GENERALIZED FINITE FOURIER COSINE TRANSFORMS

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1. THE BASIC OPERATIONAL PROPERTY

The characteristic functions of the Sturm-Liouville problem

(1) $$y''(x) + k^2 y(x) = 0, \quad y'(0) = 0, \quad hy(1) + y'(1) = 0,$$

where h is a prescribed constant other than zero, are the functions

(2) $$y = \cos k_n x \quad (n = 1, 2, \cdots),$$

corresponding to the characteristic numbers $k = k_n$, which are the positive roots of the equation

(3) $$h \cos k_n = k_n \sin k_n.$$

The operational mathematics of the linear integral transformation

(4) $$T\{F(x)\} = \int_0^1 F(x) \cos k_n x \, dx = f(n) \quad (n = 1, 2, \cdots),$$

where $F(x)$ denotes any bounded integrable function on the interval $(0,1)$, will be developed here. According to the Sturm-Liouville theory, the inverse of the transformation (4) is given by the generalized Fourier series

(5) $$F(x) = \sum_{n=1}^{\infty} N_n^{-1} f(n) \cos k_n x \quad \left(0 < x < 1, \; N_n = \int_0^1 \cos^2 k_n x \, dx\right),$$

whenever $F(x)$ satisfies conditions under which it is represented by its ordinary Fourier cosine series on the interval $(0, 1)$.

Now let $F(x)$ denote any function of class C' whose derivative $F''(x)$ is bounded and integrable on the interval. Then the basic operational property

(6) $$T\{F''(x)\} = -k_n^2 f(n) - F'(0) + [hF(1) + F'(1)] \cos k_n$$

can be obtained from the integral $T\{F''(x)\}$ by two successive integrations by parts. Thus the transformation resolves the differential form $F''(x)$ in terms of the transform $f(n)$ and the two boundary values $F'(0)$ and $hF(1) + F'(1)$ that appear in the Sturm-Liouville problem (1). This is a characteristic operational property of the transformation (4) in the sense that the kernel $\cos k_n x$ is prescribed by that resolution [1].

Boundary values of the type $hF(1) + F'(1)$ appear in a broad class of physical problems. In particular, they arise in connection with elastically supported boundaries of elastic bodies and in connection with surface transfer of a diffusing

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substance into an adjacent medium. Aside from its utility in solving such p however, our transformation represents one of the simplest finite integral formations whose kernel has no common period for all values of the param The operational properties of the common finite Fourier transformations d upon such periodicity. Methods used in developing operational properties c transformation (4) suggest procedures that may apply to other finite integr formations, such as Hankel transformations, whose kernels are not periodi

We present an elementary convolution property of our transformation, formula for the inverse transform of the product of two transforms. Our t tion is a special case of the finite Fourier transformations treated by Roet whose convolution property is written in terms of almost periodic extensio functions from the finite interval to the infinite interval. Our methods app applicable to those more general finite Fourier transformations.

2. DUAL TRANSFORMATIONS

Throughout this section $F(x)$ will denote any bounded integrable functic interval $(0, 1)$. Elementary operational properties of the transformation T written conveniently in terms of certain duals of that transformation.

Let the dual transformation S be defined as follows:

$$S\{F(x)\} = \int_0^1 F(x) \sin k_n x \, dx, \tag{7}$$

where the k_n are again the positive roots of equation (3). The kernel $z =$ and the numbers k_n are the characteristic functions and numbers of the m Sturm-Liouville problem [3]

$$z''(x) + k^2 z(x) = 0, \quad z(0) = 0, \quad h z'(1) + z''(1) = 0.$$

The transformations (7) and (4) are easily shown to be related by the form

$$S\{F(x)\} = k_n T\left\{\int_x^1 F(t)\,dt\right\}. \tag{8}$$

A second dual transformation is one with the kernel $w = \cos k_n(1 - x)$, represents the characteristic functions of the problem

$$w''(x) + k^2 w(x) = 0, \quad h w(0) - w'(0) = 0, \quad w'(1) = 0 \quad (k = k_n).$$

This transformation can be written

$$\int_0^1 F(x) \cos k_n(1 - x)\,dx = T\{F(1 - x)\}. \tag{9}$$

From the addition formula for $\cos k_n(1 - x)$, it follows that

$$T\{F(1 - x)\} = T\{F(x)\} \cos k_n + S\{F(x)\} \sin k_n,$$

and from equations (8) and (3) we can then obtain the formula

$$T\{F(1-x)\} = \cos k_n\, T\left\{F(x) + h\int_x^1 F(t)\,dt\right\}. \tag{10}$$

If we write $G(x)$ for the function inside the braces on the right of equation (10) then, formally, $G'(x) = F'(x) - hF(x)$ and $G(1) = F(1)$, from which we can solve for $F(1-x)$. This leads to the operation on a function that corresponds to the multiplication of the transform by $\cos k_n$. We replace $G(x)$ by $F(x)$ and write that relation as

$$\cos k_n T\{F(x)\} = T\{F(1-x) - h e^{-hx} * F(1-x)\}, \tag{11}$$

where the asterisk denotes the convolution used with Laplace transforms:

$$F_1(x) * F_2(x) = \int_0^x F_1(x-t)F_2(t)\,dt = \int_0^x F_2(x-t)F_1(t)\,dt. \tag{12}$$

By writing out the transform on the right of equation (11), inverting the order of the iterated integral and using relations (8) and (3), we can show that formula (11) is correct whenever $F(x)$ is bounded and integrable.

It is convenient to introduce a third dual transformation

$$U\{F(x)\} = \int_0^1 F(x)\cos k_n(1+x)\,dx. \tag{13}$$

Since

$$\begin{aligned} 2\cos k_n \int_0^1 F(x)\cos k_n x\,dx &= \int_0^1 F(x)\cos k_n(1+x)\,dx + \int_0^1 F(x)\cos k_n(1-x)\,dx \\ &= U\{F(x)\} + T\{F(1-x)\}, \end{aligned}$$

it follows from formula (11) that

$$U\{F(x)\} = T\{F(1-x) - 2h e^{-hx} * F(1-x)\}. \tag{14}$$

The transformation T is a special case of the Sturm-Liouville transformation [1]; therefore it has the properties of that transformation. In particular, the image of the function $-f(n)/k_n^2$ is the function $Y(x)$ that satisfies the conditions

$$Y''(x) = F(x), \quad Y'(0) = 0, \quad hY(1) + Y'(1) = 0.$$

Thus

$$\frac{f(n)}{k_n^2} = T\left\{\int_x^1 \int_0^r F(t)\,dt\,dr + \frac{1}{h}\int_0^1 F(t)\,dt\right\}. \tag{15}$$

3. CONVOLUTION

Let $F(x)$ and $G(x)$ denote bounded integrable functions on the interval
Then the product of their transforms can be written

$$f(n)g(n) = \int_0^1 \int_0^1 F(x)G(y)\cos k_n x \cos k_n y \, dx\,dy = \frac{1}{2}\sum_{i=1}^{4} I_i, \tag{16}$$

where the integrals I_i are defined as follows:

$$I_i = \begin{cases} \displaystyle\iint_{R_i} F(x)G(y)\cos k_n(x - y)\,dx\,dy & \text{when } i = 1, 2, \\ \displaystyle\iint_{R_i} F(x)G(y)\cos k_n(x + y)\,dx\,dy & \text{when } i = 3, 4, \end{cases}$$

and the triangular regions R_i are those halves of the unit square $0 < x <$
that lie on either side of one of the diagonals. The order of numeration is as follows.

$$R_1\colon y < x; \quad R_2\colon y > x; \quad R_3\colon x + y < 1; \quad R_4\colon x + y > 1.$$

Upon introducing new variables of integration in each of the iterated in
we find that

$$\begin{aligned} I_1 &= \int_0^1 \cos k_n s \int_0^{1-s} F(s + t)G(t)\,dt\,ds = T\left\{\int_0^{1-x} F(x + t)G(t)\,dt\right\}, \\ I_2 &= \int_0^1 \cos k_n s \int_s^1 F(t - s)G(t)\,dt\,ds = T\left\{\int_x^1 F(t - x)G(t)\,dt\right\}, \\ I_3 &= \int_0^1 \cos k_n s \int_0^s F(s - t)G(t)\,dt\,ds = T\left\{\int_1^x F(x - t)G(t)\,dt\right\}, \\ I_4 &= \int_1^2 \cos k_n s \int_{s-1}^1 F(s - t)G(t)\,dt\,ds. \end{aligned} \tag{17}$$

In order to write the integral I_4 as a transform, we introduce a new varial
$x = s - 1$; then, in view of the definition (13) of the transformation U,

$$I_4 = \int_0^1 \cos k_n(1 + x) \int_x^1 F(1 + x - t)G(t)\,dt\,dx = U\left\{\int_x^1 F(1 + x - t)G(t\right.$$

Formula (14) now enables us to write

$$I_4 = T\left\{\int_{1-x}^1 F(2 - x - t)G(t)\,dt - 2he^{-hx} * \int_{1-x}^1 F(2 - x - t)G(t)\,dt\right. \tag{18}$$

and, in view of equations (16) and (17), $f(n)g(n)$ can be written as a transform.

To simplify the result, we introduce the function $F_0(x)$ as the even periodic extension of $F(x)$, with period 2; that is,

(19) $F_0(x) = F(x)$ when $0 < x < 1$, $F_0(-x) = F_0(x)$ and $F_0(x+2) = F_0(x)$ for all x.

Then in equation (17), $F(t - x) = F_0(x - t)$, since $0 \leq x \leq 1$. In equation (18), $F(2 - x - t) = F_0(x + t)$, and equation (16) can be written

$$f(n)g(n) = T\left\{\frac{1}{2}\int_0^1 [F_0(x+t) + F_0(x-t)]G(t)dt - he^{-hx} * \int_{1-x}^1 F_0(x+t)G(t)dt\right\}. \tag{20}$$

This is one form of the convolution property for the transformation T. Elementary changes in the form of the last term in the braces lead to other forms of the convolution corresponding to the transformation T, two of which are given in the following theorem.

THEOREM. *Under the transformation* (4) *the image of the product of the transforms of two bounded integrable functions* $F(x)$ *and* $G(x)$ *is the convolution* $X[F, G]$, *that is,*

$$f(n)g(n) = T\{X[F, G]\}, \tag{21}$$

where the convolution of $F(x)$ *and* $G(x)$ *can be written*

$$\begin{aligned} X[F, G] &= \frac{1}{2}\int_0^1 [F_0(x+t) + F_0(x-t)]G(t)dt - he^{-hx} * F(1-x) * G(1-x) \\ &= \frac{1}{2}\int_0^1 [F_0(x+t) + F_0(x-t)]G(t)dt - h\int_{1-x}^1 F_0(x+t)G_h(t)dt. \end{aligned} \tag{22}$$

Here $F_0(x)$ *is the even periodic extension* (19) *of* $F(x)$ *with period* 2, *the asterisk denotes the convolution* (12), *and*

$$G_h(x) = e^{hx}\int_x^1 e^{-ht}G(t)dt. \tag{23}$$

4. TRANSFORMS OF ELEMENTARY FUNCTIONS

From the theory of Sturm-Liouville transforms [1], it follows that $-k_n^{-2}\cos k_n c$, where c is a parameter such that $0 \leq c \leq 1$, is the transform of Green's function $G(x, c)$ of problem (1) with $k = 0$. Thus

$$T\{G(x, c)\} = -\frac{\cos k_n c}{k_n^2} \quad (0 \leq c \leq 1),$$

where

(24) $G(x, c) = c - 1 - 1/h$ when $x \leq c$, $G(x, c) = x - 1 - 1/h$ when $x \geq c$.

The operational properties presented in the preceding sections are usef writing transforms of particular functions. The application of formula (6), ample, to the function $F(x) = \cos cx$, where c is a constant $(c^2 \neq k_n^2)$, gives

$$-c^2 f(n) = -k_n^2 f(n) + (h \cos c - c \sin c)\cos k_n,$$

or

$$T\{\cos cx\} = (h \cos c - c \sin c)\frac{\cos k_n}{k_n^2 - c^2} \quad (c^2 \neq k_n^2).$$

By means of such procedures and elementary integrations, we can cons table of transforms. A short table is shown here. Formulas (6), (11), (15) serve to extend any such table.

TABLE OF TRANSFORMS

$F(x)$	$f(n) = T\{F(x)\}$
1. 1	$k_n^{-1} \sin k_n \quad (= h k_n^{-2} \cos k_n)$
2. $J(x, c) = 1 \quad (0 \leq x < c)$, $J(x, c) = 0 \quad (c < x \leq 1)$	$k_n^{-1} \sin k_n c$
3. $1 + h^{-1} - x$	k_n^{-2}
4. $G(x, c)$ [see eq. (24), $0 \leq c \leq 1$]	$-k_n^{-2} \cos k_n c$
5. e^{-hx} [h in eq. (3)]	$h(h^2 + k_n^2)^{-1}$
6. $\sinh c \quad (c^2 \neq -k_n^2)$	$[(h \sinh c + c \cosh c)\cosh k_n - c]$
7. $\cosh c \quad (c^2 \neq -k_n^2)$	$(h \cosh c + c \sinh c)(c^2 + k_n^2)^{-1}$ cc
8. $h \sinh c(1 - x) + c \cosh c(1 - x)$ $(c^2 \neq -k_n^2)$	$(h \cosh c + c \sinh c)c(c^2 + k_n^2)^{-1}$
9. $G_1(x, c) = 1 \quad (x \leq c)$, $G_1(x, c) = e^{h(c-x)} \quad (c \leq x \leq 1)$	$h k_n^{-2} \cos k_n \cos k_n(1 - c)$
10. $2G_1(x, c) + hG(x, c) \quad (0 \leq c \leq 1)$	$h k_n^{-2} \cos k_n(2 - c)$
11. $G_2(x, c) = 0 \quad (x < c)$, $G_2(x, c) = e^{h(c-x)} \quad (c < x \leq 1)$	$k_n^{-1} \cos k_n \sin k_n(1 - c)$
12. $2G_2(x, c) + J(x, c) \quad (0 \leq c \leq 1)$	$k_n^{-1} \sin k_n(2 - c)$

5. A BOUNDARY VALUE PROBLEM

As an illustration of the use of the preceding operational mathematics, consider the following problem in the longitudinal displacements $Y(x, t)$ of an elastic bar. One end, $x = 0$, of the bar is free; the other end, taken as $x = 1$, is elastically supported and subject to a force that may vary with time. If a general longitudinal body force is allowed for and the bar is initially at rest and unstrained, and if the unit of time t is properly chosen, the displacements $Y(x, t)$ satisfy the conditions

$$(25)\qquad \begin{aligned} &Y_{tt}(x, t) = Y_{xx}(x, t) + F(x, t), \quad Y(x, 0) = Y_t(x, 0) = 0, \\ &Y_x(0, t) = 0, \quad Y_x(1, t) = -hY(1, t) + P(t), \end{aligned}$$

where the positive constant h depends on the coefficients of elasticity of the support and the bar.

The transformation T, with respect to x, can be applied to this problem because the differential form and boundary values with respect to x are $Y_{xx}(x, t)$, $Y_x(0, t)$ and $hY(1, t) + Y_x(1, t)$. Let

$$y(n, t) = T\{Y(x, t)\} = \int_0^1 Y(x, t)\cos k_n x \, dx .$$

In view of formula (6), the image of problem (25) is, formally,

$$(26)\qquad \ddot{y}(n, t) = -k_n^2 y(n, t) + P(t)\cos k_n + f(n, t), \quad y(n, 0) = \dot{y}(n, 0) = 0,$$

where the dots denote derivatives with respect to t, and where $f(n, t)$ is the transform of $F(x, t)$. The solution of problem (26) is

$$(27)\qquad y(n, t) = \int_0^t \left[\frac{\sin k_n(t - \tau)}{k_n} f(n, \tau) + \cos k_n \frac{\sin k_n(t - \tau)}{k_n} P(\tau)\right] d\tau \quad (t \geq 0).$$

One form of the solution of problem (25) follows from the inversion formula (5); that is,

$$Y(x, t) = \sum_{n=1}^{\infty} N_n^{-1} y(n, t)\cos k_n x .$$

But for each finite range of the variable t, the properties of our transformation T enable us to write $Y(x, t)$ in closed forms, and these forms can be verified as solutions when the prescribed functions $F(x, t)$ and $P(t)$ are sectionally continuous.

When $0 \leq t \leq 1$, then the variable $t - \tau$ in the integrand in equation (27) is confined to that range, and the coefficients of $f(n, \tau)$ and $P(\tau)$ are transforms of functions shown in entries No. 2 and No. 11 of the Table of Transforms. With the aid of the convolution property (21), it follows from equation (27) that

$$(28)\qquad Y(x, t) = \int_0^t \{X[J(x, t - \tau), F(x, \tau)] - G_2(x, 1 - t + \tau)P(\tau)\} d\tau \quad (0 \leq t \leq 1).$$

When $1 \leq t \leq 2$, let the integral of the first term inside the brackets in (27) be written

$$\int_0^{t-1} \frac{\sin k_n(t-\tau)}{k_n} f(n, \tau)d\tau + \int_{t-1}^{t} \frac{\sin k_n(t-\tau)}{k_n} f(n, \tau)d\tau .$$

In the first integrand $1 \leq t - \tau \leq t$; thus the inverse transform of $k_n^{-1} \sin k$ is given by No. 12 in the table. In the second integrand $0 \leq t - \tau \leq 1$, and the table applies again. The convolution therefore gives the inverse transf these integrals. The integral of the second term in equation (27) can be br the corresponding sum, whose inverse transforms are given by No. 11 and the table and by property (11). Thus $Y(x, t)$ can be written in closed form $1 \leq t \leq 2$.

For higher integral ranges of t, the additional transforms needed to wı can be found by first writing

$$2 \cos k_n \frac{\sin k_n(2-c)}{k_n} = \frac{\sin k_n(3-c)}{k_n} + \frac{\sin k_n(1-c)}{k_n} \qquad (0 \leq c \leq 1).$$

From No. 2, No. 12, and property (11), the inverse transform of $k_n^{-1} \sin k_n$ follows. Iterations of the procedure give the inverse transform of k_n^{-1} sin $m \leq t \leq m + 1$ $(m = 3, 4, \cdots)$.

Let us write $Y(x, t)$ for $0 \leq t \leq 1$ in the case where $P(t) = 0$ and $F(x,$ Then the transform (27) reduces to

$$y(n, t) = \frac{f(n)}{k_n^2} - \frac{\cos k_n t}{k_n^2} f(n), \tag{29}$$

and formula (15) gives the inverse transform of the first term on the right convolution property and No. 4 in the table give the inverse of the second.

$$Y(x, t) = \int_x^1 \int_0^r F(s)ds\,dr + \frac{1}{h}\int_0^1 F(s)ds + X[G(x, t), F(x)] \qquad (0 \leq t \leq 1 \tag{30}$$

where $G(x, t)$ is defined by equation (24). When $1 \leq t \leq 2$, the function G((30) should be replaced by $2h^{-1}G_1(x, 2-t) + G(x, 2-t)$, according to entr in the table. When $F(x) = 1$, formula (30) reduces to

$$Y(x, t) = \begin{cases} \frac{1}{2}t^2 & \text{when } x \leq 1 - t, \\ -\frac{1}{2}x^2 + \frac{x}{h}(1 + h - ht) + \frac{t}{h}(1 + h) - \frac{1}{h^2}\left(1 + h + \frac{h^2}{2}\right) + \frac{1}{h^2}e^{h(1-x-t)} & \text{wher} \end{cases}$$

where $0 \leq t \leq 1$.

6. OBSERVATIONS

The preceding development of operational properties of the transformation (4) depended on the Sturm-Liouville problem (1) that generates the kernel $\cos k_n x$, and on addition and product properties of trigonometric kernels. These properties themselves can be found directly from the Sturm-Liouville equation and one of the boundary conditions. The function $y = \cos kx$ is the solution of the system

$$y''(x) + k^2 y(x) = 0, \quad y'(0) = 0, \quad y(0) = 1, \tag{31}$$

where the last condition is a matter of convenient normalization of $y(x)$.

If s denotes any parameter, it follows from the differential equation in the system (31) that

$$y''(x + s) + k^2 y(x + s) = 0, \tag{32}$$

and upon eliminating k^2 between the two differential equations, we find that

$$y(x)y''(x + s) - y''(x)y(x + s) = \frac{d}{dx}[y(x)y'(x + s) - y'(x)y(x + s)] = 0.$$

Since $y'(0) = 0$ and $y(0) = 1$, then

$$y(x)y'(x + s) - y'(x)y(x + s) = y'(s).$$

In the last equation we differentiate all terms with respect to s, and we use equation (32) to obtain the relation

$$k^2 y(x)y(x + s) + y'(x)y'(x + s) = k^2 y(s).$$

The addition property for the function $y = \cos kx$ follows when we substitute $r - x$ for s:

$$y(r - x) = y(r)y(x) + \frac{1}{k^2} y'(r)y'(x). \tag{33}$$

Formula (33), with $r = 0$, shows that $y(x)$ is an even function, and hence $y'(x)$ is odd; that is,

$$y(-x) = y(x), \quad y'(-x) = -y'(x). \tag{34}$$

Then, in view of the addition formula (33),

$$2y(r)y(x) = y(r + x) + y(r - x), \tag{35}$$

which is the product formula for the function $y = \cos kx$.

The addition and product formulas for $\sin kx$ and for the product $\sin kr \cos kx$ follow from formulas (33) and (35) by differentiation with respect to r and x.

The simplicity of the derivations here is due, of course, to the fact that the coefficients in the differential equation (31) are constants. But if the development of the operational properties for the transformation T could be described in a more unified way, in terms of the Sturm-Liouville equation and boundary conditions, it seems plausible that the program may apply to other integral transformati even to general Sturm-Liouville transformations [1].

The advantages of a convolution property depend on the simplicity of th tion and on the existence of a satisfactory table of transforms. Although th volution derived in Section 3 is elementary, its application is likely to be te If the inverse transform of certain exponential functions such as $k_n^{-2}\exp(-c$ known in closed forms, then some quite general boundary value problems i son's equation, including problems that are adapted only to the transformat could be solved in closed form with the aid of the operational mathematics in this paper.

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