FUNCTIONS OF BOUNDED CHARACTERISTIC WITH PRESCRIBED AMBIGUOUS POINTS

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Let f(z) be a complex-valued uniform function defined for $|z| < 1$. We shall call a point ζ an *ambiguous point* for f(z), if $|\zeta| = 1$ and there exist two Jordan arcs J_1 and J_2, terminating in ζ and lying, except for ζ, in $|z| < 1$, such that

$$\lim_{\substack{z \to \zeta \\ z \in J_1}} f(z) \quad \text{and} \quad \lim_{\substack{z \to \zeta \\ z \in J_2}} f(z)$$

both exist and are unequal.

It has recently been shown [1, p. 382] that even if no further conditions are imposed on f(z), there are at most enumerably many ambiguous points for f(z), and it is well-known [2, p. 66] that if f(z) is regular and bounded in $|z| < 1$, there are no ambiguous points for f(z). On the other hand, corresponding to every enumerable set E on $|z| = 1$ there exist [1, p. 381] regular functions in $|z| < 1$ for which every point of E is an ambiguous point, and it is thus natural to ask whether such regular functions can, in some sense, be "nearly" bounded. Now, regular functions of bounded characteristic possess [2, pp. 185, 208, 209] some of the important boundary properties of bounded regular functions. The following result shows, however, that the two classes of functions are quite different in respect to the existence of ambiguous points.

THEOREM. *Let*

$$E = \{\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots\}$$

be an enumerable set of points on $|z| = 1$. *Then there exists a function* f(z), *regular and of bounded characteristic in* $|z| < 1$, *for which every element of* E *is an ambiguous point.*

Proof. A function g(z)/h(z), where g(z) and h(z) are bounded and regular in $|z| < 1$ and $h(z) \neq 0$ in $|z| < 1$, is a regular function of bounded characteristic in $|z| < 1$ [2, p. 189]; we shall obtain an f(z) which is of this form and satisfies the conclusion of the theorem. To this end, it is obviously sufficient to construct g(z) and h(z) so that they satisfy the following conditions:

(I) *There exists a constant* $b > 0$ *such that, in* $|z| < 1$,

$$|g(z)| > b(1 - |z|)^2,$$

and $g(z) \to 0$ *as* z *tends to an arbitrary point of* E.

Received September 6, 1955.
W. Seidel's contribution to this paper was made under the sponsorship of the National Science Foundation.

(II) *For every* $\zeta_n \in E$ *there exist two circular arcs* C_n *and* D_n *in* $|z|$ *terminating in* ζ_n, *and two positive constants* c_n *and* d_n, *such that, for* z *ciently close to* ζ_n,

$$|h(z)| < \exp[c_n/(|z| - 1)] \tag{1}$$

on C_n, *whereas on* D_n

$$|h(z)| > d_n. \tag{2}$$

For then, for every $\zeta_n \in E$,

$$\lim_{\substack{z \to \zeta_n \\ z \in C_n}} f(z) = \infty \quad \text{and} \quad \lim_{\substack{z \to \zeta_n \\ z \in D_n}} f(z) = 0.$$

To obtain g(z), we modify an argument appearing in [3, pp. 294-295]; w it in some detail, since [3] is rather inaccessible. For this part of our con it is not necessary to assume that E is enumerable, but merely that it is o zero.

Let $E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots \supset E$, where E_0 is the interval $0 \leq \theta$ for $n = 1, 2, 3, \cdots$, E_n is open and meas $E_n < n^{-3}$. Denote by $\chi_{E_n}(\theta)$ the c istic function of E_n, and set

$$G(\theta) = \sum_{n=0}^{\infty} \chi_{E_n}(\theta).$$

Then it is clear that, for every $\theta_0 \in E$, $G(\theta_0) = +\infty$ and $G(\theta) \to +\infty$ as $\theta \to \theta_0$. over, $G(\theta) = +\infty$ only if $\theta \in \bigcap_{n=0}^{\infty} E_n$, and this set is of measure 0. Since $\sum_{n=1}^{\infty} n \cdot \text{meas}(E_{n-1} - E_n) < +\infty$, $G(\theta)$ is integrable. Let us form

$$u(r\,e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)}d\theta \qquad (0 \leq r < 1,\ 0 \leq \phi < 2$$

Since $G(\theta) \geq 1$ $(0 \leq \theta \leq 2\pi)$, the function u(z) $(z = re^{i\phi})$ is harmonic and in $|z| < 1$. We have

$$u(re^{i\phi}) \leq \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\frac{1 + r}{1 - r}d\theta \leq \beta/(1 - r),$$

where β is a positive constant. If $\theta_0 \in E$, θ_0 is a point of continuity (in the sense) of $G(\theta)$, and therefore $u(z) \to +\infty$ as $z \to e^{i\theta_0}$. A harmonic conjugate in $|z| < 1$ is

$$v(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\frac{2r\sin(\phi - \theta)}{1 + r^2 - 2r\cos(\phi - \theta)}d\theta,$$

and from this representation of $v(z)$ it readily follows that

$$|v(re^{i\phi})| \leq \gamma/(1 - r)^2,$$

where γ is a positive constant. The function $g(z) = [u(z) + i\,v(z)]^{-1}$ now obviously satisfies (I).

To obtain $h(z)$, we first map $|z| < 1$, by means of a linear transformation $w = L(z)$, onto $\Im w > 0$, in such a manner that some point which, together with its negative, lies in the complement of E, is carried into the point $w = \infty$. The set E is thereby transformed into an enumerable set of real numbers $S = \{\omega_1, \omega_2, \cdots, \omega_n, \cdots\}$.

By induction on n, we shall set up a one-to-one correspondence between S and a set of nonzero rational numbers $T = \{r_1, r_2, \cdots, r_n, \cdots\}$ such that, for every n, r_n corresponds to ω_n, the correspondence preserves the natural order of the elements of S and the elements of T, respectively, and, for every pair of distinct natural numbers j and k,

$$|\omega_j - \omega_k| > |r_j - r_k|. \tag{3}$$

We choose r_1 to be 1. Let $n > 1$, and suppose that we have already defined the distinct nonzero rational numbers $r_1, r_2, \cdots, r_{n-1}$ so that they are in the same natural order as the corresponding numbers

$$\omega_1, \omega_2, \cdots, \omega_{n-1} \tag{4}$$

and so that (3) holds for $j < k \leq n - 1$. Now precisely one of the following holds: (a) $\omega_n > \omega_k$ $(k = 1, \cdots, n - 1)$; (b) $\omega_n < \omega_k$ $(k = 1, \cdots, n - 1)$; (c) neither (a) nor (b). If (a) or (b) holds, we choose r_n to be a nonzero rational number satisfying

$$0 < r_n - \max(r_1, \cdots, r_{n-1}) < \omega_n - \max(\omega_1, \cdots, \omega_{n-1})$$

or

$$0 < \min(r_1, \cdots, r_{n-1}) - r_n < \min(\omega_1, \cdots, \omega_{n-1}) - \omega_n,$$

respectively. If (c) holds, and if ω_m denotes the largest of the numbers in (4) that are less than ω_n, and ω_M denotes the smallest of the numbers in (4) that are greater than ω_n, we choose r_n to be a nonzero rational number satisfying

$$0 < r_n - r_m < \omega_n - \omega_m \quad \text{and} \quad 0 < r_M - r_n < \omega_M - \omega_n.$$

This completes the induction.

For every natural number n, let $r_n = p_n/q_n$, where p_n and q_n are integers, $q_n > 0$, and $(p_n, q_n) = 1$; set

$$A_n = 1/n^2 q_n^4; \tag{5}$$

and form the function

$$H(w) = \exp\left(- i \sum_{n=1}^{\infty} \frac{A_n}{w - \omega_n}\right).$$

The series in parentheses is obviously absolutely and uniformly convergen half-plane $\Im w \geq \delta > 0$, so that H(w) is regular for $\Im w > 0$. Furthermore every n, $\Re \frac{i}{w - \omega_n} > 0$ in $\Im w > 0$, and hence

(7) $$|H(w)| < 1$$

in this half-plane.

Now fix a natural number k. Then

(8) $$|H(\omega_k + iv)| < \exp(-A_k/v) \quad (v > 0).$$

Consider the circle Q_k given by the equation

$$w = \omega_k + e^{i\tau} \sin \tau \quad (0 < \tau < \pi),$$

lying in the upper half-plane and tangent to the real axis at ω_k. We shall s H(w) is bounded away from 0 on Q_k.

Still keeping k fixed, we divide the natural numbers other than k into t N_1 consists of those n for which $|\omega_n - \omega_k| \leq \sqrt{3}/2$ and $q_n < q_k$ (N_1 is ev finite set); N_2 consists of those $n \neq k$ for which $|\omega_n - \omega_k| \leq \sqrt{3}/2$ and q_n N_3 consists of those n for which $|\omega_n - \omega_k| > \sqrt{3}/2$. If we set

$$\Sigma_j = \sum_{n \in N_j} A_n \Re\left(\frac{i}{w - \omega_n}\right) \quad (j = 1, 2, 3),$$

then, for $w \in Q_k$,

(9) $$\sum_{n=1}^{\infty} A_n \Re\left(\frac{i}{w - \omega_n}\right) = A_k + \Sigma_1 + \Sigma_2 + \Sigma_3.$$

On Q_k, the function Σ_1 remains bounded, because it consists of a finite nu terms each of which has this property. If $n \in N_2$, $w \in Q_k$, and $\delta_{n,k}$ denote tance between ω_n and Q_k, we have, in view of (3),

$$|w - \omega_n| \geq \delta_{n,k} \geq \frac{2}{3}(\omega_k - \omega_n)^2 \geq \frac{2}{3}(r_k - r_n)^2 \geq \frac{2}{3q_k^2 q_n^2} \geq \frac{2}{3q_n^4},$$

so that, because of (5),

$$\frac{A_n}{|w - \omega_n|} < \frac{3}{2n^2},$$

which implies the boundedness of Σ_2. Finally, on Q_k, we obviously have

$$\Sigma_3 < \frac{2}{\sqrt{3} - 1} \sum_{n=1}^{\infty} A_n.$$

Thus the expression in (9) is bounded on Q_k, which means, by (6), that the constant $d_k > 0$ for which

(10) $$|H(w)| > d_k \quad (w \in Q_k).$$

Now define the regular function

$$h(z) = H(L(z)) \qquad (|z| < 1).$$

Then, because of (6) and (7), $h(z) \neq 0$ and $|h(z)| < 1$ in $|z| < 1$. Let C_n and D_n denote the pre-images, under $w = L(z)$, of the half-line $w = \omega_n + iv$ $(v > 0)$ and the circle Q_n, respectively. According to (10), we have (2) on D_n. Since

$$\lim_{\substack{z \to \zeta_n \\ z \in C_n}} \frac{v}{|z - \zeta_n|} = |L'(\zeta_n)| \quad \text{and} \quad \lim_{\substack{z \to \zeta_n \\ z \in C_n}} \frac{|z - \zeta_n|}{1 - |z|} = 1,$$

(1) follows from (8).

This completes the proof of the theorem.

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