\varepsilon), b\}$ is an arc tract. First consider the case where $a = \infty$. Then $f \in \mathscr{A}_m$, $N(r, \infty, f) = O(1)$, and we can find an arc $\gamma \subset \mathfrak{C}$ and a sequence of continuous arcs γ_n, compact (in $\mathfrak{D}$), such that $\gamma_n \rightarrow \gamma$ and

(7.1) $$\liminf_{n \rightarrow \infty} \mu_n = 0 \qquad \left(\mu_n = \max_{z \in \gamma_n} |f(z) - b|\right).$$

But (7.1) contradicts Theorem 10, so that $\{\mathfrak{T}(\varepsilon), b\}$ must be a point tract.

Now consider the case where $a \neq \infty$. Here $g = 1/(f - a) \in \mathscr{A}_m$ and $N(r, \infty, g) = N(r, a, f) = O(1)$. Using the same argument as above, we obtain a contradiction. Hence we see that in all cases $\{\mathfrak{T}(\varepsilon), b\}$ is a point tract.

Some results about the nonexistence of arc tracts follow easily from Theorem 11. The following theorem extends [6, Theorem 4].

THEOREM 12. *Let* $f \in \mathscr{A}_m$, *and suppose that there exist two complex numbers* a, b (*one of which may be* ∞) *such that* $a \neq b$, $N(r, a, f) = O(1)$, *and* $N(r, b, f) = O(1)$. *Then* f *has no arc tracts.*

Proof. By Theorem 11, all tracts $\{\mathfrak{T}(\varepsilon), c\}$ $(c \neq a)$ must be point tracts, and again by Theorem 11, all tracts $\{\mathfrak{T}(\varepsilon), d\}$ $(d \neq b)$ must be point tracts. Hence all tracts of f are point tracts.

COROLLARY. *Let* $f \in \mathscr{A}$, *and suppose* $N(r, a, f) = O(1)$ *for some finite complex number* a. *Then* f *has no arc tracts.*

Bagemihl and Seidel [3, Corollary 1 to Theorem 3] proved that a *nonconstant, meromorphic, normal function in* $\mathfrak{D}$ *has no arc tracts;* in particular, a *function in* $\mathscr{A}_m$ *that is normal has no arc tracts.*

The next theorem concerns conditions for the existence of global tracts. It is a generalization of [12, Theorem 6B].

THEOREM 13. *Let* $f \in \mathscr{A}_m$. *Then* f *has a global tract for* a *if and only if* f *is not bounded away from* a *on any curve* Γ *in* $\mathfrak{D}$ *on which* $|z| \to 1$.

Proof. Suppose first that $a = \infty$. We must prove that f has a global tract for ∞ if and only if f is unbounded on every curve Γ in $\mathfrak{D}$ on which $|z| \to 1$. Suppose that f has a global tract for ∞. Since $f \in \mathscr{A}_m$, we know that A_∞ is dense in $\mathfrak{C}$. It follows easily that $f \in \mathscr{L}_m$. Suppose next that some level curve $C(\lambda)$ is not compact. Then it ends at some point ζ of $\mathfrak{C}$. There exist a subarc γ of $\mathfrak{C}$ and a sequence $\{\gamma_n\}_{n=1}^\infty$ of continuous arcs, compact in $\mathfrak{D}$, such that $\zeta \in \gamma^0$, $\gamma_n \to \gamma$, and

$$(7.2) \qquad \lim_{n\to\infty} \inf_{z\in\gamma_n} |f| = \infty .$$

The curve $C(\lambda)$ must intersect all but a finite number of the γ_n, and we have a contradiction ($|f| = \lambda$ on $C(\lambda)$). Thus all level curves of f are compact. Hence f satisfies the hypotheses of Theorem 2, and one can easily prove that (3.5) cannot happen. From the proof of (3.6) it is clear that there exists a sequence of closed Jordan curves $J(\lambda_n)$ such that

$$(7.3) \qquad J(\lambda_n) \to \{|z| = 1\} \qquad (n \to \infty)$$

and

$$(7.4) \qquad \lim_{n\to\infty} \inf_{z\in J(\lambda_n)} |f(z)| = \infty .$$

Any curve Γ on which $|z| \to 1$ must cross all but a finite number of the $J(\lambda_n)$, and thus by (7.4) f is unbounded on Γ.

Now suppose f is unbounded on each curve Γ on which $|z| \to 1$. Then A_∞ is dense in $\mathfrak{C}$. Again we see that $f \in \mathscr{L}_m$. Also, all level curves of f must be compact, since f is unbounded on every curve Γ on which $|z| \to 1$. Therefore f

satisfies the hypotheses of Theorem 2. Again, conclusion (3.5) of Theorem 2 cannot hold, so that (3.6) must hold. It follows from the proof of (3.6) that the arc tract of (3.6) is actually a global tract.

If a is finite, the result follows if we apply the argument above to $1/(f - a)$.

If we recall the generalization of Theorem 2 mentioned in Section 3, then it is clear that the proof of Theorem 13 may be generalized so that it yields a corresponding result for arc tracts.

THEOREM 14. *Let* $f \in \mathscr{A}_m$, *and let* γ *be a subarc of* $\mathfrak{C}$. *Then* f *has an arc tract for the value* a *with end* $K \supset \gamma$ *if and only if* f *is not bounded away from* a *on any curve* Γ *on which* $|z| \to 1$ *and whose closure meets the interior of* γ.

We end this section by extending a theorem that MacLane proved for $f \in \mathscr{A}$ [12, Theorem 7]. Since MacLane's proof works for $f \in \mathscr{A}_m$, it is sufficient to state the extension.

THEOREM 15. *Let* $f \in \mathscr{A}_m$, *and let* $\{\mathfrak{T}(\varepsilon), \infty\}$ *be an arc tract of* f *with end* K. *Let* ζ *be any point of* K, *let* $\delta > 0$, *and let*

$$U(\delta, \zeta) = \{|z| < 1\} \cap \{|z - \zeta| < \delta\}.$$

Then the following three conclusions hold.

(A) $f(z)$ *assumes every finite value infinitely often in* $U(\delta, \zeta)$.

(B) *Let* $w = f(z)$ *map* $\mathfrak{D}$ *onto the Riemann surface* $\mathscr{S}$ *over the* w-*sphere. For any* $r > 0$, *let the components of* $\mathscr{S}$ *over* $\{|w| < r\}$ *be* $\Delta(r, 1), \Delta(r, 2), \cdots$. *Let* $G(r, n)$ *be the domain in* $\mathfrak{D}$ *corresponding to* $\Delta(r, n)$. *Then, for each* $r > 0$, *there exist infinitely many integers* n_k *such that* $\Delta(r, n_k)$ *is relatively compact and* $G(r, n_k) \subset U(\delta, \zeta)$.

(C) *Each* $\mathfrak{T}(\varepsilon)$ *has infinite connectivity.*

Remark. We can obtain information about an arc tract $\{\mathfrak{T}(\varepsilon), a\}$ for finite values of a by applying Theorem 15 to $1/(f - a)$.

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Rice University
and
Ballistic Research Laboratories