COMPACT SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In this paper we investigate differential equations of the form

$$y' + Uy = F(\cdot, y, \mu) \tag{*}$$

in a complex Banach space B. We assume that either $U \in L(B, B)$ and the spectrum of U (denoted by sp U) is in the right half-plane, or else U is a semigroup generator. Our main objective is the study of compact solutions of (*), that is, solutions whose range has a compact closure. This problem seems interesting since it includes periodic and almost-periodic solutions, and since it leads to the approximation of compact solutions to (*) by solutions of equations in finite-dimensional spaces. The continuity of compact solutions with respect to a parameter has been investigated by Taam [4]. We shall also investigate the continuity and analyticity of these solutions as functions of the parameter μ, where μ lies in a complex Banach space X.

The paper is divided into three sections. In Section 1 we study compact solutions of (*) in an arbitrary complex Banach space B. In Section 2, we let B be a Banach space with a basis, and we prove approximation theorems for the compact solutions of (*). In Section 3 we seek compact solutions to (*) for the case where U is a semigroup generator, and then we use the results of Sections 1 and 2 to get approximation theorems for this case.

1. SOLUTIONS IN A COMPLEX BANACH SPACE

Let R denote the real line. The norm of a vector x in B is written as $\|x\|$. For a function f on R into B, we write

$$\|f\|_\infty = \sup \{ \|f(t)\| : t \in R \}.$$

The above is called the *uniform norm* of f.

We say that a function f from R into B is *compact* if f(R) has a compact closure.

The family of functions from R into B that are Bochner integrable on every interval of unit length, and for which

$$\|f\|_s = \sup \left\{ \int_t^{t+1} \|f(s)\| \, ds : t \in R \right\}$$

is finite will be designated by BUL. We call $\| \ \|_s$ the *uniform* L_1-*norm*.

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Each of the following spaces of functions from R into B is a complex Banach space when addition and multiplication by scalars are defined in the usual way.

(a) The space $(P(R), \| \|_\infty)$ of all compact functions,

(b) the space $(C(R), \| \|_\infty)$ of all bounded continuous functions,

(c) the space $(CC(R), \| \|_\infty)$ of all continuous compact functions,

(d) the space $(BUL, \| \|_s)$ (see [4]),

(e) the space $((BULC)^-, \| \|_s)$ (this is the closure in $(BUL, \| \|_s)$ of the set BULC of compact functions of BUL).

We also need the Banach space $(L(X, Y), \| \|)$ of linear continuous operators from the Banach space X into the Banach space Y. The symbol $\| \|$ denotes the usual uniform operator norm.

If $f \in P(R)$, then for each $\varepsilon > 0$ there exists a function

$$f_\varepsilon = \sum_{i=1}^{n} x_i \chi_{E_i} \quad \left(x_1, \cdots, x_n \in B; \text{ the } E_i \text{ are disjoint, and } \bigcup E_i = R\right)$$

such that $\|f - f_\varepsilon\|_\infty < \varepsilon$. Here χ_{E_i} denotes the characteristic function of the set E_i. Such f_ε will be called *simple functions*. If $f \in BULC$, then we can assume that $f_\varepsilon \in BULC$, since we can take E_i that are measurable.

LEMMA 1. *Assume that* $U \in L(B, B)$ *and that there exist positive constants* η *and* δ *such that*

$$\|\exp(Ut)\| < \eta e^{\delta t} \quad \text{for } t \leq 0; \tag{1}$$

then the operator K *defined by the Bochner integral*

$$g(t) = (Kf)(t) = \int_{-\infty}^{0} \exp(Us) f(s+t)\, ds \qquad (t \in R,\ f \in BUL)$$

is in L(BUL, C(R)), *with* $\|K\| \leq \eta(1 - e^{-\delta})^{-1}$. *The image function* g *is absolutely continuous on every finite interval, and differentiable almost everywhere; it satisfies the differential equation*

$$g'(t) + Ug(t) = f(t) \tag{2}$$

for almost all t. (For a definition of exp(Ut) and a proof of Lemma 1, see [4].)

Remark. If $U \in L(B, B)$ and the spectrum of U lies in the right half-plane, then there exist positive constants η and δ such that U satisfies (1).

We now investigate the existence and uniqueness of compact solutions of the differential equation (*) under the following conditions:

(3) $U \in L(B, B)$, sp U is in the right half-plane, and (1) holds.

(4) $F(t, x, \mu)$ is a mapping of $R \times B \times D$ into B $(D \subset X)$ such that

(a) $F(\cdot, x, \mu) \in (BULC)^-$ for each $x \in B$ and each $\mu \in D$,

(b) for each $\mu \in D$, $F(t, \cdot, \mu)$ is continuous for almost all t from B into B,

(c) there exists a mapping $\theta(t, \rho, \mu)$ from $R \times R^+ \times D$ into R^+ such that $\theta(\cdot, \rho, \mu)$ is a real-valued BUL-function for each fixed ρ and μ; moreover, for fixed ρ and μ, and for each pair $x, y \in B$ ($\|x\| \leq \rho$, $\|y\| \leq \rho$), the inequality

$$\|F(t, x, \mu) - F(t, y, \mu)\| \leq \theta(t, \rho, \mu)\|x - y\|$$

holds for almost all t.

LEMMA 2. *For each fixed* $\mu \in D$, $F(t, x, \mu)$ *defines a mapping from* BULC *into* $(\text{BULC})^-$, *given by*

$$(Ff)(t) = F(t, f(t), \mu),$$

such that if $f, g \in \text{BULC}$ ($\|f\|_\infty \leq \rho$, $\|g\|_\infty \leq \rho$ *for some* ρ), *then*

$$\|(Ff - Fg)(t)\| \leq \theta(t, \rho, \mu)\|f(t) - g(t)\| \tag{5}$$

for almost all t.

Proof of Lemma 2. From [4, Lemma 3] we see that F is a mapping from the space of bounded BUL-functions into BUL such that inequality (5) is satisfied. Take $f \in \text{BULC}$; then for each $\varepsilon > 0$ there exists a simple function $f_\varepsilon = \sum_{i=1}^n x_i \chi_{E_i}$ satisfying $\|f - f_\varepsilon\|_\infty < \varepsilon$.

Set $F_\varepsilon(t) = \sum_{i=1}^n \chi_{E_i}(t) F(t, x_i, \mu)$. It follows from condition (4a) on $F(t, x, \mu)$ that $F_\varepsilon \in (\text{BULC})^-$. Since $f \in \text{BULC}$, there exists a ρ such that $\|f\|_\infty \leq \rho$. Thus the inequality $\|Ff - F_\varepsilon\|_s < \varepsilon\|\theta(\cdot, \rho, \mu)\|_s$ implies that $Ff \in (\text{BULC})^-$.

Definition. A function y on an interval I into B is called a solution of (*) on I if and only if

(i) y is absolutely continuous on every finite subinterval of I,

(ii) $y'(t)$ exists almost everywhere in I,

(iii) y satisfies (*) for almost all $t \in I$.

We are interested in the values of ρ and μ that satisfy the conditions

$$(6)\quad \begin{aligned} &\sup\left\{\eta \int_{-\infty}^0 e^{\delta s}\, \theta(t+s, \rho, \mu)\,ds : t \in R\right\} < r < 1, \\ &\sup\left\{\left\|\int_{-\infty}^0 \exp(Us)\, F(t+s, 0, \mu)\,ds\right\| : t \in R\right\} < \rho(1-r)/2. \end{aligned}$$

The following lemma will be fundamental in our study of compact solutions.

LEMMA 3. *If* E *is a measurable subset of* R, *if* $y \in B$, *and if* $g = y\chi_E$, *then* $Kg \in CC(R)$.

Proof. Since $g \in \text{BUL}$, Lemma 1 implies that $Kg \in C(R)$. For each $r < 0$,

$$(Kg)(t) = \int_{-\infty}^{0} \exp(Us)\, y\, \chi_E(s+t)\, ds$$

$$= \int_{-\infty}^{r} \exp(Us)\, y\, \chi_E(s+t)\, ds + \int_{r}^{0} \exp(Us)\, y\, \chi_E(s+t)\, ds\,.$$

Corresponding to each $\varepsilon > 0$, choose r so that

$$\eta \|y\| \int_{-\infty}^{r} e^{\delta s}\, ds < \varepsilon/2\,.$$

Since $\exp(Us)y$ is uniformly continuous on $[r, 0]$, it is possible to partition $[r, 0]$ into n disjoint subintervals I_i with points $a_i \in I_i$ $(i = 1, \cdots, n)$ such that

$$\left\| \exp(Us)\, y - \sum_{i=1}^{n} \chi_{I_i}(s) \exp(Ua_i)\, y \right\| < \varepsilon/2|r|$$

for all $s \in [r, 0]$. Set

$$y_\varepsilon(t) = \sum_{i=1}^{n} b_i(t) \exp(Ua_i)\, y\,,$$

where $b_i(t) = \int_{I_i} \chi_E(s+t)\, ds$. Since $|b_i(t)| \leq |r|$ $(i = 1, \cdots, n;\ t \in R)$ and since $y_\varepsilon(R)$ lies in a bounded set in the space spanned by $\exp(Ua_i)$ $(i = 1, \cdots, n)$, we see that $y_\varepsilon \in CC(R)$. It follows immediately that $\|Kg - y_\varepsilon\|_\infty < \varepsilon$. Therefore $Kg \in CC(R)$.

THEOREM 1. *If* $f \in (BULC)^-$, *then* $Kf \in CC(R)$.

Proof. Take $f \in BULC$, so that f is the uniform limit of a sequence f_n of simple functions and

$$\|Kf - Kf_n\|_\infty \leq \|K\|\, \|f - f_n\|_s \leq \|K\|\, \|f - f_n\|_\infty\,.$$

By Lemma 3, $Kf_n \in CC(R)$; therefore $Kf \in CC(R)$. The theorem follows from the density of BULC in $(BULC)^-$, since $K \in L(BUL, C(R))$.

THEOREM 2. *If* μ *and* ρ *satisfy* (6), *then there exists exactly one solution* x *of* (*) *satisfying the conditions* $\|x\|_\infty \leq \rho$, $x \in CC(R)$, *and* $\|x'\|_s < \infty$.

Proof. By (4a), Lemmas 1 and 2, and Theorem 1, the operator

$$(Ty)(t) = \int_{-\infty}^{0} \exp(Us)\, F(t+s, y(t+s), \mu)\, ds = (KF(\cdot, y(\cdot), \mu))(t) = (KFy)(t)$$

defines a mapping T from $CC(R)$ into itself.

Let ρ and μ satisfy (6). If $f \in CC(R)$ with $\|f\|_\infty \leq \rho$, then, by (4c) and (6), $\|Tf\|_\infty < \rho$. If $g \in CC(R)$ with $\|g\|_\infty \leq \rho$, then by (4c) and (6)

$$\|(Tf - Tg)(t)\| \leq \eta \int_{-\infty}^{0} e^{\delta s}\, \theta(t+s, \rho, \mu)\, ds\, \|f - g\|_\infty \leq r\|f - g\|_\infty,$$

and therefore $\|Tf - Tg\|_\infty \leq r\, \|f - g\|_\infty$. Since $r < 1$, T is a contraction mapping on the closed sphere

$$S = \{z \in CC(R) : \|z\|_\infty \leq \rho\}.$$

Since this closed sphere is a complete metric space, there exists a unique point $g \in CC(R)$ such that $Tg = g$ and $\|g\|_\infty \leq \rho$. By Lemma 1, g satisfies (*) and $g' \in BUL$. If h is a solution of (*) and $\|h\|_\infty \leq \rho$, then one can show that $Th = h$ (for details, see [4, Theorem 1]). Therefore it follows that $\|g - h\|_\infty \leq r\, \|g - h\|_\infty$, and therefore $g = h$. Thus the only bounded solution g of (*) with $\|g\|_\infty \leq \rho$ is compact.

We now consider a solution of (*) as a function of t and of the parameter $\mu \in D$; for convenience, we denote it by $x(t, \mu)$ or $y(t, \mu)$. In the next theorem, we let D be a domain, and we investigate the analyticity of the solutions $x(\cdot, \mu)$ in the domain D.

We say that a function f defined in a complex Banach space X with range in a complex Banach space Y is analytic in the domain $D \subset X$ if it is Fréchet differentiable in D, that is, if for each $x \in D$ there exists an $f'(x) \in L(X, Y)$ such that

$$\lim_{\|h\| \to 0} \|f(x+h) - f(x) - f'(x)h\| / \|h\| = 0 \qquad (h \in X).$$

Let D' be any domain contained in a complex Banach space X. Set

$$H(Y) = \{f : f \text{ is bounded and analytic from } D' \text{ into } Y\}.$$

It follows from [2, p. 113] that if for each $f \in H(Y)$

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in D'\},$$

then $(H(Y), \|\ \|_\infty)$ is a complex Banach space.

If f is analytic from D' into BUL, let us set

$$(K'F)(t, \mu) = (Kf(\cdot, \mu))(t) = \int_{-\infty}^{0} \exp(Us) f(s+t, \mu)\, ds \qquad (t \in R,\ \mu \in D').$$

Since $K \in L(BUL, C(R))$, $K \in L((BULC)^-, CC(R))$, and $K \in L(CC(R), CC(R))$, it follows that

$$K' \in L(H(BUL), H(C(R))), \qquad K' \in L(H(BULC)^-, H(CC(R))),$$

$$K' \in L(H(CC(R)), H(CC(R))).$$

If f is analytic from D' into $(BULC)^-$, then K'f is analytic from D' into CC(R).

THEOREM 3. *Let* $F(t, x, \mu)$ *satisfy* (4). *In addition, let* $D \subset X$ *be a domain such that* $F(\cdot, x, \mu)$ *is analytic from* $B \times D$ *into* BUL, *and such that for each* ρ,

$\theta(\cdot, \rho, \mu)$ is continuous from D into a real BUL-space. Then for each fixed ρ and each $\mu = \mu' \in D$ that satisfy (6) there exists a subdomain $D' \subset D$ such that for each $\mu \in D'$ the equation (*) has a unique compact solution $x(\cdot, \mu)$, and $x(\cdot, \mu)$ is analytic from D' into CC(R).

Proof. By the assumptions in the theorem, the supremum of each integral in (6) is continuous on D into R, for each fixed ρ. Thus if (6) is satisfied for some fixed ρ and $\mu = \mu'$, then there exists an open sphere N about μ' such that (6) is satisfied for each $\mu \in N$. Therefore Theorem 2 implies that for each $\mu \in N$ there exists a unique compact solution $x(\cdot, \mu)$ of (*).

If $x \in H(CC(R))$, we conclude from the assumptions on $F(t, x, \mu)$ that $F(\cdot, x(\cdot, \mu), \mu)$ is analytic from D' into $(BULC)^-$. Thus $K'Fx$ is analytic from D' into CC(R). Let $D' = N$; then

$$(7)\quad \begin{aligned} &\sup\left\{ \int_{-\infty}^{0} \eta e^{\delta s}\, \theta(t+s, \rho, \mu)\,ds \colon (t, \mu) \in R \times D' \right\} < r < 1, \\ &\sup\left\{ \left\| \int_{-\infty}^{0} \exp(Us)\, F(t+s, 0, \mu)\,ds \right\| \colon (t, \mu) \in R \times D' \right\} < \rho(1-r)/2. \end{aligned}$$

Set $S' = \{f \in H(CC(R)) \colon \|f\|_\infty \leq \rho\}$. S' is a complete metric space. By (7), $\|(K'Ff)(t, \mu)\| < \rho$ for all $(t, \mu) \in R \times D'$. Therefore $K'Ff \in H(CC(R))$, and $K'F$ maps S' into S'.

Using the same inequalities, we find that $K'F$ is a contraction mapping on S'. Thus we have an $x \in S'$ such that $x(\cdot, \mu)$ is a solution of (*) for each $\mu \in D'$. From Theorem 2 it follows that x is unique.

THEOREM 4. *Let $F(t, x, \mu)$ satisfy (4). Suppose moreover that D is open, and that for each x and each ρ the functions $F(\cdot, x, \mu)$ and $\theta(\cdot, \rho, \mu)$ are continuous mappings from D into BUL and into a real BUL-space, respectively. If ρ and μ' satisfy (6), then μ' has a neighborhood N in D such that for each $\mu \in N$ the equation (*) has a unique compact solution $x(\cdot, \mu)$; $x(\cdot, \mu)$ is continuous on N into CC(R), and $x'(\cdot, \mu)$ is continuous on N into BUL.*

Proof. As in the proof of Theorem 3, it follows that if (6) is satisfied for a fixed ρ and $\mu = \mu'$, then there exists a neighborhood N of μ' such that (6) is satisfied for each $\mu \in N$. Again from Theorem 2 we see that for each $\mu \in N$ there exists a unique compact solution $x(\cdot, \mu)$ of (*). The rest of the proof follows from [4, Theorem 2], since for each $\mu \in N$ the conditions of that theorem are met.

2. SOLUTIONS IN A BANACH SPACE WITH A BASIS

Now let B be a Banach space with a basis $\{x_i\}$. For each n, P_n denotes the projection operator that sends B into the subspace Y_n spanned by $x_1, \cdots, x_n$. From [3, pp. 134-136] we see that the P_n are continuous. Since $P_n x$ converges to x in B, for each $x \in B$, it follows from the Steinhaus-Banach theorem that there exists an $M > 0$ such that $\|P_n\| \leq M$ for all n. It is easy to see that in this case $f \in P(R)$ if and only if $P_n f$ converges to f in the uniform norm.

LEMMA 4. *If $f \in (BULC)^-$, then $P_n f \to f$ in $(BULC)^-$ as $n \to \infty$, and if $f_n, f \in (BULC)^-$ and $f_n \to f$ in $(BULC)^-$, then $P_n f_n \to f$ in $(BULC)^-$.*

Proof. If $f \in (BULC)^-$, then there exist $f_n \in BULC$ such that $f_n \to f$ in BUL. But for each pair of positive integers k and n,

$$P_k f_n \in BULC \quad \text{and} \quad \|P_k f_n - P_k f\|_s \leq M \|f_n - f\|_s.$$

Therefore $P_k f \in (BULC)^-$, for each k. The inequalities

$$\|P_k f - f\|_s \leq \|P_k f - P_k f_n\|_s + \|P_k f_n - f\|_s \leq (M+1)\|f - f_n\|_s + \|P_k f_n - f_n\|_s$$

yield the first part of the lemma. Now, if $f_n \to f$ in $(BULC)^-$, then

$$\|P_n f_n - f\|_s \leq \|P_n f_n - P_n f\|_s + \|P_n f - f\|_s \leq M \|f_n - f\|_s + \|P_n f - f\|_s,$$

and the remainder of the lemma follows.

We now assume that $M = 1$, as is the case if B is a separable Hilbert space with orthogonal basis $\{x_i\}$.

Denote by $C(R, Y_n)$ the space of all bounded continuous functions from R into $Y_n = P_n B$. Then $(C(R, Y_n), \|\ \|_\infty)$ is a complex Banach space.

THEOREM 5. *Let* $M = 1$, *and let the assumptions of Theorem* 2 *hold. Then there exist contractive operators* T_n *for all* $n \geq N$, *for some* $N > 0$, *mapping closed spheres of the space* $C(R, Y_n)$ *into themselves, such that for the unique fixed point* x_n *of* T_n,

$$x_n' \in (BULC)^-, \qquad \|x - x_n\|_\infty \to 0, \qquad \|x' - x_n'\|_s \to 0,$$

where x *is the compact solution of* (*) *given by Theorem* 2.

Proof. For $y \in C(R, Y_n)$, define

$$(T_n y)(t) = \int_{-\infty}^{0} P_n \exp(Us) P_n F(t+s, y(t+s), \mu)\, ds$$

$$= (P_n K P_n F(\cdot, y(\cdot), \mu))(t) = (P_n K P_n Fy)(t),$$

where $(Fy)(t) = F(t, y(t), \mu)$. It follows from Lemmas 1 and 2 that T_n maps $C(R, Y_n)$ into itself and that $(T_n y)' \in BUL$ $(n = 1, \cdots)$. We also see that T_n maps BULC into $C(R, Y_n)$ $(n = 1, \cdots)$.

If $y \in BULC$, then Lemmas 2 and 4 imply that $P_n Fy \to Fy$ in $(BULC)^-$. Therefore it follows from Lemma 1 and Theorem 1 that $KP_n Fy \to KFy$ in CC(R). Consequently, $P_n KP_n Fy \to KFy$ in CC(R); that is, $T_n y \to Ty$ in CC(R) for each $y \in BULC$.

From the above it follows that there exists an $N > 0$ such that

$$\sup\left\{ \left\| P_n \int_{-\infty}^{0} \exp(Us) P_n F(t+s, 0, \mu)\, ds \right\| : t \in R \right\} < \rho(1-r)/2 \quad \text{for } n \geq N.$$

Also, by the definition of T_n,

$$\|T_n x - T_n y\|_\infty \leq r \|x - y\|_\infty \quad (x, y \in C(R, Y_n)), \ \|x\|_\infty \leq \rho, \ \|y\|_\infty \leq \rho, \ n = 1, 2, \cdots).$$

The two preceding inequalities imply that for $n \geq N$, T_n is a contraction mapping on the closed sphere

$$S_n = \{y \in C(R, Y_n): \|y\|_\infty \leq \rho\}.$$

Let $x_n = T_n x_n$ be the resulting fixed points for $n \geq N$, and let $x = Tx$ be the solution of (*) given by Theorem 2. Set $f_n = Fx_n$ and $f = Fx$. Then

$$\begin{aligned}\|x - x_n\|_\infty &\leq \|x - P_n x\|_\infty + \|P_n Kf - P_n KP_n f\|_\infty + \|P_n KP_n f - P_n KP_n f_n\|_\infty \\ &\leq \|x - P_n x\|_\infty + \|K\|\,\|f - P_n f\|_s + \|KP_n f - KP_n f_n\|_\infty .\end{aligned}$$

Applying the first inequality in (6) to the last term of the above inequality, we see that

$$(1 - r)\|x - x_n\|_\infty \leq \|x - P_n x\|_\infty + \|K\|\,\|f - P_n f\|_s .$$

The first term on the right of the above inequality tends to 0, since $x \in CC(R)$; the second term also tends to 0, since $P_n f \to f$ in $(BULC)^-$. Thus $x_n \to x$ in $CC(R)$.

It follows from Lemma 1 that for almost all $t \in R$

$$x_n'(t) = -(P_n UKP_n f_n)(t) + P_n f_n(t)$$

and

$$x'(t) = -(UKF)(t) + f(t).$$

Therefore

$$\|x' - x_n'\|_s \leq \|P_n f_n - f\|_s + \|P_n UKP_n f_n - UKf\|_\infty .$$

Since

$$\|f_n - f\|_s \leq \|\theta(\cdot, \rho, \mu)\|_s \|x_n - x\|_\infty ,$$

Lemma 4 implies that $P_n f_n \to f$ in $(BULC)^-$. Thus $KP_n f_n \to Kf$ in $CC(R)$, and hence $P_n UKP_n f_n \to UKf$ in $CC(R)$. Therefore the second part of the theorem is proved.

Let us now consider the case $M \geq 1$. If $x, y \in S_n$ and the assumptions of Theorem 2 are satisfied, then $\|T_n x - T_n y\|_\infty \leq rM^2 \|x - y\|_\infty$. If $M > 1$, we cannot in general conclude that T_n is a contraction on the set S_n. But if $y \in BULC$, then, as we saw in the proof of Theorem 5, $P_n KP_n Fy \to KFy$ in $CC(R)$. If $y, y_n \in CC(R)$ and $y_n \to y$ in $CC(R)$, then there exists a $\rho > 0$ such that $\|y_n\|_\infty \leq \rho$ for all n. From the inequality

$$\|Fy_n - Fy\|_s \leq \|\theta(\cdot, \rho, \mu)\|_s \|y_n - y\|_\infty$$

and the preceding statements we see that $P_n KP_n Fy_n \to KFy$. Induction implies that $(P_n KP_n F)^m y \to (KF)^m y$ in $CC(R)$ for each $y \in CC(R)$ and each integer m. If $y \in CC(R)$ and $\|y\|_\infty \leq \rho$, then, since T is a contraction on S, we can conclude that $(KF)^m y \to x$ in $CC(R)$ as $m \to \infty$, where x is the compact solution of (*). Therefore, since for a fixed integer m and a fixed $y \in CC(R)$, $(P_n KP_n F)^m y \to (KF)^m y$ in $CC(R)$ as $n \to \infty$, we have proved the following result.

THEOREM 6. *Let* $M \geq 1$, *and let the assumptions of Theorem* 2 *hold. Then, for each* $y \in S$ *and each* $\varepsilon > 0$, *there exist positive integers* m_ε *and* n_ε *such that*

$$\|(P_{n_\varepsilon} K P_{n_\varepsilon} F)^{m_\varepsilon} y - x\|_\infty < \varepsilon,$$

where x *is the compact solution of* (*).

Thus we see that for each $M \geq 1$ we can approximate the compact solution to (*) by successive approximations in finite-dimensional spaces.

In the two preceding theorems, the only assumptions on U are that sp U is in the right half-plane and (1) is satisfied. Let us also assume the following:

$$U = E + C, \text{ where } E, C \in L(B, B), \ E \text{ is compact}, \ CP_n = P_n C, \quad \|P_n\| = 1 \text{ for all positive integers } n. \tag{8}$$

We see that $E_n = P_n E \to E$ in L(B, B). Thus $U_n = E_n + C \to U$ in L(B, B), and therefore we can conclude from [1, Lemma 3, p. 585] that

$$\|\exp(U_n t)\| < [\eta + O(1/n)] e^{\delta t} \quad \text{for all large } n \text{ and all } t \leq 0. \tag{9}$$

Thus it follows from Lemma 1 that for large n the operator K_n defined by

$$(K_n f)(t) = \int_{-\infty}^{0} \exp(U_n s) f(s+t)\, ds \qquad (t \in R, \ f \in BUL) \tag{10}$$

is in L(BUL, C(R)). From Theorem 1 it follows that $K_n \in L((BULC)^-, CC(R))$.

Since $U_n \in L(Y_n, Y_n)$, we also see that $K_n \in L(C(R, Y_n), C(R, Y_n))$. Since $U_n \to U$ in L(B, B), it follows that

$$\exp(U_n t) \to \exp(Ut) \quad \text{in } L(B, B) \tag{11}$$

uniformly for t in any bounded set.

From (9) and (10) we deduce the existence of a constant $N > 0$ such that

$$\|K_n\| \leq N \text{ for all large } n. \tag{12}$$

THEOREM 7. *Let the assumptions of Theorem* 2 *hold, and assume in addition that* U *satisfies* (8). *Then for large* n *the differential equations*

$$y' + U_n y = P_n F(\cdot, y, \mu) \tag{**}$$

have unique solutions $y_n \in C(R, Y_n)$ *such that*

$$\|y_n\|_\infty \leq \rho, \qquad \|y - y_n\|_\infty \to 0, \qquad \|y' - y_n'\|_s \to 0,$$

where y *is the unique compact solution of* (*).

Proof. Using (9), we see that for large n the operators

$$T_n = K_n P_n F,$$

which map $C(R, Y_n)$ into itself, satisfy the condition

(13) $$\|T_n x - T_n y\|_\infty \leq r\,\|x - y\|_\infty \qquad (x, y \in S_n).$$

If $f \in$ BULC, then it follows from (9) and (11) that $K_n f \to Kf$ in CC(R). From (12) it follows that $K_n f \to Kf$ in CC(R) for each $f \in (\text{BULC})^-$, and this implies that if $f_n \to f$ in $(\text{BULC})^-$, then $K_n f_n \to Kf$ in CC(R). If $y \in$ BULC, then, as we saw before, $P_n Fy \to Fy$ in $(\text{BULC})^-$. Thus $T_n y \to Ty$ in CC(R), for each $y \in$ BULC. Therefore, from the second inequality in (6) we get

(14) $$\|(T_n)(0)\|_\infty < \rho(1 - r)/2 \qquad \text{for large } n.$$

Proceeding as in the proof of Theorem 2, and using (13), (14), and the other properties of T_n, we see that for all large n the operators T_n map $(\text{BULC})^-$ into $C(R, Y_n)$, that they are contraction operators on $S_n = \{y \in C(R, Y_n): \|y\|_\infty \leq \rho\}$, and that their unique fixed points y_n are the unique solutions to (**) such that $y_n \in S_n$ and $y_n' \in$ BUL.

Let $y = Ty = KFy$ be the unique compact solution to (*), and let $y_n = T_n y_n$. Then

$$\|y - y_n\|_\infty = \|Ty - T_n y_n\|_\infty \leq \|Ty - T_n y\|_\infty + \|T_n(y_n - y)\|_\infty.$$

The first term in the last member tends to 0, since $T_n y \to Ty$ in CC(R) for $y \in$ BULC. Using (9) and the properties of the function $F(t, x, \mu)$, we see that $\|T_n(y_n - y)\|_\infty \leq r\,\|y - y_n\|_\infty$ for large n. Thus, since $r < 1$, we have proved that $y_n \to y$ in CC(R).

Since $y_n' + U_n y_n = P_n Fy_n$, where $Fy_n = F(\cdot, y_n, \mu)$, we have the inequality

$$\|y' - y_n'\|_s \leq \|U_n y_n - Uy\|_\infty + \|P_n Fy_n - Fy\|_s.$$

The first term on the right side tends to 0, since $U_n \to U$ in L(B, B) and $y_n \to y$ in CC(R). Since $y_n \to y$ in CC(R), we know from the proof of Theorem 5 that $P_n Fy_n \to Fy$ in $(\text{BULC})^-$.

3. THE CASE WHERE U GENERATES A STRONGLY CONTINUOUS SEMIGROUP

Let U be an unbounded closed linear operator on B into B such that

(15) D(U), the domain of U, is dense in B,

(16) there exist positive numbers δ and η such that for every real z ($z > -\delta$), z is in the resolvent set of $-U$, and the resolvent of $-U$ satisfies the condition

$$\|(R(z, -U))^n\| \leq \eta(z + \delta)^{-n} \qquad (n = 0, 1, \cdots).$$

Then (see [4]) there exists a strongly continuous semigroup $E(t)$ of bounded linear operators for $t \in (-\infty, 0]$ such that $E(t) = e(-t, -U)$, where $e(t, -U)$ is the semigroup generated by $-U$ on $[0, \infty)$.

LEMMA 5. *Let* U *be an unbounded closed linear operator satisfying* (15) *and* (16). *Then* $G \in L(\text{BUL}, C(R))$, *where the operator* G *is defined by the Bochner integral*

$$g(t) = (Gf)(t) = \int_{-\infty}^{0} E(s)f(t + s)\,ds \qquad (t \in R,\ f \in \text{BUL}).$$

If $f(R) \subset D(U)$ *and* f, $Uf \in BUL$, *and* $g = Gf$, *then* $g(R) \subset D(U)$, g *is absolutely continuous on every finite interval,* g *is differentiable almost everywhere and satisfies the condition* $g'(t) + Ug(t) = f(t)$ *for almost all* t, $Ug \in C(R)$, $g' \in BUL$, *and* $E(t - b)g(t)$ *is absolutely continuous on every finite interval* $[a, b]$.

For a proof of Lemma 5, see [4, Lemma 5].

We shall now investigate the compact solutions of

$$(***) \qquad g' + Ug = F(\cdot, g, \mu),$$

where $F(t, x, \mu)$ satisfies (4), $F(R \times B \times D) \subset D(U)$, U satisfies the conditions of Lemma 5, and where we also assume that $UF(\cdot, y, \mu) \in BUL$ for all $\mu \in D \subset X$ and all $y \in C(R)$.

Definition. A function f on a finite closed interval $I = [a, b]$ into B is called a solution of (***) on I if and only if

(i) $f(I) \subset D(U)$,

(ii) f is absolutely continuous on $[a, b]$,

(iii) $E(t - s)f(t)$ is absolutely continuous in t on $[a, s]$, for each $s \in [a, b]$,

(iv) $f'(t)$ exists almost everywhere on $[a, b]$ and satisfies (***) for almost all $t \in [a, b]$.

A function f is called a solution of (***) in an interval if and only if it is a solution of (***) in every compact subinterval.

LEMMA 6. *If* f *is a solution of* (***) *on* $[a, b]$, *then*

$$f(t) = E(a - t)f(a) + \int_a^t E(s - t)F(s, f(s), \mu)\,ds \qquad (t \in [a, b]).$$

The proof of Lemma 6 follows from the definition of a solution. (For details, see [4, Theorem 6].)

We are interested in those ρ, μ for which

$$(17) \qquad \begin{aligned} &\sup\left\{\eta \int_{-\infty}^0 e^{\delta s}\,\theta(t + s, \rho, \mu)\,ds : t \in R\right\} < r < 1, \\ &\sup\left\{\left\|\int_{-\infty}^0 E(s)F(t + s, 0, \mu)\,ds\right\| : t \in R\right\} < \rho(1 - r)/2. \end{aligned}$$

THEOREM 8. *Assume* (17) *is satisfied for some fixed* ρ *and* μ. *Then there exists exactly one solution* $g \in CC(R)$ *of* (***) *such that* $\|g\|_\infty \le \rho$.

To prove this theorem, we need the following. For small $b > 0$, set

$$D_b = -b^{-1}(E(-b) - I), \qquad B_b = b^{-1}(I - b^{-1}R(b^{-1}, -U)),$$

and let U_b represent either D_b or B_b. It is known (see [1, pp. 621, 625]) that

$$\|\exp(U_b t)\| \le \eta \exp[(\delta + O(b))t] \qquad (t \le 0).$$

Thus by Lemma 1 the operators K_b are in L(BUL, C(R)) for small $b > 0$, where

$$(K_b f)(t) = \int_{-\infty}^{0} \exp(U_b s) f(s+t)\,ds \qquad (t \in R,\ f \in BUL),$$

and $\|K_b\| \leq \frac{\eta}{1 - \exp[-(\delta + O(b))]}$. Therefore there exists an $M < \infty$ such that $\|K_b\| \leq M$ for small $b > 0$.

LEMMA 7. *If* $g \in (BULC)^-$, *then* $K_b g \to Gg$ *in* C(R) *as* $b \to 0^+$.

Proof. See Lemma 6 in [4], and use the condition $\|K_b\| \leq M$ and the density of BULC in $(BULC)^-$.

LEMMA 8. $G \in L((BULC)^-, CC(R))$.

The proof of Lemma 8 follows immediately from Theorem 1 and Lemma 7.

Proof of Theorem 8. Take fixed ρ and μ such that (17) is satisfied. Then by Lemmas 2, 5, and 8 it follows that $T = GF$ is a contraction mapping on the closed sphere S in CC(R). From Lemma 5 it follows that the fixed point f of GF is a solution of (***) in R. If $g \in CC(R)$ is another solution of (***) in R such that $\|g\|_\infty \leq \rho$, then it follows from Lemma 6 that $f = g$.

Now consider the differential equation

$$(****) \qquad y' + U_b y = F(\cdot, y, \mu)$$

for small $b > 0$. From Lemma 7 we deduce that if (17) is satisfied for fixed ρ and μ, then

$$\|(K_b F)(0)\|_\infty < \rho(1 - r)/2 \qquad \text{for small } b > 0.$$

Since

$$\sup\left\{\eta \int_{-\infty}^{0} \exp[(\delta + O(b))s]\ \theta(t+s, \rho, \mu)\,ds \colon t \in R\right\} < r < 1$$

for small $b > 0$, we can conclude from Theorem 2 that there exists a $g_b \in CC(R)$ satisfying the conclusions of Theorem 2 for the differential equation (****). Thus we get the following result.

THEOREM 9. *If* (17) *is satisfied for fixed* ρ *and* μ, *then for small* $b > 0$, (****) *has a compact solution satisfying all the conclusions of Theorem* 2.

THEOREM 10. *Let* g_b *be the solution of* (****) *given by Theorem* 9, *and let* g *be the solution of* (***) *given by Theorem* 8; *then* $g_b \to g$ *in* CC(R), *as* $b \to 0^+$.

Proof. Since $g_b = K_b F g_b$ and $g = GFg$, we have the inequality

$$\|g - g_b\|_\infty \leq \|K_b(Fg_b - Fg)\|_\infty + \|K_b Fg - GFg\|_\infty.$$

For small $b > 0$, $\|K_b(Fg_b - Fg)\|_\infty \leq r\|g_b - g\|_\infty$. Therefore

$$(1 - r)\|g - g_b\|_\infty \leq \|K_b Fg - GFg\|_\infty.$$

The right side of the last inequality tends to 0 by Lemma 7. Since $0 < r < 1$ the theorem is proved.

Theorems 5 and 10 can be combined:

THEOREM 11. *Let* B *be a complex Banach space with basis* $\{x_i\}$ *such that the projection operators* P_n *satisfy the condition* $\|P_n\| = M = 1$. *Let the assumptions of Theorem* 8 *be satisfied. Then for each* $\varepsilon > 0$ *there exist a positive integer* n_ε *and a small positive number* b_ε *such that the operator*

$$T_{n_\varepsilon b_\varepsilon} = P_{n_\varepsilon} K_{b_\varepsilon} P_{n_\varepsilon} F$$

is a contraction operator on the closed sphere $S_{n_\varepsilon} \subset C(R, Y_{n_\varepsilon})$, *and such that* $\|g - g_{n_\varepsilon b_\varepsilon}\|_\infty < \varepsilon$. *Here* $g_{n_\varepsilon b_\varepsilon}$ *is the unique fixed point of* $T_{n_\varepsilon b_\varepsilon}$, *and* g *is the compact solution of* (***).

Combining Theorems 6 and 10, we get the following result.

THEOREM 12. *Let all the assumptions of Theorem* 11 *be satisfied, with* $M = 1$ *replaced by* $M \geq 1$. *Then for each* $f \in S$ *and each* $\varepsilon > 0$, *there exist positive integers* n_ε *and* m_ε *and a small positive number* b_ε *such that*

$$\|(P_{n_\varepsilon} K_{b_\varepsilon} P_{n_\varepsilon} F)^{m_\varepsilon} f - g\|_\infty < \varepsilon,$$

where g *is the compact solution of* (***).

We thus see that for each $M \geq 1$ we can approximate the compact solution to (***) by successive approximations in finite-dimensional spaces.

Proceeding as in the proofs of Theorems 3 and 4, we get the following two theorems.

THEOREM 13. *Assume that* $F(t, x, \mu)$, $\theta(t, \rho, \mu)$, *and* U *satisfy the conditions in Theorem* 8, *and that in addition* $F(t, x, \mu)$ *and* $\theta(t, \rho, \mu)$ *satisfy the assumptions in Theorem* 3, *Then, for each* ρ *and each* $\mu = \mu' \in D$ *for which* (17) *is satisfied, there exists a subdomain* $D' \subset D$ *such that for each* $\mu \in D'$ *the equation* (***) *has a unique compact solution* $x(\cdot, \mu)$; *the solution* $x(\cdot, \mu)$ *is analytic from* D' *into* CC(R).

THEOREM 14. *Let* $F(t, x, \mu)$, $\theta(t, \rho, \mu)$, *and* U *satisfy the conditions in Theorem* 8, *and in addition let* $F(t, x, \mu)$ *and* $\theta(t, \rho, \mu)$ *satisfy the conditions in Theorem* 4. *Then, for each* ρ *and each* $\mu = \mu' \in D$ *such that* (17) *is satisfied, there exists a neighborhood* $N \subset D$ *of* μ *such that for each* $\mu \in N$ *the equation* (***) *has a unique compact solution* $x(\cdot, \mu)$, *and* $x(\cdot, \mu)$ *is continuous on* N *into* CC(R).

In conclusion, we note that the stability properties of the unique bounded solutions corresponding to (*) and (***) developed in [4] apply without change to the compact solutions given by the present paper, since these compact solutions are also the unique bounded solutions.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators. I. General theory,* Interscience Publishers, Inc., New York, 1958.

2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups,* Amer. Math. Soc. Colloquium Publications 31, Revised Edition, 1957.

3. L. A. Liusternik and V. J. Sobolev, *Elements of functional analysis,* Frederick Ungar Publishing Co., New York, 1961.

4. C. T. Taam, *Stability, periodicity, and almost periodicity of solutions of nonlinear differential equations in Banach spaces,* J. Math. Mech. 15 (1966), 849-876.

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