A SYSTEM OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We consider the system of equations

$$(1.1) \qquad u'(t) = -\int_{-\infty}^{\infty} \alpha(x)\, T(x, t)\, dx, \qquad T_t(x, t) = T_{xx}(x, t) + \eta(x)\, g(u(t))$$

on $-\infty < x < \infty$, $0 < t < \infty$ with the initial conditions

$$(1.2) \qquad u(0) = u_0, \qquad T(x, 0) = f(x) \quad (-\infty < x < \infty).$$

The real-valued functions $\alpha(x)$, $\eta(x)$, f(x), and g(u) as well as the real constant u_0 are prescribed; u(t) and T(x, t) are the unknowns. We determine the asymptotic behavior of solutions of (1.1) as $t \to \infty$, under broad assumptions on α, η, f, and g. We also obtain existence and uniqueness results.

In the special case

$$(1.3) \qquad g(u) = -1 + \exp u,$$

the system (1.1) describes the behavior of a continuous-medium nuclear reactor. Roughly speaking, u(t) and T(x, t) denote the deviations of the reactor power and temperature from their equilibrium values, and thus they vanish at equilibrium. The precise physical interpretation of the various quantities involved in (1.1) as well as references to the physical literature may be found in [4]. Comparison of (1.1) with the equations of [4] shows that for notational simplicity we have set two prescribed constants equal to 1. Since these constants do not affect the present considerations and can easily be reintroduced, this causes no loss of generality.

In addition to possessing physical significance, (1.1) is also of interest because of its intimate connection with a class of Volterra equations that have been investigated elsewhere (for example, in [1], [2], [6], [7]). Most relevant to the present investigation is the lemma below concerning the Volterra equation (1.8). It is a special case of a more general result obtained in [7]. However, as will be seen, there is much in the analysis of (1.1) that is not contained in the lemma. It may be noted that for several reasons the results of [1], [2], and [6] are not applicable here. The hypothesis $a(t) \in L_1(0, \infty) \cap L_2(0, \infty)$, which is required in [1] (see [7]), is too restrictive for the present application. In [2] and [6] it is assumed that $b(t) \equiv 0$, and in [1], that $b(t) \in L_1(0, \infty) \cap L_2(0, \infty)$; both of these hypotheses are much too restrictive for our present purposes. A comparison of various conditions under which the asymptotic behavior of nonlinear Volterra equations has been investigated is given in [3].

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We now state the hypotheses under which we shall investigate (1.1).

We suppose that g(u) merely satisfies the conditions

$$(1.4) \quad g(u) \in C(-\infty, \infty), \quad u g(u) > 0 \quad (u \neq 0), \quad G(u) = \int_0^u g(\xi)\, d\xi \to \infty \quad (|u| \to \infty).$$

This clearly includes the actual physical case (1.3) and the linearized case

$$(1.5) \qquad g(u) = u,$$

which was considered in [4] and [5]. (If one is interested only in initial values u_0 and f(x) that are small in an appropriate sense, then the third condition of (1.4) may be dropped. For the sake of brevity, we do not discuss this further.)

We also suppose that

$$(1.6) \quad \alpha, \eta, f \in L_2(-\infty, \infty), \quad f \in C(-\infty, \infty), \quad \eta \textit{ is locally Hölder continuous.}$$

From the form of (1.1) and from well-known facts about the heat equation, we see clearly that the assumptions of (1.6) are mathematically very natural. Furthermore, the assumptions are not physically restrictive. If the second half of (1.2) is replaced by the condition that $T(x, t) \to f(x)$ in $L_2(-\infty, \infty)$ as $t \to 0+$, then the assumption $f \in C(-\infty, \infty)$ of (1.6) may be dropped.

In order to obtain a result on stability (or even on existence) on $-\infty < x < \infty$, $0 < t < \infty$ for (1.1), we need more assumptions than (1.4) and (1.6). In fact, one can easily construct examples where (1.4) and (1.6) are satisfied but where $u(t) \to \infty$ as $t \to \infty$. Our additional assumptions take the form of relations between α, η, and (less significantly) f —and also of a nondegeneracy condition that guarantees that the two equations of (1.1) are really coupled. Specifically, we suppose that (1.16) below is satisfied. As will shortly be seen, this hypothesis is motivated by physical considerations, and it is also rather general in the light of the already noted work on Volterra equations. Since (1.16) itself is perhaps somewhat formidable in appearance, we present the sequence of increasingly general conditions (1.7), (1.14), and (1.16), which are of independent interest.

An important physical case is represented by the condition

$$(1.7) \qquad \alpha(x) = k\eta(x) \quad (-\infty < x < \infty), \quad \int_{-\infty}^{\infty} \eta^2(x)\, dx > 0,$$

where k is a positive constant. (The relation between α and η may of course be assumed to hold only in an a. e. sense. Similar remarks apply below.)

Before we generalize (1.7), it is convenient to introduce the Volterra equation

$$(1.8) \qquad u'(t) = -\int_0^t a(t-\tau)\, g(u(\tau))\, d\tau - b(t)$$

on $0 \leq t < \infty$ with the initial condition

$$(1.9) \qquad u(0) = u_0.$$

In (1.8) and (1.9), the real-valued functions $a(t)$, $b(t)$, and $g(u)$ and the real constant u_0 are prescribed, while $u(t)$ is the unknown. Our notation for Fourier transforms is

$$\hat{f}(x) = \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} f(y) \exp\{-ixy\}\, dy . \tag{1.10}$$

As we mentioned above, and as will be seen in Section 2, the special case of (1.8) where

$$\begin{cases} a(t) = \dfrac{1}{\pi} \displaystyle\int_0^\infty h_1(x) \exp\{-x^2 t\}\, dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le t < \infty) \\ b(t) = \dfrac{1}{\pi} \displaystyle\int_0^\infty h_2(x) \exp\{-x^2 t\}\, dx, \end{cases} \tag{1.11}$$

with

$$h_1(x) = \Re(\hat{\eta}(x)\hat{\alpha}(-x)), \quad h_2(x) = \Re(\hat{f}(x)\hat{\alpha}(-x)), \tag{1.12}$$

and where the function $g(u)$ is the same in (1.8) as in (1.1), arises in a natural way in the analysis of (1.1). From (1.6) it follows that

$$h_1, h_2 \in L_1(0, \infty), \tag{1.13}$$

and since α, η, and f are real, that h_1 and h_2 are even functions.

The following condition includes (1.7) as a special case. Suppose there exist a measurable function $\rho(x)$ on $-\infty < x < \infty$ and a constant λ such that

$$\begin{cases} \hat{\alpha}(x) = \rho(x)\hat{\eta}(x), \quad \rho(x) \ge \lambda > 0 \quad (-\infty < x < \infty), \\ \displaystyle\int_{-\infty}^{\infty} \eta^2(x)\, dx > 0, \quad \rho\hat{f}^2 \in L_1(-\infty, \infty). \end{cases} \tag{1.14}$$

If ρ is also bounded from above, then the restriction $\rho\hat{f}^2 \in L_1(-\infty, \infty)$ of (1.14) is automatically satisfied as a consequence of (1.6). We note that (1.14) implies the relations

$$h_1(x) = \rho(x)\,|\hat{\eta}(x)|^2, \quad h_2(x) = \rho(x)\,\Re(\hat{f}(x)\hat{\eta}(-x)). \tag{1.15}$$

It seems inherent in the problem that the hypothesis involves the Fourier transforms $\hat{\alpha}$, $\hat{\eta}$, $\hat{f}$ as well as α, η, f. The advantage of (1.14) over (1.7) is that in (1.14) the ratio $\alpha(x)/\eta(x)$ need not be a constant. The next condition, (1.16), under which we actually prove our main result, includes (1.14) as a special case. Its advantage over (1.14) is that it does not require the ratio $\hat{\alpha}(x)/\hat{\eta}(x)$ to be real.

Suppose there exist a measurable function $h_3(x)$ on $0 \le x < \infty$ and a constant Λ such that

$$(1.16)\quad \begin{cases} h_2^2(x) \leq h_1(x)h_3(x), \quad h_1(x) \geq 0, \quad h_3(x) \geq 0 \qquad (0 \leq x < \infty), \\ \int_0^\infty h_1(x)\,dx > 0, \qquad h_3 \in L_1(0,\infty), \qquad \Lambda > 0, \\ h_1(x)\xi^2 + 2h_2(x)\xi + h_3(x) \geq \Lambda\left[\,|\hat{\eta}(x)|^2 \xi^2 + 2\,\Re(\hat{f}(x)\hat{\eta}(-x))\xi + |\hat{f}(x)|^2\,\right] \\ \qquad\qquad (0 \leq x < \infty, \; -\infty < \xi < \infty). \end{cases}$$

If (1.14) holds, then, by defining $h_3(x) = \rho(x)\,|\hat{f}(x)|^2$ and $\Lambda = \lambda$, one readily sees that (1.16) also holds.

The following special case of (1.16) has the virtue of being much more explicit than the latter and of not imposing any additional hypothesis (beyond (1.6)) on f(x). It is interesting to compare it directly with (1.7). Suppose there exist real measurable functions $\rho(x)$ and $\theta(x)$ and a constant λ such that

$$(1.17)\quad \begin{cases} \hat{\alpha}(x) = [\rho(x) + i\theta(x)]\hat{\eta}(x), \quad 1 > \lambda > 0, \quad \int_{-\infty}^{\infty} \eta^2(x)\,dx > 0, \\ \lambda \leq \rho(x) \leq \frac{1}{\lambda}, \quad |\theta(x)| \leq \frac{1}{\lambda}, \quad \rho(x) = \rho(-x), \quad \theta(x) = -\theta(-x) \quad (-\infty < x < \infty). \end{cases}$$

By defining

$$h_3(x) = \left[\frac{2}{\lambda}\left(\frac{2}{\lambda} - \frac{\lambda}{2}\right)^2 + \frac{\lambda}{2}\right] |\hat{f}(x)|^2, \qquad \Lambda = \frac{\lambda}{2},$$

one verifies that (1.16) is satisfied. It should be noted that in (1.17) the requirements that ρ is even and that θ is odd are made for convenience in stating the condition. Thus, one may verify (with the aid of α and η real) that if these requirements are deleted from (1.17), then a similar condition still obtains with $\rho(x)$ replaced by the even function $(\rho(x) + \rho(-x))/2$ and with $\theta(x)$ replaced by the odd function $(\theta(x) - \theta(-x))/2$.

Our main result is the following theorem, which is concerned with existence and asymptotic behavior of solutions of (1.1). Uniqueness is discussed afterwards.

THEOREM 1. *Let* (1.4), (1.6), *and* (1.16) *be satisfied. Then* (1.1) *has a solution* u(t), T(x, t) *on* $-\infty < x < \infty$, $0 < t < \infty$ *that satisfies* (1.2) *and also*

$$(1.18)\qquad \lim_{t\to\infty} u^{(k)}(t) = 0 \qquad (k = 0, 1, 2),$$

$$(1.19)\qquad \lim_{t\to\infty} \sup_{-\infty<x<\infty} |T(x,t)| = 0.$$

The linearized case (1.5) of (1.1) was treated in [4]. There, under the hypothesis

$$(1.20)\qquad h_1(x) \geq 0 \quad (0 \leq x < \infty), \qquad \int_0^\infty h_1(x)\,dx > 0$$

and some other conditions (which we need not consider here), it was shown that

$$u(t) = O(t^{-3/2}), \quad T(x, t) = O(t^{-1/2}) \quad (t \to \infty). \tag{1.21}$$

The analysis revolved around a Tauberian theorem for Laplace transforms that is unavailable in the present nonlinear situation. The present qualitative methods involve "energy" considerations of the type known in ordinary differential equations as the Ljapounov second method. They permit the enormous generalization from (1.5) to (1.4) at the price of strengthening the hypothesis from (1.20) to (1.16) and weakening the conclusion from (1.21) to (1.18) and (1.19).

In [7], the following lemma concerning (1.8) was established as a consequence of a more general result. It is important to note that in the lemma the prescribed functions $a(t)$, $b(t)$, $g(u)$ are only assumed to satisfy the stated conditions. It is not assumed in the lemma that $a(t)$ and $b(t)$ have the special form (1.11), (1.12), even though that is the case in the present application.

LEMMA. *Let* $g(u)$ *satisfy* (1.4), *and let* $a(t)$, $b(t)$ *satisfy*

$$\begin{cases} a(t) \in C[0, \infty), & (-1)^k a^{(k)}(t) \geq 0 \quad (0 < t < \infty;\ k = 0, 1, 2, 3), \\ b(t) \in C[0, \infty), & b''(t) \text{ exists on } 0 < t < \infty, \end{cases} \tag{1.22}$$

where $a(t) \not\equiv a(0)$. *Further, let there exist a function* $c(t)$ *such that*

$$\begin{cases} c(t) \in C[0, \infty), \quad c''(t) \text{ exists on } 0 < t < \infty, \\ [b^{(k)}(t)]^2 \leq a^{(k)}(t)\, c^{(k)}(t) \quad (0 < t < \infty;\ k = 0, 1, 2). \end{cases} \tag{1.23}$$

Then for each u_0 *there exists a solution* $u(t)$ *of* (1.8) *on* $0 \leq t < \infty$ *that satisfies* (1.9) *and also*

$$\lim_{t \to \infty} u^{(k)}(t) = 0 \quad (k = 0, 1, 2). \tag{1.24}$$

In Section 2 we give a complete proof of Theorem 1. We show how the lemma may be invoked to establish (1.18) as a consequence of (1.24). This is probably the quickest way to obtain (1.18). In the proof we then introduce the energy function $V(t)$ of (2.13) and use it to establish (1.19).

In the proof of the lemma (see [7]), the energy function

$$E(t) = G(u(t)) + \frac{1}{2} a(t) \left[\int_0^t g(u(s))\, ds \right]^2 + b(t) \int_0^t g(u(s))\, ds + \frac{1}{2} c(t) - \frac{1}{2} \int_0^t a'(t - \tau) \left[\int_\tau^t g(u(s))\, ds \right]^2 d\tau$$

played a vital role. However, even in the special case $b(t) = c(t) \equiv 0$, the functions $E(t)$ and $V(t)$ are not the same (see [6]). The essential difference between $E(t)$ and $V(t)$ is the following. $E(t)$ generalizes a function that was introduced in [2] for the study of the special case of (1.8) where $b(t) \equiv 0$ and $a(t)$ satisfies the monotonicity conditions (1.22). The function $V(t)$ generalizes a function introduced in [6] for the study of (1.8), again with $b(t) \equiv 0$, but under the stronger requirement of complete monotonicity on $a(t)$. From the condition $h_1(x) \geq 0$ of (1.16) and the first equation of (1.11), it follows that $a(t)$ is completely monotonic in the present problem.

In Section 3, we show briefly how to avoid the use of $E(t)$ (and therefore of the lemma), and we use only $V(t)$. The proof of (1.19) is the same as in Section 2, but in order to obtain (1.18), we need a much more detailed analysis of $V(t)$ than in Section 2. However, we only carry this analysis to a point where the technique of [6] completes the proof.

The proofs of Sections 2 and 3 rest partially on the somewhat involved Tauberian analysis of [6] and [7]. We find it of some interest that because of the special nature of (1.1), these arguments can be replaced by others that involve (1.1) more directly. This is done in Section 4, where the additional hypothesis

$$\alpha' \in L_2(-\infty, \infty) \tag{1.25}$$

is adjoined to Theorem 1.

The following theorem concerns the uniqueness problem for (1.1), (1.2). The linear case, (1.5), of this result was obtained in [5] (Theorem E). Since the present proof is an obvious modification of that of the linear case, we omit it.

THEOREM 2. *Suppose that* (1.6) *is satisfied and* $0 < t_0 < \infty$. *Suppose further that the Volterra equation* (1.8), *with* $a(t)$ *and* $b(t)$ *defined by* (1.11) *and* (1.12), *has at most one solution on* $0 \leq t \leq t_0$ *that satisfies the initial condition* (1.9). *Then there exists at most one solution* $u(t)$, $T(x, t)$ *of* (1.1) *on* $-\infty < x < \infty$, $0 < t \leq t_0$ *such that*

$$\begin{cases} u'(t) \text{ exists on } 0 < t \leq t_0, \quad \lim_{t \to 0+} u(t) = u_0, \\ T(x, t), T_t(x, t), T_{xx}(x, t) \in C \text{ on } -\infty < x < \infty, \ 0 < t \leq t_0, \\ T(x, t) \in L_2(-\infty, \infty) \quad \text{and} \quad \sup_{0 < t \leq t_0} \int_{-\infty}^{\infty} T^2(x, t)\,dx < \infty, \\ \lim_{t \to 0+} \int_{-\infty}^{\infty} |T(x, t) - f(x)|^2\,dx = 0. \end{cases} \tag{1.26}$$

We can of course guarantee the uniqueness supposition for (1.8) of Theorem 2 by assuming that $g(u)$ is locally Lipschitzian. However, more general uniqueness conditions for (1.8) are possible (see for example Theorem 3.1 of [8]). It is easily seen from the proofs of Theorem 1 that the solution $u(t)$, $T(x, t)$ of Theorem 1 satisfies the conditions (1.26) for all t_0.

We conclude with a few remarks concerning the perturbed version of (1.1) given by

$$u'(t) = -\int_{-\infty}^{\infty} \alpha(x)\,T(x, t)\,dx + \sigma(t), \qquad T_t(x, t) = T_{xx}(x, t) + \eta(x)\,g(u(t)),$$

where $\sigma(t) \in L_1(0, \infty)$ denotes the perturbation, and where the other quantities remain the same as before. This system, which permits the incorporation of some physical effects ignored in (1.1), can be treated by essentially the same methods as above. Instead of the above lemma, we require the more involved Theorem 1 of [7]. The technical details are considerable, but they follow readily from the present work together with [7], and we omit them.

2. PROOF OF THEOREM 1

(i) Consider (1.8) with a(t) and b(t) defined by (1.11) and (1.12) and with g(u) as in (1.1). This is a well-defined equation, regardless of whether (1.1) possesses any solutions. (We can motivate our approach by supposing that (1.1) has a solution that satisfies (1.2) and the well-known formula (2.6), which will be used shortly.)

From (1.11) and (1.13) it follows that

$$(2.1)\quad \begin{cases} a^{(k)}(t) = (-1)^k \dfrac{1}{\pi} \displaystyle\int_0^\infty x^{2k} h_1(x) \exp\{-x^2 t\}\, dx, \\ \\ b^{(k)}(t) = (-1)^k \dfrac{1}{\pi} \displaystyle\int_0^\infty x^{2k} h_2(x) \exp\{-x^2 t\}\, dx. \end{cases} \qquad (0 < t < \infty;\ k = 0, 1, \cdots)$$

From (2.1) and the condition $h_1(x) \geq 0$ of (1.16) it follows that

$$(2.2)\qquad a(t) \in C[0, \infty), \qquad (-1)^k a^{(k)}(t) \geq 0 \quad (0 < t < \infty;\ k = 0, 1, \cdots).$$

Define

$$(2.3)\qquad c(t) = \frac{1}{\pi} \int_0^\infty h_3(x) \exp\{-x^2 t\}\, dx \qquad (0 \leq t < \infty).$$

Then, from the condition $h_3 \in L_1(0, \infty)$ of (1.16), we obtain the equation

$$(2.4)\qquad c^{(k)}(t) = (-1)^k \frac{1}{\pi} \int_0^\infty x^{2k} h_3(x) \exp\{-x^2 t\}\, dx \qquad (0 < t < \infty;\ k = 0, 1, \cdots).$$

From (1.16), (2.1), (2.4), and the Schwarz inequality we obtain for $k = 0, 1, \cdots$ the relations

$$\begin{aligned}(b^{(k)}(t))^2 &\leq \left(\frac{1}{\pi} \int_0^\infty x^{2k} |h_2(x)| \exp\{-x^2 t\}\, dx\right)^2 \\ &\leq \left(\frac{1}{\pi} \int_0^\infty [x^{2k} h_1(x) \exp\{-x^2 t\}]^{1/2} [x^{2k} h_3(x) \exp\{-x^2 t\}]^{1/2} dx\right)^2 \\ &\leq \left(\frac{1}{\pi} \int_0^\infty x^{2k} h_1(x) \exp\{-x^2 t\}\, dx\right)\left(\frac{1}{\pi} \int_0^\infty x^{2k} h_3(x) \exp\{-x^2 t\}\, dx\right).\end{aligned}$$

Thus

$$(b^{(k)}(t))^2 \leq a^{(k)}(t)\, c^{(k)}(t) \qquad (0 < t < \infty;\ k = 0, 1, \cdots). \tag{2.5}$$

Now, by the lemma of Section 1, the equation (1.8) has a solution $u(t)$ on $0 \leq t \leq \infty$ that satisfies (1.9) and (1.24). If more than one such solution exists, we choose one and call it $u(t)$.

Define $T(x, t)$ on $-\infty < x < \infty$, $0 < t < \infty$ by

$$T(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t) f(\xi)\, d\xi + \int_0^t \int_{-\infty}^{\infty} G(x - \xi, t - \tau)\eta(\xi) g(u(\tau))\, d\xi\, d\tau, \tag{2.6}$$

where

$$G(x, t) = [4\pi t]^{-1/2} \exp\left\{-\frac{x^2}{4t}\right\}, \tag{2.7}$$

and where $u(t)$ is the function defined in the preceding paragraph. Classical results concerning the inhomogeneous heat equation (for their statements, see for example Lemmas B_1 and B_2 of [5]) give the following information: $T(x, t)$, $T_t(x, t)$, $T_{xx}(x, t)$ are continuous in (x, t) for $-\infty < x < \infty$, $0 < t < \infty$, the second equation of (1.1) is satisfied there, and $T(x, t)$ is continuous in (x, t) for $-\infty < x < \infty$, $0 \leq t < \infty$, with $T(x, 0) = f(x)$. Thus, the pair $u(t)$, $T(x, t)$ satisfies (1.2) and the second half of (1.1). It is well known (and follows from (2.6)), that for each fixed t, $T(x, t)$ belongs to $L_2(-\infty, \infty)$ as a function of x, and that its Fourier transform $\hat{T}(x, t)$ with respect to x is given by the formula

$$\hat{T}(x, t) = \hat{f}(x) \exp\{-x^2 t\} + \hat{\eta}(x) \int_0^t g(u(\tau)) \exp\{-x^2(t - \tau)\}\, d\tau. \tag{2.8}$$

Using (2.8), Parseval's relation, and the definitions (1.11) and (1.12), one can verify that

$$\int_{-\infty}^{\infty} \alpha(x) T(x, t)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\alpha}(-x) \hat{T}(x, t)\, dx = \int_0^t a(t - \tau) g(u(\tau))\, d\tau + b(t), \tag{2.9}$$

where the necessary change of order of integration is easily justified by Fubini's theorem. However, from (2.9) and the fact that $u(t)$ is a solution of (1.8), we see that the pair $u(t)$, $T(x, t)$ also satisfies the first half of (1.1). The assertion (1.18) follows, of course, from (1.24).

(ii) For the completion of the proof of Theorem 1, it remains only to establish (1.19). Somewhat surprisingly, (1.18) will not be needed. It follows from (1.6) and (2.8) that T_x, $T_{xx} \in L_2(-\infty, \infty)$ for each $t > 0$. We need the elementary inequality (of Sobolev type)

$$\begin{aligned} \sup_{-\infty < x < \infty} T^4(x, t) &\leq 4 \int_{-\infty}^{\infty} T^2(x, t)\, dx \int_{-\infty}^{\infty} T_x^2(x, t)\, dx \\ &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} |\hat{T}(x, t)|^2\, dx \int_{-\infty}^{\infty} x^2 |\hat{T}(x, t)|^2\, dx \qquad (0 < t < \infty), \end{aligned} \tag{2.10}$$

whose simple proof we include for completeness. The relation

$$\frac{\partial}{\partial x} T^2(x, t) = 2T(x, t)\, T_x(x, t)$$

implies that

$$T^2(x_2, t) - T^2(x_1, t) = 2 \int_{x_1}^{x_2} T(x, t)\, T_x(x, t)\, dx$$

for all x_1, x_2. Hence

$$|T^2(x_2, t) - T^2(x_1, t)|^2 \leq 4 \int_{-\infty}^{\infty} T^2(x, t)\, dx \int_{-\infty}^{\infty} T_x^2(x, t)\, dx .$$

Let $t > 0$ be fixed. Then, letting x_1 pass through a sequence of values on which $T(x, t)$ becomes small (such a sequence must exist, because $T \in L_2(-\infty, \infty)$), we obtain (2.10).

Using Fubini's theorem and (1.11), we may write (1.8) in the equivalent form

$$u'(t) = -\frac{1}{\pi} \int_0^{\infty} [h_1(x)\gamma(x, t) + h_2(x)] \exp\{-x^2 t\}\, dx, \tag{2.11}$$

where

$$\gamma(x, t) = \int_0^t g(u(\tau)) \exp\{x^2 \tau\}\, d\tau . \tag{2.12}$$

Define the energy function $V(t)$ by

$$V(t) = G(u(t)) + \frac{1}{2\pi} \int_0^{\infty} [h_1(x)\gamma^2(x, t) + 2h_2(x)\gamma(x, t) + h_3(x)] \exp\{-2x^2 t\}\, dx \geq 0 \tag{2.13}$$

(the inequality is implied by (1.4) and (1.16)).

Differentiating (2.13) and using (2.11) to simplify the resulting expression, we obtain (on $0 < t < \infty$) the formula

$$V'(t) = -\frac{1}{\pi} \int_0^{\infty} [h_1(x)\gamma^2(x, t) + 2h_2(x)\gamma(x, t) + h_3(x)]\, x^2 \exp\{-2x^2 t\}\, dx \leq 0; \tag{2.14}$$

the inequality follows from (1.16).

From (2.8) and (2.12) it follows that

$$|\hat{T}(x, t)|^2 = [|\hat{\eta}(x)|^2 \gamma^2(x, t) + 2\,\Re(\hat{f}(x)\hat{\eta}(-x))\gamma(x, t) + |\hat{f}(x)|^2] \exp\{-2x^2 t\}, \tag{2.15}$$

which together with (1.16) and (2.14) implies that

$$2\pi V(0) \geq 2\pi V(t) \geq \int_0^\infty [h_1(x)\gamma^2(x, t) + 2h_2(x)\gamma(x, t) + h_3(x)] \exp\{-2x^2 t\}\, dx$$

$$\geq \Lambda \int_0^\infty [|\hat{\eta}(x)|^2 \gamma^2(x, t) + 2\Re(\hat{f}(x)\hat{\eta}(-x))\gamma(x, t) + |\hat{f}(x)|^2] \exp\{-2x^2 t\}\, dx$$

$$= \Lambda \int_0^\infty |\hat{T}(x, t)|^2\, dx = \frac{\Lambda}{2} \int_{-\infty}^\infty |\hat{T}(x, t)|^2\, dx.$$

Thus

$$\int_{-\infty}^\infty |\hat{T}(x, t)|^2\, dx \leq \frac{4\pi}{\Lambda} V(0) \qquad (0 \leq t < \infty). \tag{2.16}$$

Similarly, from (1.16), (2.14), and (2.15) we deduce that

$$-\pi V'(t) \geq \Lambda \int_0^\infty x^2 |\hat{T}(x, t)|^2\, dx = \frac{1}{2}\Lambda \int_{-\infty}^\infty x^2 |\hat{T}(x, t)|^2\, dx,$$

that is,

$$\int_{-\infty}^\infty x^2 |\hat{T}(x, t)|^2\, dx \leq -\frac{2\pi}{\Lambda} V'(t) \qquad (0 < t < \infty). \tag{2.17}$$

From (2.10), (2.16), and (2.17) we obtain the inequality

$$\sup_{-\infty < x < \infty} T^4(x, t) \leq -\frac{8}{\Lambda^2} V(0) V'(t) \qquad (0 < t < \infty). \tag{2.18}$$

It follows from (1.18) that there exists a constant $K_1 < \infty$ such that

$$|u(t)| \leq K_1 \qquad (0 \leq t < \infty). \tag{2.19}$$

Alternatively (and we use this in Sections 3 and 4) formulas (2.13) and (2.14) imply that

$$G(u(t)) \leq V(t) \leq V(0) = G(u_0) + \frac{1}{2\pi} \int_0^\infty h_3(x)\, dx,$$

which together with (1.4) implies that (2.19) holds, with K_1 depending only on $V(0)$.

The positive constants K_j ($j = 2, 3, \cdots$) appearing below depend only on $V(0)$.

Observe that (1.4) and (2.19) imply

$$|x^2 \gamma(x, t)| \leq K_2 \exp\{x^2 t\} \qquad (0 \leq x < \infty,\ 0 \leq t < \infty). \tag{2.20}$$

Differentiating (2.14), one obtains the relation

$$
(2.21)\quad \begin{aligned} V''(t) = &- \frac{2}{\pi} g(u(t)) \int_0^\infty [h_1(x)\gamma(x, t) + h_2(x)] x^2 \exp\{-x^2 t\}\, dx \\ &+ \frac{2}{\pi} \int_0^\infty [h_1(x)\gamma^2(x, t) + 2h_2(x)\gamma(x, t) + h_3(x)] x^4 \exp\{-2x^2 t\}\, dx \end{aligned}
$$

on $0 < t < \infty$. Since $h_1, h_2, h_3 \in L_1(0, \infty)$, conditions (2.19), (2.20), and (2.21) imply directly that

$$
(2.22)\qquad |V''(t)| \le K_3 \quad (1 \le t < \infty).
$$

(Of course, we may replace 1 in $1 \le t < \infty$ by any positive constant, provided we make an appropriate change in K_3.)

From the conditions $V(t) \ge 0$, $V'(t) \le 0$, (2.22), and the mean-value theorem it follows (a proof is given in Lemma 1 of [2]) that

$$
(2.23)\qquad \lim_{t \to \infty} V'(t) = 0;
$$

together with (2.18), this implies (1.19), and the proof is complete.

3. ANOTHER PROOF OF THEOREM 1

This proof, unlike that of Section 2, rests entirely on the function $V(t)$ of (2.13). Its starting point is the same as that of Section 2, namely (1.8) with $a(t)$ and $b(t)$ defined by (1.11) and (1.12) and with $g(u)$ as in (1.1). From (1.13) it follows that

$$
(3.1)\qquad a(t), b(t) \in C[0, \infty).
$$

By a result of [8], the condition $g \in C(-\infty, \infty)$ of (1.4) and condition (3.1) imply the existence (but not the uniqueness) of a solution $u(t)$ of (1.8) on $0 \le t \le t_0$ (for some $t_0 > 0$) that satisfies (1.9). (If g is locally Lipschitzian, the existence and uniqueness of $u(t)$ follow from the usual successive approximations of Picard.)

Define $V(t)$ on $0 \le t \le t_0$ by (2.13). The calculation of Section 2 yields (2.14) on $0 < t \le t_0$. From (2.13) and (2.14), we obtain (2.19) on $0 \le t \le t_0$ (as noted in Section 2 for the interval $0 \le t < \infty$), where K_1 does not depend on t_0. However, the *a priori* bound (2.19), together with a result of [8], implies that $u(t)$ may be continued as a solution of (1.8) onto $0 \le t < \infty$. (If g is locally Lipschitzian, this continuation can also be accomplished by an argument that is classical for ordinary differential equations.)

Having established (2.19) on $0 \le t < \infty$, and with it also (2.13) and (2.14) on $0 \le t < \infty$, we proceed as in Section 2 to obtain (2.23). Inspection of (2.14) shows that (2.23), together with (2.20) and the condition $h_1, h_2, h_3 \in L_1(0, \infty)$, implies

$$
(3.2)\qquad \lim_{t \to \infty} \int_0^\infty x^2 h_1(x)\, \Gamma^2(x, t)\, dx = 0,
$$

where

$$\Gamma(x, t) = \gamma(x, t) \exp\{-x^2 t\} = \int_0^t g(u(\tau)) \exp\{-x^2(t-\tau)\} d\tau .$$

Differentiation of (2.11) yields a formula for $u''(t)$, which together with (2.19) implies $|u''(t)| \leq K_4$ $(1 \leq t < \infty)$. The latter inequality, together with (2.19) and the mean-value theorem, implies

$$|u'(t)| \leq K_5 \qquad (0 \leq t < \infty). \tag{3.3}$$

It may now be shown that (1.16), (3.2), and (3.3) imply the result (1.18). As noted in Section 1, a proof of this last assertion is contained in [6].

$T(x, t)$ is now defined by (2.6), and the proof is completed as in Section 2.

4. STILL ANOTHER PROOF OF THEOREM 1

Here we follow the method of Section 3 through the verification of both (2.19), by way of $V(t)$, on $0 \leq t < \infty$ and the limit (2.23). Thus, $T(x, t)$ is defined by (2.6), and the relations (2.15), (2.16), (2.17), and (2.18), together with the result (1.19), are established as before. We note that this much of the proof is self-contained. Only (1.18) remains to be established. This we now do directly from (1.1) without any appeal to earlier asymptotic results for the Volterra equation. Here it is convenient to make the additional integrability assumption (1.25).

First we show that

$$\lim_{t \to \infty} u'(t) = 0, \tag{4.1}$$

which is (1.18) for $k = 1$. Let $\varepsilon > 0$. Then (1.16) implies that

$$\int_X^\infty \alpha^2(x)\,dx \leq \varepsilon, \qquad \int_{-\infty}^{-X} \alpha^2(x)\,dx \leq \varepsilon \tag{4.2}$$

for some $X = X(\varepsilon) < \infty$. From (2.16) and (4.2), together with Schwarz's inequality and Parseval's theorem, we get the bounds

$$\left|\int_X^\infty \alpha(x) T(x, t)\,dx\right|^2 \leq \frac{2}{\Lambda} V(0)\varepsilon, \qquad \left|\int_{-\infty}^{-X} \alpha(x) T(x, t)\,dx\right|^2 \leq \frac{2}{\Lambda} V(0)\varepsilon \tag{4.3}$$

for $0 \leq t < \infty$. On the other hand,

$$\left|\int_{-X}^{X} \alpha(x) T(x, t)\,dx\right| \leq \left[2X \int_{-\infty}^{\infty} \alpha^2(x)\,dx\right]^{1/2} \sup_{-\infty < x < \infty} |T(x, t)| . \tag{4.4}$$

From (1.19), (4.3), (4.4), and the first equation of (1.1) one obtains the relations

$$|u'(t)| = \left|\int_{-\infty}^{\infty} \alpha(x) T(x, t)\,dx\right| \leq 3\left[\frac{2}{\lambda} V(0)\varepsilon\right]^{1/2} \qquad (t_0 \leq t < \infty)$$

for some $t_0 = t_0(\varepsilon) < \infty$; this establishes (4.1).

Use of the Parseval relation in the first equation of (1.1) and differentiation of (2.8) leads to the equations

$$(4.5) \qquad u'(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\alpha}(-x)\hat{T}(x, t)\,dx, \qquad \hat{T}_t(x, t) = -x^2\hat{T}(x, t) + \hat{\eta}(x)g(u(t)).$$

Differentiating both sides of the first equation of (4.5), we find that

$$u''(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\alpha}(-x)\hat{T}_t(x, t)\,dx \qquad (0 < t < \infty),$$

which together with the second equation of (4.5) implies that

$$(4.6) \qquad u''(t) + \nu g(u(t)) = \frac{1}{2\pi} I(t) \qquad (0 < t < \infty),$$

where

$$(4.7) \qquad \nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(x)\hat{\alpha}(-x)\,dx = \frac{1}{\pi}\int_0^{\infty} h_1(x)\,dx > 0,$$

$$(4.8) \qquad I(t) = \int_{-\infty}^{\infty} x^2\hat{\alpha}(-x)\hat{T}(x, t)\,dx = -2\pi\int_{-\infty}^{\infty} \alpha(x)T_{xx}(x, t)\,dx \qquad (0 < t < \infty).$$

From (2.17), (4.8), and (1.25) it follows that

$$I^2(t) \leq \int_{-\infty}^{\infty} x^2|\hat{\alpha}(-x)|^2\,dx \int_{-\infty}^{\infty} x^2|\hat{T}(x, t)|^2\,dx \leq -\frac{4\pi^2}{\Lambda}V'(t)\int_{-\infty}^{\infty} |\alpha'(x)|^2\,dx,$$

which together with (2.23) yields the relation

$$(4.9) \qquad \lim_{t\to\infty} I(t) = 0.$$

Suppose $u(t) \not\to 0$ as $t \to \infty$. Then there exist a real number $u^* \neq 0$ (with $|u^*| \leq K_1$) and a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $u(t_n) \to u^*$ as $n \to \infty$. Suppose $u^* > 0$ (a similar argument holds if $u^* < 0$). From (4.1) and the mean-value theorem it now follows that there exist a sequence $\{\Delta_n\}$ ($\Delta_n \to \infty$ as $n \to \infty$) and an integer N_1 such that

$$(4.10) \qquad u^*/2 \leq u(t) \leq K_1 \qquad (t_n \leq t \leq t_n + \Delta_n,\ n \geq N_1).$$

Define

$$(4.11) \qquad \mu = \min_{u^*/2 \leq u \leq K_1} g(u) > 0$$

(the inequality follows from (1.4)). Clearly, (4.10) and (4.11) imply

$$\nu g(u(t)) \geq \nu\mu \qquad (t_n \leq t \leq t_n + \Delta_n,\ n \geq N_1),$$

which together with (4.6) and (4.9) yields the inequality

$$u''(t) \leq -\nu\mu/2 \qquad (t_n \leq t \leq t_n + \Delta_n,\ n \geq N_2)$$

for some integer $N_2 \geq N_1$. Integrating, we find that

$$u'(t_n + \Delta_n) - u'(t_n) \leq -\frac{1}{2}\nu\mu\Delta_n \qquad (n \geq N_2),$$

which is obviously incompatible with (4.1) and the condition that $\Delta_n \to \infty$ as $n \to \infty$. Hence $u(t) \to 0$ as $t \to \infty$, which is (1.18) for $k = 0$. From (1.18) for $k = 0$, (4.6), and (4.9), we obtain (1.18) for $k = 2$; this completes the proof.

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