ON THE REPRESENTATION OF A FUNCTION BY A POISSON TRANSFORM

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Recently, Pollard [1] has obtained a real inversion formula for the convolution transforms

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(y)}{1 + (x - y)^2}, \tag{1}$$

with $\alpha(y)$ of bounded variation on every finite interval, and

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(y)dy}{1 + (x - y)^2}, \tag{2}$$

with $g(y)$ integrable on every finite interval. The integrals are interpreted in the sense

$$\int_{-\infty}^{\infty} = \lim_{R \to \infty, S \to \infty} \int_{-S}^{R}$$

In this paper we shall obtain necessary and sufficient conditions for $f(x)$ to be representable in the form (1) with

(I) $\alpha(y)$ of bounded variation on $(-\infty, \infty)$,

or (II) $\alpha(y)$ nondecreasing and bounded on $(-\infty, \infty)$;

and for $f(x)$ to be representable in the form (2) with

(III) $g(y)$ integrable on $(-\infty, \infty)$,

or (IV) $g(y)$ bounded on $(-\infty, \infty)$,

or (V) $g(y) \in L_p$ on $(-\infty, \infty)$, for some $p > 1$.

Our conditions will be phrased in terms of the operator T_t defined by the formula

$$T_t f(x) = (\cos tD) f(x) + D^{-1}(\sin tD)\hat{f}(x),$$

where

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$$\hat{f}(x) = -\frac{1}{\pi}\int_0^\infty u^{-2}[f(x+u)+f(x-u)-2f(x)]\,du,$$

$$(\cos tD)f(x) = \sum_{k=0}^\infty \frac{(-1)^k t^{2k} f^{(2k)}(x)}{(2k)!}$$

$$D^{-1}(\sin tD)\hat{f}(x) = \sum_{k=0}^\infty \frac{(-1)^k t^{2k+1}}{(2k+1)!}\hat{f}^{(2k)}(x).$$

This operator was employed in constructing the inversion theory of [1].

THEOREM 1. *The following set of conditions is necessary and sufficie[nt for]* $f(x)$ *to be the Poisson transform of a function of bounded variation:*

(a) $f(x)$ *and* $\hat{f}(x) \in C^\infty$ *for all real* x;

(b) $|f^{(k)}(x)| \le Bk!$, $|\hat{f}^{(k)}(x)| \le B(k+1)!$,

where B *is independent of* k *and* x;

(c) $\int_{-\infty}^{\infty} |T_t f(x)|\,dx < B$ *for* $|t| < 1$.

Proof. The necessity of (a) and (b) is demonstrated in [1, pp. 543-547]. [It is] also shown in [1, p. 548] that if f is of the form (1), then

$$T_t f(x) = \int_{-\infty}^{\infty} \frac{(1-t)\,d\alpha(y)}{(1-t)^2+(x-y)^2}. \tag{3}$$

Since

$$\int |d\alpha(y)| \le B < \infty, \tag{4}$$

$$\int |T_t f(x)|\,dx \le \frac{1}{\pi}\int |d\alpha(y)| \int \frac{(1-t)\,dx}{(1-t)^2+(x-y)^2}$$
$$\le \frac{1}{\pi}\int |d\alpha(y)| \le B$$

(when limits of integration are omitted, the range is understood to be $(-\infty, \infty)$). [To] establish the sufficiency we first note that (a) and (b) guarantee the existen[ce of the] transform $T_t f(x)$ for $|t| < 1$. In the next three lemmas we suppose that $f(x)$ [and] $\hat{f}(x)$ satisfy conditions (a), (b) and (c) of Theorem 1.

LEMMA 1.1.

$$\int \frac{(\cos tD)f(y)\,dy}{1+(x-y)^2} = \frac{1}{2\pi}\int \left\{\frac{1-t}{(1-t)^2+(x-y)^2} + \frac{1+t}{(1+t)^2+(x-y)^2}\right\} f(y)\,dy \tag{5}$$

Proof. The left member of (5) is equal to

$$\int \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \cdot \frac{f^{(2k)}(y)}{1+(x-y)^2} dy.$$

The estimates furnished by (b) enable us to integrate by parts $2k$ times and to write

$$\int \frac{f^{(2k)}(y)\,dy}{1+(x-y)^2} = \int f(y) \frac{d^{2k}}{dy^{2k}} \left(\frac{1}{1+(x-y)^2}\right) dy$$

$$= \int f(y)\,dy \int (iu)^{2k} e^{iu(x-y)} e^{-|u|} du.$$

The rest of the proof parallels [1, p. 547].

LEMMA 1.2.

$$\int \frac{D^{-1}(\sin tD)\hat{f}(y)\,dy}{1+(x-y)^2} = \frac{1}{2\pi} \int \left\{ \frac{1-t}{(1-t)^2+(x-y)^2} - \frac{1+t}{(1+t)^2+(x-y)^2} \right\} f(y)\,dy.$$

Proof. As in the lemma above,

$$\int \frac{\hat{f}^{(2k)}(y)\,dy}{1+(x-y)^2} = \int \hat{f}(y) \frac{d^{2k}}{dy^{2k}} \left(\frac{1}{1+(x-y)^2}\right) dy.$$

Now let

$$\hat{f}_\varepsilon(x) = \int_\varepsilon^\infty u^{-2}[f(x+u)+f(x-u)-2f(x)]\,du.$$

Since by (c) $f(x)$ is in L, $\hat{f}_\varepsilon(x)$ is in L and has a Fourier transform

$$F(x) \int_\varepsilon^\infty \left(\frac{\sin xu}{u}\right)^2 du,$$

where $F(x)$ is the Fourier transform of $f(x)$. Furthermore $\hat{f}_\varepsilon(x)$ has a bound which is independent of ε, for

$$|\hat{f}_\varepsilon(x)| \le \left(\int_0^1 + \int_1^\infty \right) u^{-2} |f(x+u)+f(x-u)-2f(x)|\,du.$$

The boundedness of $f(x)$ implies that the second integral is less than $3B$, and by the mean-value theorem of the differential calculus, there exist values $\xi_1(x, u)$ and $\xi_2(x, u)$ $(x < \xi_1 < x+u,\ x-u < \xi_2 < x)$ such that

$$u^{-2}|[f(x+u)-f(x)]-[f(x)-f(x-u)]| = u^{-1}|f'(\xi_1(x,u)) - f'(\xi_2(x,u))|$$

$$\le u^{-1}|f''(\xi_3(x,u))(\xi_2-\xi_1)| \le 2B.$$

Thus the first integral is less than $2B$, $|\hat{f}_\varepsilon(x)| \le 5B$, and by dominated convergence

$$\hat{f}(y)\frac{d^{2k}}{dy^{2k}}\left(\frac{1}{1+(x-y)^2}\right)dy = \lim_{\varepsilon\to 0}\int \hat{f}_\varepsilon(y)\frac{d^{2k}}{dy^{2k}}\left(\frac{1}{1+(x-y)^2}\right)dy$$

$$= \lim_{\varepsilon\to 0}\int f(y)dy\int e^{iu(x-y)}\left(\int_\varepsilon^\infty\left(\frac{\sin uv}{uv}\right)^2 dv\right) e^{-|}$$

$$= \int f(y)dy\int e^{iu(x-y)}(iu)^{2k}|u|e^{-|u|}du,$$

the last step being justified by the fact that $f(x)$ is in L. We may now trea fashion analogous to (5) [1, p. 547] and obtain our result.

LEMMA 1.3.

$$\lim_{t\to 1}\frac{1}{\pi}\int\frac{T_t f(y)dy}{1+(x-y)^2} = f(x). \tag{7}$$

Proof. Adding (5) and (6), we find that the left member of (7) is equal t

$$\lim_{t\to 1}\frac{1}{\pi}\int\frac{(1-t)f(y)dy}{(1-t)^2+(x-y)^2},$$

and it is well known that this integral tends to $f(x)$ [3, p. 31].

We now set

$$\alpha_t(y) = \int_0^y T_t f(u)du.$$

By (c),

$$|\alpha_t(y)| \le \int_{-\infty}^{\infty}|T_t f(u)|du \le B,$$

hence by Helly's selection theorem [2, p. 29] there exists a sequence

$$\{t_k\} \quad (t_k\to 1 \text{ as } k\to\infty)$$

and a function $\alpha(x)$ of bounded variation on $(-\infty, \infty)$ such that

$$\lim_{k\to\infty}\alpha_{t_k}(x) = \alpha(x).$$

By virtue of Lemma 1.3 and the definition of $\alpha_t(x)$,

$$\lim_{k\to\infty}\frac{1}{\pi}\int\frac{T_{t_k}f(y)dy}{1+(x-y)^2} = \lim_{k\to\infty}\frac{1}{\pi}\int\frac{d\alpha_{t_k}(y)}{1+(x-y)^2}$$

$$= \lim_{k\to\infty}\frac{1}{\pi}\int\alpha_{t_k}(y)\frac{d}{dy}\left(\frac{1}{1+(x-y)^2}\right)dy =$$

Since the $\alpha_{t_k}(y)$ are uniformly bounded in y and k, we may take the limit under the integral sign; this yields the relation

$$\frac{1}{\pi}\int \alpha(y)\frac{d}{dy}\left(\frac{1}{1+(x-y)^2}\right)dy = f(x).$$

Integrating by parts once more, we have

$$f(x) = \frac{1}{\pi}\int \frac{d\alpha(y)}{1+(x-y)^2}.$$

THEOREM 2. *The following set of conditions is necessary and sufficient for* $f(x)$ *to be representable in the form* (2) *with* $g(y)$ *integrable:*

(a) $f(x)$ *and* $\hat{f}(x) \in C^\infty$ *for all real* x;

(b) $|f^{(k)}(x)| \leq Bk!$, $|\hat{f}^{(k)}(x)| \leq B(k+1)!$, *where* B *is independent of* k *and* x;

(c) $\int |T_t f(x)|dx \leq B$ *when* $|t| < 1$;

(d) $\lim_{t \to 1,\, t' \to 1} \int |T_{t'} f(x) - T_t f(x)|dx = 0$.

Proof. The demonstration of the necessity of conditions (a) to (c) involves only trivial modification of the proofs in Theorem 1. To establish the necessity of (d), we write

$$T_t f(y) - g(y) = \frac{1}{\pi}\int \frac{1-t}{(1-t)^2 + (y-\xi)^2}[g(\xi) - g(y)]\,d\xi.$$

Let $\xi - y = w$; when

$$\int |T_t f(y) - g(y)|dy \leq \frac{1}{\pi}\int dy \int \frac{1-t}{(1-t)^2 + w^2}|g(y+w) - g(y)|dw$$

$$= \frac{1}{\pi}\int \frac{1-t}{(1-t)^2 + w^2}dw \int |g(y+w) - g(y)|dy.$$

Let $\phi(w) = \int |g(y+w) - g(y)|dy$; then, since $g(y) \in L$, we have $\lim_{w \to 0} \phi(w) = 0$ and

$$\lim_{t \to 1} \int \frac{(1-t)\phi(w)dw}{(1-t)^2 + w^2} = \phi(0) = 0.$$

Hence $\lim_{t \to 1} \int |T_t f(y) - g(y)|dy = 0$, and this implies (d).

For the sufficiency we observe that conditions (a) to (c) guarantee a representation in the form (1), and that we may assume $\alpha(y)$ to be normalized [2, p. 16]. By the inversion formula for (1) [1, p. 549],

$$\lim_{t \to 1} \int_0^x T_t f(u)du = \alpha(x),$$

and by (d), $T_t f(y)$ converges in the mean to a function $g(y)$; hence

$$\lim_{t\to 1} \int_0^x T_t f(u)\,du = \int_0^x g(u)\,du .$$

Since $\alpha(x)$ is an integral, it has a derivative, almost everywhere, which is $g(x)$. Accordingly,

$$f(x) = \frac{1}{\pi}\int \frac{g(y)\,dy}{1 + (x - y)^2}.$$

THEOREM 3. *The following set of conditions is necessary and sufficie* $f(x)$ *to be representable in the form* (2) *with* $|g(y)| \le B$:

(a) $f(x)$ *and* $\hat{f}(x) \in C^\infty$ *for all real* x;

(b) $|f^{(k)}(x)| \le Bk!$, $|\hat{f}^{(k)}(x)| \le B(k + 1)!$, *where* B *is independent of* k *a*

(e) $|T_t f(x)| \le B$.

Proof. The necessity of (a) and (b) are shown in [1, pp. 543-547]. Since

$$T_t f(x) = \frac{1}{\pi}\int \frac{(1 - t)g(y)\,dy}{(1 - t)^2 + (x - y)^2},$$

$$|T_t f(x)| \le \frac{B}{\pi}\int \frac{(1 - t)\,dy}{(1 - t)^2 + (x - y)^2} = B.$$

For the sufficiency we need two lemmas.

LEMMA 3.1. *Let* $f(x)$ *satisfy* (a) *and* (b) *and let* $\{f_n(x)\}$ *be a sequence tions satisfying the conditions*

$$|f_n(x)| \le B, \quad |f_n{}^{('')}(x)| \le B,$$
$$f_n(x) = 0 \quad (|x| > n),$$
$$\lim_{n\to\infty} f_n(x) = f(x);$$

then $\hat{f}_n(x)$ *exists,* $|\hat{f}_n(x)| \le B$, *and* $\lim_{n\to\infty} \hat{f}_n(x) = \hat{f}(x)$.

Proof. We first observe that there exists a sequence $\{f_n(x)\}$ satisfying hypotheses of the lemma. An example is furnished by the functions

$$f_n(x) = \begin{cases} f(x) & (|x| \le n - 1), \\ 0 & (|x| > n), \\ a_{n0} + a_{n1}(x - n + 1) + a_{n2}(x - n + 1)^2 + a_{n3}(x - n + 1)^3 + a_{n4}(x \cdot & \\ \quad + a_{n5}(x - n + 1)^5 & (n - 1 \le x \le n), \\ b_{n0} + b_{n1}(x + n - 1) + b_{n2}(x + n - 1)^2 + b_{n3}(x + n - 1)^3 + b_{n4}(x + & \\ \quad + b_{n5}(x + n - 1)^5 & (-n \le x \le -n + 1), \end{cases}$$

where the a_{nk} are determined by the conditions

$$f_n(n-1) = f(n-1),\ f_n'(n-1) = f'(n-1),\ f_n^{('')}(n-1) = f^{('')}(n-1),$$
$$f_n(n) = f_n'(n) = f_n''(n) = 0,$$

and the b_{nk} are determined by replacing $n - 1$ by $1 - n$ and n by $-n$ in the arguments of the functions above. Utilizing again the techniques which were employed in demonstrating the boundedness of $\hat{f}_\varepsilon(x)$ in the previous theorem, we can show that $|\hat{f}_n(x)| < 5B$. By dominated convergence,

$$\lim_{n\to\infty} \int_0^\infty u^{-2}[f_n(x+u) + f_n(x-u) - 2f_n(u)]du = \hat{f}(x),$$

since the integral is majorized by

$$F(u) = \begin{cases} 2B & (0 \le u < 1), \\ 3B/u^2 & (1 \le u < \infty). \end{cases}$$

LEMMA 3.2. *Let* $\hat{f}(x)$ *satisfy* (a) *and* (b) *of Theorem* 3; *then*

$$\int \frac{\hat{f}^{(k)}(y)\,dy}{1+(x-y)^2} = \int f(y)dy \int e^{iu(x-y)} (iu)^k |u| e^{-|u|} du.$$

Proof. By successive integrations by parts, the left-hand side of the equation becomes

$$(-1)^k \int \hat{f}(y) \frac{d^k}{dy^k}\left(\frac{1}{1+(x-y)^2}\right) dy.$$

But since $\hat{f}_n(x)$ is bounded and $f_n(x)$ is integrable,

$$\lim_{n\to\infty} \int \hat{f}_n(y) \frac{d^k}{dy^k}\left(\frac{1}{1+(x-y)^2}\right) dy$$
$$= \lim_{n\to\infty} \int f_n(y)dy \int e^{iu(x-y)} (iu)^k |u|\, e^{-|u|} du$$
$$= \int f(y)dy \int e^{iu(x-y)} (iu)^k |u|\, e^{-|u|} du,$$

the last step being justified since $|f(y)| < B$ and the function

$$\int e^{iu(x-y)} (iu)^k |u|\, e^{-|u|} du$$

is in L [1, p. 546].

We may now establish, by arguments similar to those employed in Theorem 1, that

$$\lim_{t \to 1} \frac{1}{\pi} \int \frac{T_t f(y) dy}{1 + (x - y)^2} = f(x).$$

To complete the proof of the sufficiency, we observe that (e) guarantees the of a sequence $\{t_k\}$ ($t_k \to 1$ as $k \to \infty$) and a bounded function $g(x)$ such that

$$\lim_{k \to \infty} \frac{1}{\pi} \int \frac{T_{t_k} f(y) dy}{1 + (x - y)^2} = \frac{1}{\pi} \int \frac{g(y) dy}{1 + (x - y)^2}.$$

Hence

$$\frac{1}{\pi} \int \frac{g(y) dy}{1 + (x - y)^2} = f(x).$$

THEOREM 4. *The following set of conditions is necessary and sufficie* $f(x)$ *to be representable in the form* (1) *with* $\alpha(y)$ *nondecreasing and bounde*

(a) $f(x)$ *and* $\hat{f}(x) \in C^{\infty}$ *for all real* x;

(b) $|f^{(k)}(x) \leq Bk!$, $|\hat{f}^{(k)}(x)| \leq B(k + 1)!$, *where* B *is independent of* k *a*

(f) $f(x) \in L \; (-\infty, \infty)$;

(g) $T_t f(x) \geq 0 \quad (|t| < 1)$.

Proof. Since a bounded nondecreasing function is of bounded variation, and (f) are necessary for the representation. Since

$$T_t f(x) = \frac{1}{\pi} \int \frac{(1 - t) d\alpha(y)}{(1 - t)^2 + (x - y)^2}$$

with $d\alpha(y) \geq 0$, (g) is necessary. For the sufficiency, we may employ the used in Theorem 3 to obtain the relation

$$\int \frac{T_t f(y) dy}{1 + (x - y)^2} = \int \frac{(1 - t) f(y) dy}{(1 - t)^2 + (x - y)^2}.$$

We need now only establish that $\int T_t f(y) dy < B$, and the result will foll Theorem 1; but

$$(8) \qquad \begin{aligned} \int T_t f(y) dy &= \frac{1}{\pi} \int T_t f(y) dy \int \frac{dx}{1 + (x - y)^2} \\ &= \frac{1}{\pi} \int dx \int \frac{T_t f(y) dy}{1 + (x - y)^2} \\ &= \frac{1}{\pi} \int dx \int \frac{(1 - t) f(y) dy}{(1 - t)^2 + (x - y)^2} \\ &= \frac{1}{\pi} \int f(y) dy \int \frac{(1 - t) dx}{(1 - t)^2 + (x - y)^2} \\ &= \int f(y) dy < B, \text{ by (f)}. \end{aligned}$$

THEOREM 5. *The following set of conditions is necessary and sufficient for* f(x) *to be representable in the form* (2) *with* g(x) *in* L_p $(-\infty, \infty)$:

(a) f(x) *and* $\hat{f}(x) \in C^\infty$ *for all real* x;

(b) $|f^{(k)}(x)| \leq Bk!$, $|\hat{f}^{(k)}(x)| \leq B(k+1)!$, *where* B *is independent of* k *and* x;

(h) $\int |T_t f(y)|^p dy < B \quad (|t| < 1)$.

Proof. For the necessity, we need only establish (h), since the necessity of (a) and (b) may be deduced from [1, pp. 543-547].

$$\int |T_t f(y)|^p dy \leq \int dy \left| \int \frac{(1-t)g(u)du}{(1-t)^2 + (y-u)^2} \right|^p$$

$$\leq \int dy \left| \int \left(\frac{1-t}{(1-t)^2 + (y-u)^2} \right)^{1/q} \left(\frac{1-t}{(1-t)^2 + (y-u)^2} \right)^{1/p} |g(u)| \, du \right|^p,$$

and by Hölder's inequality the integral above is not greater than

$$\int dy \left(\int \frac{(1-t)|g(u)|^p du}{(1-t)^2 + (y-u)^2} \right) \left(\int \frac{(1-t)du}{(1-t)^2 + (y-u)^2} \right)^{p/q};$$

but

$$\int \frac{(1-t)du}{(1-t)^2 + (y-u)^2} \leq \pi,$$

and hence

$$\int |T_t f(y)|^p dy \leq \int |g(u)|^p du \int \frac{(1-t)dy}{(1-t)^2 + (y-u)^2} \leq B.$$

For the sufficiency we may proceed as in Theorems 3 and 4 to establish

$$\lim_{t \to 1} \frac{1}{\pi} \int \frac{T_t f(y) dy}{1 + (x-y)^2} = f(x). \tag{9}$$

By a selection theorem [4, p. 130], (h) guarantees the existence of a function g(x) in L_p and a sequence $\{t_k\}$ $(t_k \to 1)$ such that

$$\lim_{k \to \infty} \frac{1}{\pi} \int \frac{T_{t_k} f(y) dy}{1 + (x-y)^2} = \frac{1}{\pi} \int \frac{g(y) dy}{1 + (x-y)^2};$$

but by (9) the expression above is equal to f(x).

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